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h-Fourier Integral Operators with Complex Phase

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Abstract
We study in this work a class of h-Fourier integral operators with complex phase. These operators are continuous on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$.

Keywords: *h-admissible Fourier integral operators; complex phase; boundedness ; compactness; amplitude; composition.*

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1. Introduction

A Fourier integral operator is an operator that can be written in the form

$$(I(a,\phi)f)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\theta,y)}a(x,\theta,y)f(y)\,dyd\theta, \tag{1.1}$$

$f \in S(\mathbb{R}^n)$ (the Schwartz space). The function $a(x,\theta,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n)$ is called the amplitude, the function $\phi(x,y,\theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n;\mathbb{R})$ is called the phase function. The study of these operators, which are intimately connected to the theory of linear partial differential operators, has a long history and there is a large body of results made by a several authors (see, e.g.,[2, 5–12]). The first works on Fourier integral operators deal with local properties. We note that, K. Asada and D. Fujiwara [2] have studied for the first time a class of Fourier integral operators defined on $\mathbb{R}^n$.

In this paper we consider one of the most important problems in the theory of differential equations which is the study of the h-Fourier integral operators with a complex phase, this type of operator is represented by formula of the type

$$(I(a,\phi;h)f)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{\frac{i}{h}\phi(x,\theta,y)}a(x,\theta,y)f(y)\,dyd\theta, \tag{1.2}$$

in which appear two C^∞-functions, the phase function $\phi(x,y,\theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n)$ and the amplitude $a(x,\theta,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n)$ and a semiclassical parameter $h \in]0,h_0]$.

The purpose of this work is to generalize the notion of h-Fourier integral operators defined in [8] by considering the phase function ϕ with complex values, and appling the same technique of [2] to show that the h-Fourier integral operators with complex phase are well defined and they are continuous on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$ (the space of tempered distributions). We give also a result where it is shown that these types of operators are stable by composition.

When the phase function $\phi(x,y,\theta) = S(x,\theta) - y\theta$, where $S \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\theta;\mathbb{C})$, the operator (1.2) will be a particular case of h-Fourier integral operators with complex phase. In this case we will also give some hypothesis on the phase function ϕ and the amplitude a.

Let us now describe the plan of this article. In the second section we recall the continuity of some general class of Fourier integral operators on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$. The composition of h-Fourier integral operators with complex phase is given in the third section. The last section is devoted to study the particular case.

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2. A general class of h-Fourier integral operators with complex phase

In this section we define the class of integral transformations of type

$$\left(I\left(a,\phi;h\right),f\right)(x)=\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^N}e^{\frac{i}{h}\phi(x,\theta,y)}a\left(x,\theta,y\right)f\left(y\right)dyd\theta, \tag{2.1}$$

where $f\in S\left(\mathbb{R}^n\right), x\in\mathbb{R}^n, h\in\left]0,h_0\right]$, and $\phi:\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n\to\mathbb{C}$.

In general the integral (2.1) is not absolutely convergent, so we can use the technique of oscillatory integral developed by Hörmander in [10].

Notation 2.1. *For $(x,\theta,y)\in\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n$, we set*

$$\lambda\left(x,\theta,y\right)=\left(1+\left|x\right|^2+\left|y\right|^2+\left|\theta\right|^2\right)^{1/2}.$$

The phase function $\phi=\varphi+i\psi$ and the amplitude a are assumed to satisfy the following conditions:

(H_1) $\phi:\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n\to\mathbb{C}$ is a C^∞ application.

(H_2) $\forall\left(\alpha,\beta,\gamma\right)\in\mathbb{N}^n\times\mathbb{N}^N\times\mathbb{N}^n, \exists C_{\alpha\beta\gamma}\geq 0:$

$$\left|\partial_x^\alpha\partial_y^\beta\partial_\theta^\gamma\phi\left(x,\theta,y\right)\right|\leq C_{\alpha\beta\gamma}\left[\lambda\left(x,\theta,y\right)\right]^{2-(|\alpha|+|\beta|+|\gamma|)}.$$

(H_3) There exist real numbers $K_1,K_2>0$ such that

$$K_1\lambda\left(x,\theta,y\right)\leq\lambda\left(\partial_y\phi,\partial_\theta\phi,y\right)\leq K_2\lambda\left(x,\theta,y\right), \forall\left(x,\theta,y\right)\in\mathbb{R}_x^n\times\mathbb{R}_\theta^N\times\mathbb{R}_y^n.$$

