**Abstract**
We study in this work a class of $h$-Fourier integral operators with complex phase. These operators are continuous on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$.

**Keywords:** $h$-admissible Fourier integral operators; complex phase; boundedness; compactness; amplitude; composition.

**AMS Subject Classification (2010):** Primary: 35S30; Secondary: 35S05; 47G30.

**1. Introduction**

A Fourier integral operator is an operator that can be written in the form

$$ (I(a, \phi) f)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{ih\phi(x,\theta,y)} a(x,\theta,y) f(y) \, dy \, d\theta, $$

(1.1)

where $f \in S(\mathbb{R}^n)$ (the Schwartz space). The function $a(x,\theta,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n)$ is called the amplitude, the function $\phi(x,y,\theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n; \mathbb{R})$ is called the phase function. The study of these operators, which are intimately connected to the theory of linear partial differential operators, has a long history and there is a large body of results made by several authors (see, e.g., [2, 5–12]). The first works on Fourier integral operators deal with local properties. We note that, K. Asada and D. Fujiwara [2] have studied for the first time a class of Fourier integral operators defined on $\mathbb{R}^n$.

In this paper we consider one of the most important problems in the theory of differential equations which is the study of the $h$-Fourier integral operators with a complex phase, this type of operator is represented by formula of the type

$$ (I(a, \phi; h) f)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{\frac{i}{h}\phi(x,\theta,y)} a(x,\theta,y) f(y) \, dy \, d\theta, $$

(1.2)

in which appear two $C^\infty$-functions, the phase function $\phi(x,y,\theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n)$ and the amplitude $a(x,\theta,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n)$ and a semiclassical parameter $h \in [0, h_0]$.

The purpose of this work is to generalize the notion of $h$-Fourier integral operators defined in [8] by considering the phase function $\phi$ with complex values, and applying the same technique of [2] to show that the $h$-Fourier integral operators with complex phase are well defined and they are continuous on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$ (the space of tempered distributions). We give also a result where it is shown that these types of operators are stable by composition.

When the phase function $\phi(x,y,\theta) = S(x,\theta) - y\theta$, where $S \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C})$, the operator (1.2) will be a particular case of $h$-Fourier integral operators with complex phase. In this case we will also give some hypothesis on the phase function $\phi$ and the amplitude $a$.

Let us now describe the plan of this article. In the second section we recall the continuity of some general class of Fourier integral operators on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$. The composition of $h$-Fourier integral operators with complex phase is given in the third section. The last section is devoted to study the particular case.

Received: 26–09–2017, Accepted: 20–03–2018
2. A general class of $h$-Fourier integral operators with complex phase

In this section we define the class of integral transformations of type

$$
(I (a, \phi; h), f) (x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} \overline{e^{i \phi(x, \theta, y)}} a(x, \theta, y) f(y) dyd\theta,
$$

(2.1)

where $f \in S(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $h \in [0, h_0]$, and $\phi : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{C}$.

In general the integral (2.1) is not absolutely convergent, so we can use the technique of oscillatory integral developed by Hörmander in [10].

Notation 2.1. For $(x, \theta, y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$, we set

$$\lambda(x, \theta, y) = \left(1 + |x|^2 + |y|^2 + |\theta|^2\right)^{1/2}.$$

The phase function $\phi = \varphi + i\psi$ and the amplitude $a$ are assumed to satisfy the following conditions:

$(H_1)$ $\phi : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{C}$ is a $C^\infty$ application.

