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Extended Bessel Matrix Functions

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Abstract
This work is devoted to the study of some new families of matrix functions which provide a further extension of the extended Bessel matrix functions. In the sequel, some new and interesting properties of these families of k-Bessel matrix functions have been investigated and the connections between k-Bessel matrix functions and k-Laguerre matrix polynomials are indicated in the concluding section of the paper.

Keywords: *Bessel, modified Bessel and Tricomi matrix functions; Bessel's matrix differential equation; Integral representations.*

AMS Subject Classification (2010): *Primary: 15A60, 33C25, 33C45, 33C60.*

1. Introduction

It is well known that the Bessel functions play a major role in mathematics and physics. While on the one hand, they are among the most widely used tools for the application of the theory of complex functions, on the other hand, they are used for solving various problems in mathematical physics, hydrodynamics, electromagnetics and nuclear physics. Bourget's functions are the generalization of Bessel functions, so they are potentially applicable (see [4, 5, 18]).

In the scalar case, we recall that the function $J_{n,k}(z)$ which can be defined as a generalization of the integral representation of the Bessel functions

$$J_{n,k}(z) = \frac{1}{2i\pi}\int_0^1 t^{-n-1}\left(t+\frac{1}{t}\right)^k \exp\left[\frac{1}{2}z\left(t-\frac{1}{t}\right)\right]dt$$

where n is an integer and k is a positive integer.

As usual, I and $\mathbf{O}$ will denote the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. The author has earlier studied the Laguerre, modified Laguerre, Legendre and Konhauser matrix polynomials in [12, 13, 15, 16]. In [6, 7], the Bessel matrix function $J_A(z)$ of the first kind of order A satisfies Bessel's matrix differential equation

$$\left[z^2\frac{d^2}{dz^2}I + z\frac{d}{dz}I + z^2I - A^2\right]J_A(z) = \mathbf{0} \tag{1.1}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ and $J_A(z)$ is a $\mathbb{C}^{N\times 1}$-valued matrix function. Motivated by previous works Jódar et. al. [6, 7], Çekim and Altin [1], Çekim and Erkuş-Duman [2], Sastre and Jódar [11], and Shehata [14, 17], we introduce and study the extended Bessel matrix function in this paper. The important properties, matrix recurrence relations and matrix differential equations of fourth order for the extended Bessel matrix function have been worked out in this paper by employing a novel technique in Section 2. In Section 3, the definitions of the extended modified Bessel and Tricomi matrix function are introduced as a new family of matrix functions and a new generalization of the extended Bessel matrix functions has also been introduced. Finally, we introduce and study to extend the concept of a new k-Bessel matrix functions, and we give the connections between k-Bessel matrix functions of the first kind and k-Laguerre matrix polynomials in Section 4.

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1.1 Preliminaries

First of all, we recall briefly the theorems and definitions for some classes of special matrix functions that have been employed basically throughout this paper. All over this work unless otherwise stated, we assume that A is a matrix in $\mathbb{C}^{N\times N}$, its spectrum is denoted by $\sigma(A)$ where $\sigma(A)$ is the set of all the eigenvalues of A.

Theorem 1.1. *Following [3], if $f(z)$ and $g(z)$ are holomorphic functions defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$ and $\sigma(B)\subset\Omega$, such that $AB=BA$, then*

$$f(A)g(B)=g(B)f(A).$$

Definition 1.1. (Jódar and Cortés [8]) If A is a positive stable matrix in $\mathbb{C}^{N\times N}$ such that

$$Re(\mu)>0,\quad \forall\,\mu\in\sigma(A). \tag{1.2}$$

Definition 1.2. (Jódar and Cortés [9]) For $A\in\mathbb{C}^{N\times N}$, the Pochhammer symbol or shifted factorial is defined as:

$$(A)_n=A(A+I)(A+2I)\ldots(A+(n-1)I);n\geq 1,\ (A)_0=I. \tag{1.3}$$

Definition 1.3. (Jódar and Cortés [9]) Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$, then the Gamma matrix functions $\Gamma(A)$ is defined as

$$\Gamma(A)=\int_0^\infty e^{-t}t^{A-I}dt;\quad t^{A-I}=\exp\Big((A-I)\ln t\Big) \tag{1.4}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ such that $A+nI$ is an invertible matrix for all integers $n\geq 0$, then it follows that

$$(A)_n=\Gamma(A+nI)\Gamma^{-1}(A);\ n\geq 1,\ (A)_0=I. \tag{1.5}$$

Definition 1.4. Let us take $A\in\mathbb{C}^{N\times N}$ such that

$$\mu \text{ is not a negative integer for every } \mu\in\sigma(A). \tag{1.6}$$

Then the Bessel matrix function $J_A(z)$ of the first kind of order A is defined as follows [11]:

$$\begin{aligned}J_A(z)&=\sum_{k=0}^\infty\frac{(-1)^k}{k!}\Gamma^{-1}(A+(k+1)I)\Big(\frac{z}{2}\Big)^{A+2kI}\\&=\Big(\frac{z}{2}\Big)^{A}\Gamma^{-1}(A+I)\,{}_0F_1\Big(-;A+I;-\frac{z^2}{4}\Big);\quad |z|<\infty;\quad |arg(z)|<\pi.\end{aligned} \tag{1.7}$$