(H_3^*) There exist real numbers $K_1^*,K_2^*>0$ such that

$$K_1^*\lambda\left(x,\theta,y\right)\leq\lambda\left(x,\partial_\theta\phi,\partial_x\phi\right)\leq K_2^*\lambda\left(x,\theta,y\right), \forall\left(x,\theta,y\right)\in\mathbb{R}_x^n\times\mathbb{R}_\theta^N\times\mathbb{R}_y^n.$$

(H_4) $\forall\left(x,\theta,y\right)\in\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n:\psi\left(x,\theta,y\right)\geq 0.$

For any open subset Ω of $\mathbb{R}_x^n\times\mathbb{R}_\theta^N\times\mathbb{R}_y^n, \mu\in\mathbb{R}$ and $\rho\in\left[0,1\right]$, we set

$$\Gamma_\rho^\mu\left(\Omega\right)=\left\{a\in C^\infty\left(\Omega\right):\left|\partial_x^\alpha\partial_y^\beta\partial_\theta^\gamma a\right|\leq C_{\alpha\beta\gamma}\left[\lambda\left(x,\theta,y\right)\right]^{\mu-(|\alpha|+|\beta|+|\gamma|)}\right\}.$$

For $\Omega=\mathbb{R}_x^n\times\mathbb{R}_\theta^N\times\mathbb{R}_y^n$, we denote $\Gamma_\rho^\mu\left(\Omega\right)=\Gamma_\rho^\mu$.

Now if φ satisfies $\left(H_1\right),\left(H_2\right),\left(H_3\right),\left(H_4\right)$ and $a\in\Gamma_0^\mu$, we can give a meaning to the right hand side of (2.1), so we consider $g\in S\left(\mathbb{R}_x^n\times\mathbb{R}_\theta^N\times\mathbb{R}_y^n\right), g\left(0\right)=1$. If $a\in\Gamma_0^\mu$, we define

$$a_\sigma\left(x,\theta,y\right)=g\left(\frac{x}{\sigma},\frac{\theta}{\sigma},\frac{y}{\sigma}\right)a\left(x,\theta,y\right), \ \sigma>0.$$

We have the following result concerning the boundedness of h-Fourier integral operators with complex phase on $S\left(\mathbb{R}^n\right)$ and on $S'\left(\mathbb{R}^n\right)$.

Theorem 2.1. *If the phase function ϕ satisfies $\left(H_1\right),\left(H_2\right),\left(H_3\right)$ and $\left(H4\right)$ and if $a\in\Gamma_0^\mu$, then*

1. *For all $f\in S\left(\mathbb{R}^n\right)$, $\lim_{p\to\infty}\left[\left(I\left(a_p,\phi;h\right),f\right)\right](x)$ exists for every $x\in\mathbb{R}^n$ and is independent of the choice of the function g. We set then*
$$\left(I\left(a,\phi;h\right),f\right)=\lim_{p\to\infty}\left(I\left(a_p,\phi;h\right),f\right)$$
2. *$I\left(a,\phi;h\right)$ is a linear continuous operator from $S\left(\mathbb{R}^n\right)$ into itself.*
3. *Furthermore, if ϕ satisfies $\left(H_3^*\right)$, so $I\left(a,\phi;h\right)\in\mathcal{L}\left(S'\left(\mathbb{R}^n\right)\right)$ (here $S'\left(\mathbb{R}^n\right)$ is the space of all tempered distributions on $\mathbb{R}^n$)*

Proof. Let $\delta \in C_0^\infty(\mathbb{R}^n)$ such that $supp\delta \subseteq [-1,2]$ and $\delta \equiv 1$ on $[0,1]$.

For all $\varepsilon > 0$, we set

$$\omega_\varepsilon(x,\theta,y) = \delta\left(\frac{|\partial_y\phi|^2 + |\partial_\theta\phi|^2}{\varepsilon\lambda(x,\theta,y)^2}\right).$$

From (H_3) there exists $C > 0$ for which we have on the support of ω_ε

$$\lambda(x,\theta,y) \le C\left[\left(1+|y|^2\right)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}\lambda(x,\theta,y)\right].$$

Choosing ε small enough we get that there exists a constant C_0, such that the inequality

$$\lambda(x,\theta,y) \le C_0\left(1+|y|^2\right)^{\frac{1}{2}}$$

holds in the support of ω_ε.