$(H_2)$ $\forall (\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}_n$, $\exists C_{\alpha \beta \gamma} \geq 0$:

$$|\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x, \theta, y)| \leq C_{\alpha \beta \gamma} [\lambda(x, \theta, y)]^{2 - (|\alpha| + |\beta| + |\gamma|)}.$$

$(H_3)$ There exist real numbers $K_1, K_2 > 0$ such that

$$K_1 \lambda(x, \theta, y) \leq \lambda(\partial_x \phi, \partial_\theta \phi, \varphi) \leq K_2 \lambda(x, \theta, y), \forall (x, \theta, y) \in \mathbb{R}^n_x \times \mathbb{R}^n_\theta \times \mathbb{R}^n_y.$$

$(H_4)^*$ There exist real numbers $K_1^*, K_2^* > 0$ such that

$$K_1^* \lambda(x, \theta, y) \leq \lambda(\partial_x \phi, \partial_\theta \phi, \varphi) \leq K_2^* \lambda(x, \theta, y), \forall (x, \theta, y) \in \mathbb{R}^n_x \times \mathbb{R}^n_\theta \times \mathbb{R}^n_y.$$

$(H_4)$ $\forall (x, \theta, y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n : \psi(x, \theta, y) \geq 0$.

For any open subset $\Omega$ of $\mathbb{R}^n_x \times \mathbb{R}^n_\theta \times \mathbb{R}^n_y$, $\mu \in \mathbb{R}$ and $\rho \in [0, 1]$, we set

$$\Gamma^\mu_\rho(\Omega) = \{a \in C^\infty(\Omega) : |\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma a| \leq C_{\alpha \beta \gamma} [\lambda(x, \theta, y)]^{\mu - (|\alpha| + |\beta| + |\gamma|)}\}.$$

For $\Omega = \mathbb{R}_x^n \times \mathbb{R}_\theta^n \times \mathbb{R}_y^n$, we denote $\Gamma^\mu_0(\Omega) = \Gamma^\mu_0$.

Now if $\varphi$ satisfies $(H_1), (H_2), (H_3)$ and $a \in \Gamma^0_0$, we can give a meaning to the right hand side of (2.1), so we consider $g \in S(\mathbb{R}_x^n \times \mathbb{R}_\theta^n \times \mathbb{R}_y^n)$, $g(0) = 1$. If $a \in \Gamma^0_0$, we define

$$a_\sigma(x, \theta, y) = g\left(\frac{x}{\sigma}, \frac{\theta}{\sigma}, \frac{y}{\sigma}\right) a(x, \theta, y), \sigma > 0.$$

We have the following result concerning the boundedness of $h$-Fourier integral operators with complex phase on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$.

Theorem 2.1. If the phase function $\phi$ satisfies $(H_1), (H_2), (H_3)$ and $(H_4)$ and if $a \in \Gamma^0_0$, then

1. For all $f \in S(\mathbb{R}^n)$, $\lim_{\rho \to \infty} [(I (a_p, \phi; h), f)](x)$ exists for every $x \in \mathbb{R}^n$ and is independent of the choice of the function $g$. We set then

$$ (I (a, \phi; h), f) = \lim_{\rho \to \infty} (I (a_p, \phi; h), f) $$

2. $I (a, \phi; h)$ is a linear continuous operator from $S(\mathbb{R}^n)$ into itself.

3. Furthermore, if $\phi$ satisfies $(H_4^*)$, so $I (a, \phi; h) \in \mathcal{L}(S' (\mathbb{R}^n))$ (here $S' (\mathbb{R}^n)$ is the space of all tempered distributions on $\mathbb{R}^n$).
Proof. Let $\delta \in C_0^\infty (\mathbb{R}^n)$ such that $\text{supp} \delta \subseteq [-1, 2]$ and $\delta \equiv 1$ on $[0, 1]$.

For all $\varepsilon > 0$, we set
\[
\omega_\varepsilon (x, \theta, y) = \delta \left( \frac{|\partial_y \phi|^2 + |\partial_\theta \phi|^2}{\varepsilon \lambda (x, \theta, y)^2} \right).
\]

From $(H_3)$ there exists $C > 0$ for which we have on the support of $\omega_\varepsilon$
\[
\lambda (x, \theta, y) \leq C \left( 1 + |y|^2 \right)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \lambda (x, \theta, y).
\]

Choosing $\varepsilon$ small enough we get that there exists a constant $C_0$, such that the inequality
\[
\lambda (x, \theta, y) \leq C_0 \left( 1 + |y|^2 \right)^{\frac{1}{2}}
\]
holds in the support of $\omega_\varepsilon$.