Definition 1.5. [19] If $k\in\mathbb{N}$ and A is a positive stable matrix in $\mathbb{C}^{N\times N}$. Then the k-gamma matrix function is given by

$$\Gamma_k(A)=\int_0^\infty e^{-\frac{t^k}{k}}t^{A-I}dt. \tag{1.8}$$

Definition 1.6. [19] For $A\in\mathbb{C}^{N\times N}$ and $k\in\mathbb{N}$, the Pochhammer matrix k-symbol is defined as:

$$(A)_{n,k}=\begin{cases}A(A+kI)(A+2kI)\ldots(A+(n-1)kI)=\Gamma_k(A+nkI)\Gamma_k^{-1}(A), & n\in\mathbb{N}\\ I, & n=0,\end{cases} \tag{1.9}$$

where $A+nkI$ is an invertible matrix for all integers $nk\geq 0$ with $n\in\mathbb{N}_0=\{0\}\cup\mathbb{N}$ and $k\in\mathbb{N}$.

Lemma 1.1. *[19, 20] Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying $Re(z)>0$, for every eigenvalue $z\in\sigma(A)$, $n\geq 1$ be an integer, and $k>0$, then by application of matrix calculus, we have*

$$\Gamma_k(A)=\lim_{n\to\infty}n!k^n[(A)_{n,k}]^{-1}(nk)^{\frac{1}{k}A-I} \tag{1.10}$$

where $(A)_{n,k}$ is defined by (1.9).

2. Extended Bessel matrix functions

Throughout this section we assume that A and B are commuting matrices in $\mathbb{C}^{N\times N}$.

Definition 2.1. Let us suppose that A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions

$$\begin{aligned} Re(\mu) &\text{ is an integer for every eigenvalue } \mu \in \sigma(A), \\ Re(\nu) &\text{ is a positive integer for every eigenvalue } \nu \in \sigma(B). \end{aligned} \tag{2.1}$$

Then, we define the extended Bessel matrix functions by the integral representation:

$$J_{A,B}(z) = \frac{1}{2i\pi}\int_0^1 t^{-A-I}\left(t+\frac{1}{t}\right)^B \exp\left[\frac{1}{2}z\left(t-\frac{1}{t}\right)\right]dt. \tag{2.2}$$

Moreover, from (2.2), it is evident that

$$J_{A,\mathbf{0}}(z) = J_A(z),$$

and we take $B = I$ in this formula

$$zJ_{A,I}(z) = 2AJ_A(z).$$

Remark 2.1. For $B = \mathbf{0}$, (2.2) reduced to $J_A(z)$ Bessel matrix functions defined in [11].

Corollary 2.1. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (2.1). For the extended Bessel matrix functions,*

$$J_{-A,B}(z) = (-1)^{A-B}J_{A,B}(z) = e^{(A-B)\ \ln(-1)}J_{A,B}(z) \tag{2.3}$$

is satisfied.

Proof. From (2.2), we obtain (2.3). □

Now, we obtain some matrix recurrence relations for the extended Bessel matrix functions.

Theorem 2.1. *The extended Bessel matrix functions $J_{A,B}(z)$ given by the integral representation (2.2) satisfy the following matrix recurrence relations*

$$J_{A,B}(z) = J_{A-I,B-I}(z) + J_{A+I,B-I}(z), \tag{2.4}$$

$$2J'_{A,B}(z) = J_{A-I,B}(z) - J_{A+I,B}(z) \tag{2.5}$$

and

$$4J''_{A,B-2I}(z) = J_{A,B}(z) - 4J_{A,B-2I}(z) \tag{2.6}$$

where A, $A-I$, $A-2I$, B, $B-I$ and $B-2I$ are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (2.1).