From this inequality we can see that $I(\omega_\varepsilon a_p,\phi;h)f$ is an absolutely convergent integral and we have

$$\lim_{p\to\infty} I(\omega_\varepsilon a_p,\phi;h)f = I(\omega_\varepsilon a,\phi;h)f. \tag{2.2}$$

The continuity of the operator $I(\omega_\varepsilon a,\phi;h)f$ from $S(\mathbb{R}^n)$ into itself follows from (H_2).

Next we study the limit $\lim_{p\to\infty} I((1-\omega_\varepsilon)a_p,\phi;h)f$. Consider the operator

$$L = \frac{h}{i}\frac{\left(\sum_{j=1}^n(\partial_{y_j}\phi)\frac{\partial}{\partial y_j} + \sum_{j=1}^N(\partial_{\theta_j}\phi)\frac{\partial}{\partial\theta_j}\right)}{|\partial_y\phi|^2 + |\partial_\theta\phi|^2}.$$

One can show easily that

$$L\left(e^{i\phi}\right) = e^{i\phi}. \tag{2.3}$$

Let Ω_0 be the open subset of $\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n$ defined by

$$\Omega_0 = \left\{(x,\theta,y), |\partial_y\phi|^2 + |\partial_\theta\phi|^2 > \frac{\varepsilon}{2}\lambda(x,\theta,y)^2\right\}.$$

By recurrence we prove that:

For all integers $q \ge 0$, and $b \in C^\infty\left(\mathbb{R}_y^n\times\mathbb{R}_\theta^N\right)$, we have

$$\left({}^tL\right)^q((1-\omega_\varepsilon)b) = \sum_{|\alpha|+|\beta|\le q} g_{\alpha,\beta}^q\partial_y^\beta\partial_\theta^\beta((1-\omega_\varepsilon)b), \tag{2.4}$$

where the $g_{\alpha,\beta}^q$ are in $\Gamma_0^{-q}(\Omega_0)$ and depend only on ϕ. In particular for $q = 0$, we have

$${}^tL = \sum_j F_j\frac{\partial}{\partial y_j} + \sum_j G_j\frac{\partial}{\partial\theta_j} + H, \tag{2.5}$$

where $F_j \in \Gamma_0^{-1}(\Omega_0)$, $G_j \in \Gamma_0^{-1}(\Omega_0)$, $H \in \Gamma_0^{-2}(\Omega_0)$.

From (2.3) we have also for all integer $q \ge 0$,

$$I((1-\omega_\varepsilon)a_p,\phi;h)f(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{\frac{i}{h}\phi(x,\theta,y)}\left({}^tL\right)^q((1-\omega_\varepsilon)a_p,f)\,dyd\theta. \tag{2.6}$$

But

$$\left({}^tL\right)^q((1-\omega_\varepsilon)a_pf) \text{ described (when } p \text{ varies) a bound of } \Gamma_0^{\mu-q}, \tag{2.7}$$

and

$$\lim_{p\to\infty}\left({}^tL\right)^q((1-\omega_\varepsilon)a_pf)(x,\theta,y) = \left({}^tL\right)^q((1-\omega_\varepsilon)af)(x,\theta,y), \tag{2.8}$$

for all $(x,\theta,y) \in \mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^n$.

To finish, we have for all $s > n + N$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} \lambda^{-s}(x,\theta,y)\, dyd\theta \leq \gamma_s \lambda^{n+N-s}(x). \tag{2.9}$$

From (2.6)-(2.9) and using the Lebesgue's theorem we obtain

$$\lim_{p\to\infty} I\left((1-\omega_\varepsilon)a_p,\phi;h\right)f(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{\frac{i}{h}\phi(x,\theta,y)}\left({}^tL\right)^q\left((1-\omega_\varepsilon)a,f;h\right)dyd\theta, \tag{2.10}$$

where q satisfies $q > n + N + \mu$.

The first assertion of the theorem can be proved from (2.2) and (2.10).