From this inequality we can see that $I (\omega_\varepsilon a_p, \phi; h) f$ is an absolutely convergent integral and we have
\[
\lim_{p \to \infty} I (\omega_\varepsilon a_p, \phi; h) f = I (\omega_\varepsilon a, \phi; h) f.
\]

The continuity of the operator $I (\omega_\varepsilon a, \phi; h) f$ from $S (\mathbb{R}^n)$ into itself follows from $(H_2)$.

Next we study the limit $\lim_{p \to \infty} I ((1 - \omega_\varepsilon) a_p, \phi; h) f$. Consider the operator
\[
L = \frac{h}{i} \left( \sum_{j=1}^n (\partial_y \phi) \frac{\partial}{\partial y_j} + \sum_{j=1}^N (\partial_\theta \phi) \frac{\partial}{\partial \theta_j} \right).
\]

One can show easily that
\[
L (e^{i\phi}) = e^{i\phi}.
\]

Let $\Omega_0$ be the open subset of $\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$ defined by
\[
\Omega_0 = \left\{ (x, \theta, y), \ |\partial_y \phi|^2 + |\partial_\theta \phi|^2 > \varepsilon \frac{\lambda (x, \theta, y)^2}{2} \right\}.
\]

By recurrence we prove that:

For all integers $q \geq 0$, and $b \in C_0^\infty (\mathbb{R}^n \times \mathbb{R}_0^N)$, we have
\[
(tL)^q ((1 - \omega_\varepsilon) b) = \sum_{|\alpha| + |\beta| \leq q} g_{\alpha,\beta} \partial_y^\alpha \partial_\theta^\beta ((1 - \omega_\varepsilon) b),
\]

where the $g_{\alpha,\beta}$ are in $\Gamma_0^{-q} (\Omega_0)$ and depend only on $\phi$. In particular for $q = 0$, we have
\[
tL = \sum_j F_j \frac{\partial}{\partial y_j} + \sum_j G_j \frac{\partial}{\partial \theta_j} + H,
\]

where $F_j \in \Gamma_0^{-1} (\Omega_0), G_j \in \Gamma_0^{-1} (\Omega_0), H \in \Gamma_0^{-2} (\Omega_0)$.

From (2.3) we have also for all integer $q \geq 0$,
\[
I ((1 - \omega_\varepsilon) a_p, \phi; h) f (x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{i\phi (x, \theta, y)} (tL)^q ((1 - \omega_\varepsilon) a_p, f) dy d\theta.
\]

But
\[
(tL)^q ((1 - \omega_\varepsilon) a_p f) \text{ described (when } p \text{ varies) a bound of } \Gamma_0^{\mu-q},
\]

and
\[
\lim_{p \to \infty} (tL)^q ((1 - \omega_\varepsilon) a_p f) (x, \theta, y) = (tL)^q ((1 - \omega_\varepsilon) af) (x, \theta, y),
\]

for all $(x, \theta, y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$. 


To finish, we have for all $s > n + N$

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^N} \lambda^{-s} (x, \theta, y) \, dy \, d\theta \leq \gamma_s \lambda^{n+N-s} (x). \tag{2.9}
$$

From (2.6)-(2.9) and using the Lebesgue’s theorem we obtain

$$
\lim_{p \to \infty} I ((1 - \omega_{\varepsilon}) a_p, \phi; h) f (x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{\tilde{\gamma}_{\phi}(x,\theta,y)} (t L)^{q} ((1 - \omega_{\varepsilon}) a, f; h) \, dy \, d\theta, \tag{2.10}
$$

where $q$ satisfies $q > n + N + \mu$.