Proof. Using (2.2) in the right hand side of (2.4), we have

$$\begin{aligned} &\frac{1}{2i\pi}\int_0^1 t^{-A+I-I}\left(t+\frac{1}{t}\right)^{B-I} \exp\left[\frac{1}{2}z\left(t-\frac{1}{t}\right)\right]dt \\ &+\frac{1}{2i\pi}\int_0^1 t^{-A-I-I}\left(t+\frac{1}{t}\right)^{B-I} \exp\left[\frac{1}{2}z\left(t-\frac{1}{t}\right)\right]dt \\ =&\frac{1}{2i\pi}\int_0^1 t^{-A-I}\left(t+\frac{1}{t}\right)(t+\frac{1}{t})^{B-I} \exp\left[\frac{1}{2}z\left(t-\frac{1}{t}\right)\right]dt \\ =&\frac{1}{2i\pi}\int_0^1 t^{-A-I}\left(t+\frac{1}{t}\right)^{B} \exp\left[\frac{1}{2}z\left(t-\frac{1}{t}\right)\right]dt = J_{A,B}(z), \end{aligned}$$

which proves (2.4), now differentiating (2.2) with respect to z, we have

$$\begin{aligned}\frac{d}{dz}J_{A,B}(z) =&\frac{1}{2}\frac{1}{2i\pi}\int_0^1 t^{-A-I}\Big(t+\frac{1}{t}\Big)^B\Big(t-\frac{1}{t}\Big)\ \exp\Big[\frac{1}{2}z\Big(t-\frac{1}{t}\Big)\Big]dt\\ =&\frac{1}{2}\Big(J_{A-I,B}(z)-J_{A+I,B}(z)\Big).\end{aligned}$$

Using (2.2) and differentiating with respect to z, we have

$$\begin{aligned}\frac{d^2}{dz^2}J_{A,B-2I}(z) =&\frac{1}{4}\frac{1}{2i\pi}\int_0^1 t^{-A-I}\Big(t+\frac{1}{t}\Big)^{B-2I}\Big(t-\frac{1}{t}\Big)^2\ \exp\Big[\frac{1}{2}z\Big(t-\frac{1}{t}\Big)\Big]dt\\ =&\frac{1}{4}\Big(J_{A-2I,B-2I}(z)+J_{A+2I,B-2I}(z)-2J_{A,B-2I}(z)\Big)\end{aligned}$$

and in view of (2.5) leads to

$$\begin{aligned}&J_{A-I,B-I}(z)=J_{A-2I,B-2I}(z)+J_{A,B-2I}(z),\\ &J_{A+I,B-I}(z)=J_{A,B-2I}(z)+J_{A+I,B-2I}(z),\\ &J_{A-I,B-I}(z)+J_{A+I,B-I}(z)=J_{A-2I,B-2I}(z)+J_{A+I,B-2I}(z)+2J_{A,B-2I}(z),\end{aligned}$$

and

$$J_{A-2I,B-2I}(z)+J_{A+I,B-2I}(z)=J_{A,B}(z)-2J_{A,B-2I}(z).$$

According to the last expression, we get

$$\frac{d^2}{dz^2}J_{A,B-2I}(z)=\frac{1}{4}\Big(J_{A,B}(z)-4J_{A,B-2I}(z)\Big)=\frac{1}{4}J_{A,B}(z)-J_{A,B-2I}(z).$$

□

In a similar manner as in the proof of Theorem 2.1, one can easily obtain the next results.

Theorem 2.2. *Let A, $A-I$ and B are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (2.1). Then the extended Bessel matrix functions $J_{A,B}(z)$ given by the integral representation (2.2) satisfy the following pure matrix recurrence relation*

$$zJ_{A,B+2I}(z)=2AJ_{A,B+I}(z)-2(B+I)\Big(J_{A-I,B}(z)-J_{A+I,B}(z)\Big). \tag{2.7}$$

We give the matrix differential equation of extended Bessel matrix functions. This result is contained in the following.

Theorem 2.3. *The extended Bessel matrix functions $J_{A,B}(z)$ satisfy the matrix differential equation of the fourth order:*

$$\begin{aligned}\Big[&z^2\frac{d^4}{dz^4}I+(2B+5I)z\frac{d^3}{dz^3}+\Big(2z^2I+(B+2I)^2-A^2\Big)\frac{d^2}{dz^2}\\ &+(2B+5I)z\frac{d}{dz}+\Big(z^2I+B+2I-A^2\Big)\Big]J_{A,B}(z)=\mathbf{0}.\end{aligned} \tag{2.8}$$