Now let's show the continuity of $I\left((1-\omega_\varepsilon)a,\phi;h\right)$. From (2.5) and (2.10), we have

$$I\left((1-\omega_\varepsilon)a,\phi;h\right)f(x) = \sum_{|\gamma|\leq q}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{\frac{i}{h}\phi(x,\theta,y)} b_\gamma^{(q)}(x,\theta,y)\,\partial_y^\gamma f(y)\,dyd\theta, \tag{2.11}$$

with $b_\gamma^{(q)} \in \Gamma_0^{\mu-q}$. On the other hand, we have

$$x^\alpha\partial_x^\beta\left(e^{\frac{i}{h}\phi}b_\gamma^{(q)}(x,\theta,y)\right) \in \Gamma_0^{\mu-q+|\alpha|+|\beta|}. \tag{2.12}$$

This property and (2.11) imply that, for all $q > n + N + \mu + |\alpha| + |\beta|$, there exists a constant $C_{\alpha,\beta,q}$ such that

$$\left|x^\alpha\partial_x^\beta I\left((1-\omega_\varepsilon)a,\phi;h\right)f(x)\right| \leq C_{\alpha,\beta,q}\sup_{\substack{x\in\mathbb{R}^n\\|\gamma|\leq q}}\left|\partial_x^\gamma f(x)\right|,$$

which proves the continuity of $I\left((1-\omega_\varepsilon)a,\phi;h\right)$.

The last assertion of the theorem is an immediate consequence of the second one, indeed the matter is to show that the operator ${}^tF$ is continuous from $S(\mathbb{R}^n)$ to itself, where $F = I(a,\phi;h)$. But ${}^tF = I\left(\widetilde{a},\widetilde{\phi};h\right)$, with

$$\begin{aligned}\widetilde{\phi}(x,\theta,y) &= \phi(y,\theta,x),\\ \widetilde{a}(x,\theta,y) &= a(y,\theta,x).\end{aligned}$$

Since ϕ satisfies (H_3^*), then $\widetilde{\phi}$ satisfies (H_3), so we can deduce the result. □

Remark 2.1. We can obtain the same result if the hypothesis on ϕ are fulfilled only on the support of the amplitude a.

3. Composition of two h-Fourier integral operators with complex phase

In this section we prove that the composition of two h-Fourier integral operators with complex phase, have a meaning, and give an operator of same type.

Theorem 3.1. *Assume that the phase functions ϕ_1 and ϕ_2 satisfy $(H_1), (H_2), (H_3)$ and (H_4). Set*

$$\phi(x,\theta,z) = \phi_1(x,\theta_1,y) + \phi_2(y,\theta_2,z), \tag{3.1}$$

with $\theta_1 \in \mathbb{R}^{N_1}, \theta_2 \in \mathbb{R}^{N_2}, x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}^n, \theta = (\theta_1, y, \theta_2)$. Then ϕ verifies $(H_1), (H_2), (H_3)$ and (H_4), and for all $a_1 \in \Gamma_0^{\mu_1}, a_2 \in \Gamma_0^{\mu_2}$, we have

$$I(a_1,\phi_1;h)\,I(a_2,\phi_2;h) = I(a_1\times a_2,\phi;h), \tag{3.2}$$

where

$$(a_1\times a_2)(x,\theta,z) = a_1(x,\theta_1,y)\,a_2(y,\theta_2,z).$$

Proof. We first observe that $(H_1), (H_2)$ and (H_4) are trivial. So we have to prove the condition (H_3).

We can see that the first inequality is evident, so it suffices to show that ϕ satisfies the following property: there exists $K > 0$ such that

$$\lambda(x,\theta_1,y,\theta_2,z) \leq K\lambda\left(z,\partial_z\phi_2,\partial_y\phi_1+\partial_y\phi_2,\partial_{\theta_1}\phi_1,\partial_{\theta_2}\phi_2\right). \tag{3.3}$$

Applying (H_3) to ϕ_1 and ϕ_2 we get that there exists $C > 0$ such that

$$\lambda(x, \theta_1, y, \theta_2, z) \leq C\lambda(\partial_y\phi_1, \partial_{\theta_1}\phi_1, y, \partial_{\theta_2}\phi_2, \partial_z\phi_2, z), \tag{3.4}$$

but we have also

$$\lambda(y) \leq C^{'}\lambda(\partial_{\theta_2}\phi_2, \partial_z\phi_2, z), \tag{3.5}$$

from (H_3) applied to ϕ_2, and

$$|\partial_y\phi_2| \leq C''\lambda(y, \theta_2, z) \leq C'''(\lambda(\partial_{\theta_2}\phi_2, \partial_z\phi_2, z)), \tag{3.6}$$

from (H_2) and (H_3) applied to ϕ_2.