The first assertion of the theorem can be proved from (2.2) and (2.10).

Now let’s show the continuity of $I ((1 - \omega_{\varepsilon}) a, \phi; h)$. From (2.5) and (2.10), we have

$$
I ((1 - \omega_{\varepsilon}) a, \phi; h) f (x) = \sum_{|\gamma| \leq q} \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{\tilde{\gamma}_{\phi}(x,\theta,y)} b^{(q)}_{\gamma} (x, \theta, y) \partial_{\gamma}^{|\beta|} f (y) \, dy \, d\theta, \tag{2.11}
$$

with $b^{(q)}_{\gamma} \in \Gamma_0^{\mu-q}$. On the other hand, we have

$$
x^{\alpha} \partial^{\beta}_{\gamma} \left( e^{\tilde{\gamma}_{\phi}(x,\theta,y)} (x, \theta, y) \right) \in \Gamma_0^{\mu-q+|\alpha|+|\beta|}. \tag{2.12}
$$

This property and (2.11) imply that, for all $q > n + N + \mu + |\alpha| + |\beta|$, there exists a constant $C_{\alpha,\beta,q}$ such that

$$
\left| x^{\alpha} \partial^{\beta}_{\gamma} I ((1 - \omega_{\varepsilon}) a, \phi; h) f (x) \right| \leq C_{\alpha,\beta,q} \sup_{x \in \mathbb{R}^n} \left| \partial^{\gamma}_{\beta} f (x) \right|,
$$

which proves the continuity of $I ((1 - \omega_{\varepsilon}) a, \phi; h)$.

The last assertion of the theorem is an immediate consequence of the second one, indeed the matter is to show that the operator $t F$ is continuous from $S (\mathbb{R}^n)$ to itself, where $F = I (a, \phi; h)$. But $t F = I (\tilde{a}, \tilde{\phi}; h)$, with

$$
\tilde{\phi} (x, \theta, y) = \phi (y, \theta, x),
\tilde{a} (x, \theta, y) = a (y, \theta, x).
$$

Since $\phi$ satisfies $(H_3)$, then $\tilde{\phi}$ satisfies $(H_3)$, so we can deduce the result. \hfill \Box

**Remark 2.1.** We can obtain the same result if the hypothesis on $\phi$ are fulfilled only on the support of the amplitude $a$.

### 3. Composition of two $h$-Fourier integral operators with complex phase

In this section we prove that the composition of two $h$-Fourier integral operators with complex phase, have a meaning, and give an operator of same type.

**Theorem 3.1.** Assume that the phase functions $\phi_1$ and $\phi_2$ satisfy $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$. Set

$$
\phi (x, \theta, z) = \phi_1 (x, \theta_1, y) + \phi_2 (y, \theta_2, z), \tag{3.1}
$$

with $\theta_1 \in \mathbb{R}^{N_1}$, $\theta_2 \in \mathbb{R}^{N_2}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^n$, $\theta = (\theta_1, \theta_2)$. Then $\phi$ verifies $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$, and for all $a_1 \in \Gamma^{\mu_1}_0$, $a_2 \in \Gamma^{\mu_2}_0$, we have

$$
I (a_1, \phi_1; h) I (a_2, \phi_2; h) = I (a_1 \times a_2, \phi; h), \tag{3.2}
$$

where

$$
(a_1 \times a_2) (x, \theta, z) = a_1 (x, \theta_1, y) a_2 (y, \theta_2, z).
$$