Proof. Let us denote by (1.1) the differential operator $\Phi=z^2\frac{d^2}{dz^2}I+z\frac{d}{dz}I+z^2I-A^2$. Turning now to the Bessel's differential operator, we obtain

$$\begin{aligned}\Phi\ J_{A,B}(z)=&\Phi\frac{1}{2i\pi}\int_0^1 t^{-A-I}\Big(t+\frac{1}{t}\Big)^B\ \exp\Big[\frac{1}{2}z\Big(t-\frac{1}{t}\Big)\Big]dt\\ =&\frac{1}{2i\pi}\Big[t^{-A}\ \exp\Big[\frac{1}{2}z\Big(t-\frac{1}{t}\Big)\Big]\Big]\Big[\Big(t+\frac{1}{t}\Big)\frac{z}{2}I+A\Big]\Big(t+\frac{1}{t}\Big)^B|_0^1\\ &-\frac{1}{2i\pi}B\int_0^1 t^{-A-I}\ \exp\Big[\frac{1}{2}z\Big(t-\frac{1}{t}\Big)\Big]\Big[\Big(t+\frac{1}{t}\Big)\frac{z}{2}I+A\Big]\Big(t-\frac{1}{t}\Big)\Big(t+\frac{1}{t}\Big)^{B-I}dt,\end{aligned}$$

where we put in the integrated part the limits of the integration. If the limits are such that the integrated part vanishes, then

$$\begin{aligned}\Phi\ J_{A,B}(z) &= -2Bz\frac{d}{dz}J_{A,B}(z) + \frac{1}{2i\pi}B\left[t^{-A}\ \exp\left[\frac{1}{2}z\left(t-\frac{1}{t}\right)\right]\left(t-\frac{1}{t}\right)\left(t+\frac{1}{t}\right)^{B-I}|_0^1\right.\\ &\quad -\frac{1}{2i\pi}B\int_0^1 t^{-A-I}\ \exp\left[\frac{1}{2}z\left(t-\frac{1}{t}\right)\right]\left[B\left(t+\frac{1}{t}\right)^{B} - 4(B-I)\left(t+\frac{1}{t}\right)^{B-2I}\right]dt.\end{aligned}$$

Putting in the limits the integrated part again vanishes, we have

$$\Phi\ J_{A,B}(z) = -2zB\frac{d}{dz}J_{A,B}(z) - B^2 J_{A,B}(z) + 4B(B-I)J_{A,B-2I}(z).$$

Operating on this equation by $\frac{d^2}{dz^2}+1$, we get

$$\begin{aligned}&\left(\frac{d^2}{dz^2}+1\right)\left[\Phi\ J_{A,B}(z) + 2zB\frac{d}{dz}J_{A,B}(z) + B^2 J_{A,B}(z)\right]\\ &= 4B(B-I)\left[\frac{d^2}{dz^2}J_{A,B-2I}(z) + J_{A,B-2I}(z)\right].\end{aligned}$$

Using the recurrence formula (2.6), we obtain

$$\left(\frac{d^2}{dz^2}+1\right)\left[\Phi\ J_{A,B}(z) + 2Bz\frac{d}{dz}J_{A,B}(z) + B^2 J_{A,B}(z)\right] = B(B-I)J_{A,B}(z).$$

Therefore, the equation (2.8) is established. □

Finally, let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (2.1). We define the extended modified Bessel matrix functions by the relation link with extended Bessel matrix functions

$$\begin{aligned}\mathbf{I}_{A,B}(z) &= i^{-A}\ J_{A,B}(iz) = e^{-A\ln(i)}\ J_{A,B}(iz)\\ &= \frac{1}{2i\pi}\int_0^1 t^{-A-I}\left(\frac{t}{i}+\frac{i}{t}\right)^{B}\ \exp\left[\frac{1}{2}z\left(t+\frac{1}{t}\right)\right]dt.\end{aligned} \tag{2.9}$$

Next, we define the extended Tricomi matrix functions by the following link with extended Bessel matrix function

$$\begin{aligned}\mathbf{C}_{A,B}(z) &= z^{-\frac{1}{2}A}\ J_{A,B}(2\sqrt{z})\\ &= \frac{1}{2i\pi}z^{-\frac{1}{2}A}\int_0^1 t^{-A-I}\left(t+\frac{1}{t}\right)^{B}\ \exp\left[\sqrt{z}\left(t-\frac{1}{t}\right)\right]dt\end{aligned} \tag{2.10}$$

where A and B are matrices in $C^{N\times N}$ satisfying the condition (2.1).

As a similar matrix function, we define the matrix function $J(z;A,B)$ by the series

$$\begin{aligned}J(z;A,B) = \sum_{k=0}^{\infty}&\frac{(-1)^k}{k!}\Gamma^{-1}(A+2B+kI)\left(\frac{z}{2}\right)^{A+2B+2kI}\\ &2A(A+2B-2I)^{-1}(A+2B)^{-1}(A+2B+2kI+2I)^{-1}.\end{aligned} \tag{2.11}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ such that (1.6), and $A+2B-2I$, $A+2B$ and $A+2B+2I$ are invertible matrices.