Next we note that

$$|\partial_y\phi_1| \leq |\partial_y\phi_1 + \partial_y\phi_2| + |\partial_y\phi_2|. \tag{3.7}$$

The inequalities (3.4)-(3.7) imply (3.3).

It remains to show the composition formulas. Consider for $i = 1, 2$, the sequences of functions

$$\chi_p^i(x, \theta_i, y) = \exp\left(-p^{-1}\left(|x|^2 + |\theta_i|^2 + |y|^2\right)\right); (x, \theta_i, y) \in \mathbb{R}^n \times \mathbb{R}^{N_i} \times \mathbb{R}^n$$

We can show that (3.2) is satisfied for

$$a_p^1 = a_1\chi_p^1,\ a_p^2 = a_2\chi_p^2.$$

But

$$\chi_p^1(x, \theta_1, y)\,\chi_p^2(y, \theta_2, z) = \exp\left(-p^{-1}\left(|x|^2 + 2|y|^2 + |\theta_1|^2 + |\theta_2|^2 + |z|^2\right)\right).$$

Then it results that

$$\lim_{p\to\infty}\left(I\left(a_p^1a_p^2, \phi; h\right)f\right)(x) = (I(a_1a_2, \phi)f; h)(x), \tag{3.8}$$

for all $f \in S(\mathbb{R}^n)$.

On the other hand, we have seen in the proof of Theorem 2.1 that there exists, for all $l \in \mathbb{N}$ and $j = 1, 2$, an integer $M_{j,l}$ and a constant $C_{j,l} > 0$, such that, for all f in $S(\mathbb{R}^n)$ and $p \geq 1$, we have

$$\left\|I\left(a_p^j, \phi_j; h\right)f\right\|_{B^l} \leq C_{l,j}\|f\|_{B^{M_{j,l}}}, \tag{3.9}$$

where $B^l(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n), x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n), |\alpha| + |\beta| \leq l\right\}$.

So, for all fixed f_0 in $S(\mathbb{R}^n)$, $g_p = I\left(a_p^2, \phi_2; h\right)f_0$ describes a bounded subset of $S(\mathbb{R}^n)$ when p varies. Since $S(\mathbb{R}^n)$ is a Montel space, we can extract a subsequence, denoting also g_p, that converges in $S(\mathbb{R}^n)$ to $g = I(a_1, \phi_2; h)f_0$, but we have

$$\begin{aligned} & \left\|I\left(a_p^1, \phi_1; h\right)g_p - I(a_1, \phi_1; h)g\right\|_{B^l} \\ \leq\ & \left\|I\left(a_p^1, \phi_1; h\right)(g_p - g)\right\|_{B^l} + \left\|\left(I\left(a_p^1, \phi_1; h\right) - I(a_1, \phi_1; h)\right)g\right\|_{B^l}. \end{aligned} \tag{3.10}$$

Even re-extract a subsequence, we can suppose that

$$I\left(a_p^1, \phi_1; h\right)g \to I(a_1, \phi_1; h)g, \text{ in } S(\mathbb{R}^n). \tag{3.11}$$

From (3.9)-(3.11), It follows so, that for all l, leaves to extract a subsequence, we have

$$I\left(a_p^1, \phi_1; h\right)I\left(a_p^2, \phi_2; h\right)f_0 \to I(a_1, \phi_1; h)I(a_2, \phi_2; h)f_0 \text{ in } B^l.$$

□

4. The particular case

The purpose of this section is to study a particular case of the phase function ϕ which is very important in Cauchy problems, see [13]. Consider ϕ of the form

$$\phi(x,y,\theta) = S(x,\theta) - y\theta,$$

and suppose that S satisfies:

(G_1) $S \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\theta; \mathbb{C})$, where $S = S_1 + iS_2$.

(G_2) For all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exist $C_{\alpha,\beta} > 0$, such that

$$\left|\partial_x^\alpha \partial_\theta^\beta S(x,\theta)\right| \le C_{\alpha,\beta} \lambda(x,\theta)^{(2-|\alpha|-|\beta|)}.$$

(G_3) There exists $\delta_0 > 0$ such that

$$\inf_{x,\theta\in\mathbb{R}^n} \left|\det \frac{\partial^2 S}{\partial x \partial\theta}(x,\theta)\right| \ge \delta_0.$$

(G_4) $\forall (x,\theta) \in \mathbb{R}^n \times \mathbb{R}^N$, $S_2(x,\theta) \ge 0$.