**Proof.** We first observe that $(H_1)$, $(H_2)$ and $(H_4)$ are trivial. So we have to prove the condition $(H_3)$.

We can see that the first inequality is evident, so it suffices to show that $\phi$ satisfies the following property: there exists $K > 0$ such that

$$
\lambda (x, \theta_1, y, \theta_2, z) \leq K \lambda (z, \partial_z \phi_2, \partial_y \phi_1 + \partial_y \phi_2, \partial_{\theta_1} \phi_1, \partial_{\theta_2} \phi_2). \tag{3.3}
$$
Applying $(H_3)$ to $\phi_1$ and $\phi_2$ we get that there exists $C > 0$ such that
\[
\lambda(x, \theta_1, y, \theta_2, z) \leq C\lambda(\partial_{\theta_1} \phi_1, \partial_{\theta_1} \phi_1, y, \partial_{\theta_2} \phi_2, \partial_{\theta_2} \phi_2, z),
\]
but we have also
\[
\lambda(y) \leq C'\lambda(\partial_{\theta_2} \phi_2, \partial_{\theta_2} \phi_2, z),
\]
from $(H_3)$ applied to $\phi_2$, and
\[
|\partial_y \phi_2| \leq C''\lambda(y, \theta_2, z) \leq C'''(\lambda(\partial_{\theta_2} \phi_2, \partial_{\theta_2} \phi_2, z)),
\]
from $(H_2)$ and $(H_3)$ applied to $\phi_2$. Next we note that
\[
|\partial_y \phi_1| = |\partial_y \phi_1 + \partial_y \phi_2| + |\partial_y \phi_2|.
\]
The inequalities (3.4)-(3.7) imply (3.3).

It remains to show the composition formulas. Consider for $i = 1, 2$, the sequences of functions

\[
\chi^1_p(x, \theta_1, y) = \exp\left(-p^{-1}\left(|x|^2 + |\theta_1|^2 + |y|^2\right)\right); (x, \theta_1, y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n
\]

We can show that (3.2) is satisfied for

\[
a_p^1 = a_1 \chi_p^1, \quad a_p^2 = a_2 \chi_p^2.
\]

But

\[
\chi^1_p(x, \theta_1, y) \chi^2_p(y, \theta_2, z) = \exp\left(-p^{-1}\left(|x|^2 + 2|y|^2 + |\theta_1|^2 + |\theta_2|^2 + |z|^2\right)\right).
\]

Then it results that

\[
\lim_{p \to \infty} (I(a_p^1 a_p^2, \phi; h) f)(x) = (I(a_1 a_2, \phi) f; h)(x),
\]

for all $f \in S(\mathbb{R}^n)$.

On the other hand, we have seen in the proof of Theorem 2.1 that there exists, for all $l \in \mathbb{N}$ and $j = 1, 2$, an integer $M_{j,l}$ and a constant $C_{l,j} > 0$, such that, for all $f \in S(\mathbb{R}^n)$ and $p \geq 1$, we have

\[
\|I(a_p^l, \phi_j; h) f\|_{B^l} \leq C_{l,j} \|f\|_{B^{M_{j,l}}},
\]

where $B^l(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n), x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n), |\alpha| + |\beta| \leq l\}$.

So, for all fixed $f_0$ in $S(\mathbb{R}^n)$, $g_p = I(a_p^2, \phi_2; h) f_0$ describes a bounded subset of $S(\mathbb{R}^n)$ when $p$ varies. Since $S(\mathbb{R}^n)$ is a Montel space, we can extract a subsequence, denoting also $g_p$, that converges in $S(\mathbb{R}^n)$ to $g = I(a_1, \phi_2; h) f_0$, but we have

\[
\|I(a_p^l, \phi_1; h) g_p - I(a_1, \phi_1; h) g\|_{B^l} \leq \|I(a_p^l, \phi_1; h) (g_p - g)\|_{B^l} + \|(I(a_p^l, \phi_1; h) - I(a_1, \phi_1; h)) g\|_{B^l}.
\]

Even re-extract a subsequence, we can suppose that

\[
I(a_p^l, \phi_1; h) g \to I(a_1, \phi_1; h) g, \quad \text{in } S(\mathbb{R}^n).
\]