Theorem 2.4. *Let A and B are matrices in $\mathbb{C}^{N\times N}$ such that (1.6) holds for these two matrices. For the matrix functions $J(z;A,B)$ the following relation holds:*

$$(A+2B-2I)(A+2B+2I)\left[J(z;A,B) - J(z;A,B+I)\right] = 2AJ_{A+2B}(z). \tag{2.12}$$

Proof. By (2.11) and (1.7) the equation (2.12) follows directly. □

3. Generalized Bessel matrix functions

Definition 3.1. If A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions $Re(\mu) > -1$ for $\mu \in \sigma(A)$, $Re(\nu) \notin \mathbb{Z}^-$ for $\nu \in \sigma(B)$, and $AB = BA$, then

$$J_{A,B}(z) = \int_0^1 (1-t^2)^B t^{A+I}\ J_A(zt)dt. \tag{3.1}$$

When $B = \mathbf{0}$, we have

$$J_{A,\mathbf{0}}(z) = \frac{1}{z} J_{A+I}(z).$$

By (3.1), we have

$$J_{A,B}(z) = \int_0^1 (1-t^2)^B d\left(\frac{1}{z} t^{A+I}\ J_{A+I}(zt)\right) \tag{3.2}$$

and hence, after integrating by parts

$$J_{A,B}(z) = \frac{2}{z} B\ J_{A+I,B-I}(z) \tag{3.3}$$

where $Re(\nu-1) \notin \mathbb{Z}^-$ for $\nu - 1 \in \sigma(B-I)$. Applying the mathematical induction, we get

$$J_{A,B}(z) = \left(\frac{2}{z}\right)^B \Gamma(B+I)\ J_{A+B,\mathbf{0}}(z). \tag{3.4}$$

4. k-Bessel matrix functions: Definition and properties

In this section, we give definitions of the k-hypergeometric matrix functions and k-Bessel matrix functions and their properties. Furthermore, we give the extended forms of these Bessel matrix functions in [1, 2, 14].

Definition 4.1. Let A be a matrix in $\mathbb{C}^{N\times N}$ where $A + nkI$ is an invertible matrix for every integer $nk \geq 0$ with $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we define the k-hypergeometric matrix functions by the following series

$${}_0F_{1,k}(-;A;z) = \sum_{n=0}^{\infty} [(A)_{n,k}]^{-1} \frac{z^n}{n!}. \tag{4.1}$$

Theorem 4.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ such that the matrix $A + nkI$ is an invertible matrix for every integer $nk \geq 0$ with $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then k-hypergeometric matrix functions ${}_0F_{1,k}(-;A;z)$ is a solution of the k-hypergeometric matrix differential equation of second order*

$$\Big[\theta(k\theta I + A - kI) - zI\Big]\ {}_0F_{1,k}\big(-;A;z\big) = \mathbf{0}. \tag{4.2}$$

Proof. From (4.1), we find that

$$k\theta(k\theta I + A - kI)\ {}_0F_{1,k}\big(-;A;z\big) = \sum_{n=0}^{\infty} \frac{knI + A - kI}{n!} z^n [(A)_{n,k}]^{-1} = \sum_{n=1}^{\infty} \frac{kz^n}{(n-1)!} [(A)_{n-1,k}]^{-1}$$

By shifting index, we obtain that

$$k\theta(k\theta I + A - kI)\ {}_0F_{1,k}\big(-;A;z\big) = kz \sum_{n=0}^{\infty} \frac{z^n}{n!} [(A)_{n,k}]^{-1} = kz\ {}_0F_{1,k}\big(-;A;z\big).$$

i.e, equivalently

$$\Big[\theta(k\theta I + A - kI) - zI\Big]\ {}_0F_{1,k}\big(-;A;z\big) = \mathbf{0}.$$

Since

$$\theta W = zW', \quad \theta(\theta-1)W = z^2W'', \tag{4.3}$$

has $W = {}_0F_{1,k}\big(-;A;z\big)$ as one solution of the matrix differential equation. This Eq. (4.2) can also be rewritten in the form

$$kzW'' + AW' - W = \mathbf{0}. \tag{4.4}$$

□

Definition 4.2. For A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.6), we define the k-Bessel matrix function ${}_kJ_A(z)$ of the first kind as:

$$\begin{aligned} {}_kJ_A(z) &= \left(\frac{z}{2}\right)^A \Gamma_k^{-1}(A+I)\ {}_0F_{1,k}\left(;A+I;-\frac{z^2}{4}\right) \\ &= \Gamma_k^{-1}(A+I)\sum_{k=0}^{\infty}\frac{(-1)^k}{n!}[(A+I)_{n,k}]^{-1}\left(\frac{z}{2}\right)^{A+2nI} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{n!}\Gamma_k^{-1}(A+(kn+1)I)\left(\frac{z}{2}\right)^{A+2nI}. \end{aligned} \tag{4.5}$$

where the Pochhammer k-symbol $(A)_{n,k}$ is given in (1.9) and the k-gamma matrix functions $\Gamma_k(A)$ is given in (1.8).