Remark 4.1. From (G_2) and (G_3) and using the global inversion theorem we see that the mappings φ_1 and φ_2 defined by

$$\varphi_1 : (x,\theta) \to (x, \partial_x S(x,\theta)), \qquad \varphi_2 : (x,\theta) \to (\theta, \partial_\theta S(x,\theta)),$$

are global diffeomorphisms from $\mathbb{R}^{2n}$ onto $\mathbb{R}^n \times \mathbb{C}^n$. Indeed we have

$$J_{\varphi_1} = \begin{pmatrix} I_n & 0 \\ \frac{\partial^2 S}{\partial x^2} & \frac{\partial^2 S}{\partial x\partial\theta} \end{pmatrix}, \qquad J_{\varphi_2} = \begin{pmatrix} 0 & I_n \\ \frac{\partial^2 S}{\partial x\partial\theta} & \frac{\partial^2 S}{\partial\theta^2} \end{pmatrix},$$

and so

$$|J_{\varphi_1}| = |J_{\varphi_2}| = \left|\det \frac{\partial^2 S}{\partial x\partial\theta}\right| \ge \delta_0 \ne 0, \quad \text{for all } (x,\theta) \in \mathbb{R}^{2n}.$$

Furthermore

$$\left\|(\varphi_1'(x,\theta))^{-1}\right\| = \frac{1}{\left|\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\right|} \left\|{}^t A(x,\theta)\right\| \tag{4.1}$$

$$\left\|(\varphi_2'(x,\theta))^{-1}\right\| = \frac{1}{\left|\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\right|} \left\|{}^t B(x,\theta)\right\|, \tag{4.2}$$

where $A(x,\theta)$ and $B(x,\theta)$ are respectively the cofactor matrix of $\varphi_1'(x,\theta)$ and $\varphi_2'(x,\theta)$. By (G_2), we know that $\|{}^t A(x,\theta)\|$ and $\|{}^t B(x,\theta)\|$ are uniformly bounded.

Lemma 4.1. *If S satisfies $(G_1),(G_2),(G_3)$ and (G_4), then S satisfies the following inequalities:*

(i) There exist $C_1, C_2 > 0$, such that

$$\begin{cases} |x| \le C_1 \lambda(\theta, \partial_\theta S), & \text{for all } (x,\theta) \in \mathbb{R}^{2n}, \\ |\theta| \le C_2 \lambda(x, \partial_x S), & \text{for all } (x,\theta) \in \mathbb{R}^{2n}. \end{cases} \tag{4.3}$$

(ii) There exists $C_3 > 0$ such that for all $(x,\theta),(x',\theta') \in \mathbb{R}^{2n}$,

$$|x - x'| + |\theta - \theta'| \le C_3 \left[|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta')| + |\theta - \theta'|\right]. \tag{4.4}$$

The proof of the Lemma is similar to that of [12, lemma 3.3]

Remark 4.2. When $\theta = \theta'$ in (4.4), we have for all $(x,x',\theta) \in \mathbb{R}^{3n}$,

$$|x - x'| \le C_3 \left|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta)\right|. \tag{4.5}$$

Lemma 4.2. *Assume that S satisfies $(G_1),(G_2),(G_3)$ and (G_4). Then the function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ satisfies $(H_1),(H_2),(H_3),(H_3^*)$ and (H_4).*

Proof. It is clear that $(H_1), (H_2)$ and (H_4) are satisfied. Let's prove (H_3).

First observe that the second inequality in (H_3) is a consequence of (4.3). Also from (4.3) we have

$$\lambda(x,\theta,y) \le \lambda(x,\theta) + \lambda(y) \le C_4 (\lambda(\theta, \partial_\theta S) + \lambda(y)),\ C_4 > 0.$$

Further $\partial_{y_j}\phi = -\theta_j$, and $\partial_{\theta_j}\phi = \partial_{\theta_j} S - y_j$, so

$$\lambda(\theta, \partial_\theta S) = \lambda(\partial_y \phi, \partial_\theta \phi + y) \le 2\lambda(\partial_y\phi, \partial_\theta\phi, y),$$

which implies for some $C_5 > 0$,

$$\lambda(x,\theta,y) \le C_4 (2\lambda(\partial_y\phi, \partial_\theta\phi, y)) \le \frac{1}{C_5}\lambda(\partial_y\phi, \partial_\theta\phi, y).$$

The condition (H_3^*) can be shown in the same way. □

Proposition 4.1. *Assume that S satisfies $(G_1), (G_2)$ and (G_4), so there exists a constant $\varepsilon > 0$ such that the phase function ϕ given in (4.3) belongs to $\Gamma_1^2(\Omega_{\phi,\varepsilon})$, where*

$$\Omega_{\phi,\varepsilon} = \left\{(x,\theta,y) \in \mathbb{R}^{3n}; |\partial_\theta S(x,\theta) - y|^2 < \varepsilon\left(|x|^2 + |y|^2 + |\theta|^2\right)\right\}.$$

Proof. The matter is to show that:

there exists $\varepsilon > 0$, such that for all $\alpha, \beta, \gamma \in \mathbb{N}^n$, there exist $C_{\alpha,\beta,\gamma} > 0$:

$$\left|\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y)\right| \le C_{\alpha,\beta,\gamma}\lambda(x,\theta,y)^{(2-|\alpha|-|\beta|-|\gamma|)},\ (x,\theta,y) \in \Omega_{\phi,\varepsilon}. \tag{4.6}$$

For $|\gamma| = 1$, (for some $j \in \{1,\ldots,n\}$, $\gamma_j = 1$) we have

$$\left|\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y)\right| = \left|\partial_x^\alpha \partial_\theta^\beta(-\theta)\right| = \begin{cases} 0 & \text{if } |\alpha| \ne 0 \\ \left|\partial_\theta^\beta(-\theta_j)\right| & \text{if } \alpha = 0 \end{cases};$$

and for $|\gamma| > 1$, we have

$$\left|\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y)\right| = 0.$$

Then the estimate (4.6) is satisfied. It remains the case $|\gamma| = 0$.

But for all $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha| + |\beta| \le 1$, and from (G_2) there exists $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\theta^\beta \phi(x,\theta,y)\right| = \left|\partial_x^\alpha \partial_\theta^\beta S(x,\theta) - \partial_x^\alpha \partial_\theta^\beta(y\theta)\right| \le C_{\alpha,\beta}\lambda(x,\theta,y)^{(2-|\alpha|-|\beta|)}.$$

If $|\alpha| + |\beta| \ge 2$, one has $\partial_x^\alpha \partial_\theta^\beta \phi(x,\theta,y) = \partial_x^\alpha \partial_\theta^\beta S(x,\theta)$, and so in $\Omega_{\phi,\varepsilon}$ we have

$$|y| = |\partial_\theta S(x,\theta) - y - \partial_\theta S(x,\theta)| \le \sqrt{\varepsilon}\left(|x|^2 + |y|^2 + |\theta|^2\right)^{\frac{1}{2}} + C_6\lambda(x,\theta), \tag{4.7}$$

with $C_6 > 0$. Choosing ε small enough, to get a constant $C_7 > 0$ such that

$$|y| \le C_7\lambda(x,\theta),\ \forall (x,\theta,y) \in \Omega_{\phi,\varepsilon}.$$

Which prove the equivalence

$$\lambda(x,\theta,y) \simeq \lambda(x,\theta) \text{ in } \Omega_{\phi,\varepsilon}. \tag{4.8}$$

Therefore from the last property and (G_2) we obtain (4.6). □

In virtue of the equivalence (4.8), we deduce the following result.

Proposition 4.2. *If the amplitude $a : (x,\theta) \to a(x,\theta)$ is in $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n)$, then the amplitude $b : (x,\theta,y) \to a(x,\theta)$ is in $\Gamma_k^m\left(\mathbb{R}_x^n \times \mathbb{R}_\theta^n \times \mathbb{R}_y^n\right) \cap \Gamma_k^m(\Omega_{\phi,\varepsilon})$, for $k \in \{0,1\}$.*

As a consequence of the previous calculus we obtain a result of boundedness of h-admissible Fourier integral [1, 14, 15] operators with complex phase in $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$.

Theorem 4.1. *Let F_h be an integral operator of the form*

$$(F_h\psi)(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\theta)-y\theta)} a(x,\theta)\psi(y)\,dyd\theta.$$

where $a \in \Gamma_k^m\left(\mathbb{R}_{x,\theta}^{2n}\right)$, $k = 0,1$, $h \in]0,h_0]$ and S satisfies $(G_1), (G_2), (G_3)$ and (G_4). Then F_h can be extended to a linear continuous operator from $S(\mathbb{R}^n)$ into itself, and from $S'(\mathbb{R}^n)$ into itself.

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