Theorem 4.2. *The extended k-Bessel matrix functions hold the matrix differential equation of second order*

$$\begin{aligned} kz^2\frac{d^2}{dz^2}\ {}_kJ_A(z) &+ \Big[z(2-k)I + 2Az(1-k)\Big]\frac{d}{dz}\ {}_kJ_A(z) \\ &+ \Big[(k-2)A^2 + 2kA - 2A + z^2I\Big]\ {}_kJ_A(z) = \mathbf{0}. \end{aligned} \tag{4.6}$$

Proof. From (4.3), we now put $y = -\frac{z^2}{4}$ and $A = A + I$, therefore to obtain

$$\frac{dW}{dy} = -\frac{2}{z}\frac{dW}{dz}, \quad \frac{d^2W}{dy^2} = -\frac{4}{z^3}\frac{dW}{dz} + \frac{4}{z^2}\frac{d^2W}{dz^2},$$

in which primes denote differentiations with respect to z. One a solution of (4.3) is $W = {}_0F_{1,k}\left(;A+I;-\frac{z^2}{4}\right)$ in the form

$$kzW'' + (2A + 2I - kI)W' + zW = \mathbf{0}. \tag{4.7}$$

We seek an equation satisfied by $U = z^AW$. Hence in (4.7) we now put $W = z^{-A}U$ and arrive at the matrix differential equation

$$kz^2U'' + \Big[z(2-k)I + 2Az(1-k)\Big]U' + \Big[(k-2)A^2 + 2kA - 2A + z^2I\Big]U = \mathbf{0},$$

of which one solution is $U = z^AW = z^A\ {}_0F_{1,k}\left(;A+I;-\frac{z^2}{4}\right)$. Eq. (4.6) is k-Bessel matrix differential equation. □

Definition 4.3. Let A be a matrix in $\mathbb{C}^{N\times N}$ such that

$$-r \notin \sigma(A) \text{ for every integers } r > 0, \tag{4.8}$$

and λ is a complex number with $Re(\lambda) > 0$. Then we define the k-Laguerre matrix polynomials by

$$L_{n,k}^{(A,\lambda)}(x) = \sum_{r=0}^{n}\frac{(-n)_{r,k}(A+I)_{n,k}[(A+I)_{r,k}]^{-1}(\lambda x)^r}{r!n!}. \tag{4.9}$$

Theorem 4.3. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the conditions (1.6), (4.8) and $Re(\lambda)>0$. Then connection between k-Bessel matrix functions and k-Laguerre matrix polynomials is the following relation*

$$\lim_{n\to\infty}\left[n^{-A}L_{n,k}^{(A,\lambda)}\left(\frac{x}{n}\right)\right]=\left(\lambda x\right)^{-\frac{1}{2}A}\ {}_kJ_A(2\sqrt{\lambda x}). \tag{4.10}$$

Proof. Using (4.9), (4.5) and (1.10), respectively, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left[k^{-n}(nk)^{I-\frac{1}{k}(A+I)}L_{n,k}^{(A,\lambda)}\left(\frac{x}{n}\right)\right]\\
&=\lim_{n\to\infty}\left[\frac{(nk)^{I-\frac{1}{k}(A+I)}}{n!k^n}(A+I)_{n,k}\sum_{r=0}^{n}\frac{(-n)_{r,k}[(A+I)_{r,k}]^{-1}\left(\lambda\frac{x}{n}\right)^r}{r!}\right]\\
&=\Gamma_k^{-1}(A+I)\sum_{r=0}^{\infty}\frac{(-1)^r[(A+I)_{r,k}]^{-1}\left(\lambda x\right)^r}{r!}\\
&=\left(\lambda x\right)^{-\frac{1}{2}A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\Gamma_k^{-1}(A+(kr+1)I)\left(\sqrt{\lambda x}\right)^{A+2rI}\\
&=\left(\lambda x\right)^{-\frac{1}{2}A}\ {}_kJ_A(2\sqrt{\lambda x}).
\end{aligned}$$

□

Theorem 4.4. *If A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the conditions (1.6) and (4.8). Then the k-Bessel matrix functions of the first kind and k-Laguerre matrix polynomials satisfy the following relation*

$$\sum_{n=0}^{\infty}\Gamma_k^{-1}(A+(kn+1)I)L_{n,k}^{(A,\lambda)}(x)t^n=e^t\left(\lambda xt\right)^{-\frac{1}{2}A}\ {}_kJ_A(2\sqrt{\lambda xt}). \tag{4.11}$$

Proof. Using (4.9),(4.5) and (1.10), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\Gamma_k^{-1}(A+(kn+1)I)L_{n,k}^{(A,\lambda)}(x)t^n&=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-n)_{r,k}\Gamma_k^{-1}(A+I)[(A+I)_{r,k}]^{-1}}{r!n!}\left(\lambda x\right)^r t^n\\
&=\left(\lambda x\right)^{-\frac{1}{2}A}\sum_{n,r=0}^{\infty}\frac{(-1)^r}{n!r!}\Gamma_k^{-1}(A+(kr+1)I)\left(\sqrt{\lambda x}\right)^{A+2rI}t^{n+r}\\
&=e^t\left(\lambda xt\right)^{-\frac{1}{2}A}\ {}_kJ_A(2\sqrt{\lambda xt}).
\end{aligned}$$

□

Definition 4.4. For A and $B\in\mathbb{C}^{N\times N}$ which A satisfy the condition (1.6), we define the k-Bessel matrix function ${}_kJ_{A,B}(z)$ of the first kind as:

$${}_kJ_{A,B}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(B)_{n,k}}{(n!)^2}\Gamma_k^{-1}(A+(n+1)I)\left(\frac{z}{2}\right)^{A+2nI} \tag{4.12}$$

where $B+nkI$ is an invertible matrix for all integers $nk\geq 0$ with $k\in\mathbb{N}$ and $n\in\mathbb{N}_0$.

Now, we show that some elementary properties.

Theorem 4.5. *For the k-Bessel matrix function ${}_kJ_{A,B}(z)$, we obtain the following matrix recurrence relation*

$$\frac{B}{k}\ {}_kJ_{A,B+kI}(z)-\frac{1}{2k}\left(2B-kA\right)\ {}_kJ_{A,B}(z)=\frac{z}{2}\frac{d}{dz}\ {}_kJ_{A,B}(z). \tag{4.13}$$

Proof. Using the relation

$$n(B)_{n,k}=\frac{1}{k}\big(B+nkI\big)(B)_{n,k}-\frac{1}{k}B(B)_{n,k},$$

$$(B)_{n+1,k}=(B+nkI)(B)_{n,k}=B(B+kI)_{n,k},$$

we get

$$n(B)_{n,k}=\frac{1}{k}B\Big[(B+kI)_{n,k}-(B)_{n,k}\Big].$$

Using (4.12) and differentiating with respect to z, we have

$$\begin{aligned}\frac{d}{dz}\Big[z^{-A}\ {}_kJ_{A,B}(z)\Big]&=2\sum_{n=0}^{\infty}\frac{(-1)^n n(B)_{n,k}}{2^{A+2nI}(n!)^2}\Gamma_k^{-1}(A+(n+1)I)z^{(2n-1)I}\\&=\frac{2}{k}B\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{A+2nI}(n!)^2}\Big[(B+kI)_{n,k}-(B)_{n,k}\Big]\Gamma_k^{-1}(A+(n+1)I)z^{(2n-1)I}\\&=\frac{2}{k}Bz^{-A-I}\sum_{n=0}^{\infty}\frac{(-1)^n(B+kI)_{n,k}}{2^{A+2nI}(n!)^2}\Gamma_k^{-1}(A+(n+1)I)z^{A+2nI}\\&-\frac{2}{k}Bz^{-A-I}\sum_{n=0}^{\infty}\frac{(-1)^n(B)_{n,k}}{2^{A+2nI}(n!)^2}\Gamma_k^{-1}(A+(n+1)I)z^{A+2nI}\\&=\frac{2}{k}Bz^{-A-I}\ {}_kJ_{A,B+kI}(z)-\frac{2}{k}Bz^{-A-I}\ {}_kJ_{A,B}(z).\end{aligned}$$

Thus, for k-Bessel matrix functions, we give the property

$$z^{-A}\frac{d}{dz}\ {}_kJ_{A,B}(z)-Az^{-A-I}\ {}_kJ_{A,B}(z)=\frac{2}{k}Bz^{-A-I}\ {}_kJ_{A,B+kI}(z)-\frac{2}{k}Bz^{-A-I}\ {}_kJ_{A,B}(z). \tag{4.14}$$

Multiplying (4.14) by $\frac{1}{2}z^{A+I}$, we have

$$\frac{z}{2}\frac{d}{dz}\ {}_kJ_{A,B}(z)=\frac{1}{k}B\ {}_kJ_{A,B+kI}(z)-\frac{1}{2k}\Big(2B-kA\Big)\ {}_kJ_{A,B}(z).$$

□

Theorem 4.6. *If A and $A-kI$ are matrices in $\mathbb{C}^{N\times N}$ which satisfies the condition (1.6), and $k\in\mathbb{N}$. Then*

$$\frac{d}{dz}\Big[z^{\frac{1}{2}A}\ {}_kJ_{A,B}(\sqrt{z})\Big]=2^{-k}z^{\frac{1}{2}(A+kI)-I}\ \ {}_kJ_{A-kI,B}(\sqrt{z}). \tag{4.15}$$

Proof. Using (4.12) and differentiating with the respect to z, we have

$$\begin{aligned}\frac{d}{dz}\Big[z^{\frac{1}{2}A}\ {}_kJ_{A,B}(\sqrt{z})\Big]&=\sum_{n=0}^{\infty}\frac{(-1)^n(B)_{n,k}(A+nI)}{2^{n+2k}(n!)^2}\Gamma_k^{-1}(A+(n+k)I)z^{A+(n-1)I}\\&=2^{-k}z^{\frac{1}{2}(A-I)}\sum_{n=0}^{\infty}\frac{(-1)^n(B)_{n,k}}{(n!)^2}\Gamma_k^{-1}(A+nI)\Big(\frac{\sqrt{z}}{2}\Big)^{A+(2n-1)I}\\&=2^{-k}z^{\frac{1}{2}(A+kI)-I}\ \ {}_kJ_{A-kI,B}(\sqrt{z}),\end{aligned}$$

which is the required proof. □

Theorem 4.7. *For the k-Bessel matrix function ${}_kJ_{A,B}(z)$, we have the following matrix recurrence relation*

$$\Big[A+I-\frac{1}{2}(A+kI)\Big]\ {}_kJ_{A+kI,B}(z)+\frac{z}{2}\frac{d}{dz}\ {}_kJ_{A+kI,B}(z)=\Big(\frac{z}{2}\Big)^k\ {}_kJ_{A,B}(z). \tag{4.16}$$

Proof. To prove the theorem, it is enough to use the definition (4.12) details are therefore omitted. □

Definition 4.5. For A and $B \in \mathbb{C}^{N\times N}$ which A satisfy the condition (1.6), we define the k-modified Bessel matrix functions of the first kind by

$$
{}_k\mathbb{I}_{A,B}(z) = \sum_{n=0}^{\infty} \frac{(B)_{n,k}}{(n!)^2} \Gamma_k^{-1}(A + (n+1)I) \left(\frac{z}{2}\right)^{A+2nI} \tag{4.17}
$$

where $B + nkI$ is an invertible matrix for all integers $nk \geq 0$ with $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Theorem 4.8. *The k-modified Bessel matrix functions of the first kind of order A. Then holds*

$$
\frac{d}{dz}\left[z^{\frac{1}{2}A} \ {}_k\mathbb{I}_{A,B}(\sqrt{z}) \right] = 2^{-k} z^{\frac{1}{2}(A+kI)-I} \ {}_k\mathbb{I}_{A-kI,B}(\sqrt{z}) \tag{4.18}
$$

where $k \in \mathbb{N}$, A and $A - kI$ are matrices in $\mathbb{C}^{N\times N}$ satisfy the condition (1.6).

Proof. From definition (4.17), we have

$$
\begin{aligned}
\frac{d}{dz}\left[z^{\frac{1}{2}A} \ {}_k\mathbb{I}_{A,B}(\sqrt{z}) \right] &= \sum_{n=0}^{\infty} \frac{(B)_{n,k}(A+nI)}{2^{n+2k}(n!)^2} \Gamma_k^{-1}(A + (n+k)I) z^{A+(n-1)I} \\
&= 2^{-k} z^{\frac{1}{2}(A-I)} \sum_{n=0}^{\infty} \frac{(B)_{n,k}}{(n!)^2} \Gamma_k^{-1}(A + nI) \left(\frac{\sqrt{z}}{2}\right)^{A+(2n-1)I} \\
&= 2^{-k} z^{\frac{1}{2}(A+kI)-I} \ {}_k\mathbb{I}_{A-kI,B}(\sqrt{z}).
\end{aligned}
$$

□

5. Conclusions

A new approach has been introduced in this paper for studying some important properties of certain matrix special functions viz matrix recurrence relations, matrix differential recurrence relations and matrix differential equation. The method developed in this paper can also be used to study some other special matrix functions which play a vital role in Mathematical Physics.

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