From (3.9)-(3.11), It follows so, that for all $l$, leaves to extract a subsequence, we have

\[
I(a_p^1, \phi_1; h) I(a_p^2, \phi_2; h) f_0 \to I(a_1, \phi_1; h) I(a_2, \phi_2; h) f_0 \quad \text{in } B^l.
\]
4. The particular case

The purpose of this section is to study a particular case of the phase function $\phi$ which is very important in Cauchy problems, see [13]. Consider $\phi$ of the form

$$\phi (x, y, \theta) = S (x, \theta) - y\theta,$$

and suppose that $S$ satisfies:

$(G_1)$ $S \in C^\infty (\mathbb{R}_x^n \times \mathbb{R}_y^n; \mathbb{C})$, where $S = S_1 + iS_2$.

$(G_2)$ For all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exist $C_{\alpha, \beta} > 0$, such that

$$\left\| \partial_x^\alpha \partial_\theta^\beta S (x, \theta) \right\| \leq C_{\alpha, \beta} \lambda (x, \theta)^{2 - |\alpha| - |\beta|}.$$

$(G_3)$ There exists $\delta_0 > 0$ such that

$$\inf_{x, \theta \in \mathbb{R}^n} \left\| \det \frac{\partial^2 S}{\partial x \partial \theta} (x, \theta) \right\| \geq \delta_0.$$

$(G_4)$ $\forall (x, \theta) \in \mathbb{R}^n \times \mathbb{R}^N$, $S_2 (x, \theta) \geq 0$.

Remark 4.1. From $(G_2)$ and $(G_3)$ and using the global inversion theorem we see that the mappings $\varphi_1$ and $\varphi_2$ defined by

$$\varphi_1 : (x, \theta) \mapsto (x, \partial_x S (x, \theta)), \quad \varphi_2 : (x, \theta) \mapsto (\theta, \partial_\theta S (x, \theta)),$$

are global diffeomorphisms from $\mathbb{R}^{2n}$ onto $\mathbb{R}^n \times \mathbb{C}^n$. Indeed we have

$$J_{\varphi_1} = \begin{pmatrix} I_n & 0 \\ \frac{\partial^2 S}{\partial x^2} & \frac{\partial^2 S}{\partial x \partial \theta} \end{pmatrix}, \quad J_{\varphi_2} = \begin{pmatrix} 0 & I_n \\ \frac{\partial^2 S}{\partial x \partial \theta} & \frac{\partial^2 S}{\partial \theta^2} \end{pmatrix},$$

and so

$$|J_{\varphi_1}| = |J_{\varphi_2}| = \left\| \det \frac{\partial^2 S}{\partial x \partial \theta} (x, \theta) \right\| \geq \delta_0 \neq 0, \text{ for all } (x, \theta) \in \mathbb{R}^{2n}.$$

Furthermore

$$\left\| (\varphi_1' (x, \theta))^{-1} \right\| = \frac{1}{\left\| \det \frac{\partial S}{\partial x \partial \theta} (x, \theta) \right\|} \left\| A (x, \theta) \right\|,$$

(4.1)

and

$$\left\| (\varphi_2' (x, \theta))^{-1} \right\| = \frac{1}{\left\| \det \frac{\partial S}{\partial x \partial \theta} (x, \theta) \right\|} \left\| B (x, \theta) \right\|,$$

(4.2)

where $A (x, \theta)$ and $B (x, \theta)$ are respectively the cofactor matrix of $\varphi_1 (x, \theta)$ and $\varphi_2 (x, \theta)$. By $(G_2)$, we know that $\left\| A (x, \theta) \right\|$ and $\left\| B (x, \theta) \right\|$ are uniformly bounded.

Lemma 4.1. If $S$ satisfies $(G_1), (G_2), (G_3)$ and $(G_4)$, then $S$ satisfies the following inequalities:

(i) There exist $C_1, C_2 > 0$, such that

$$\begin{cases} |x| \leq C_1 \lambda (x, \partial_\theta S), & \text{for all } (x, \theta) \in \mathbb{R}^{2n}, \\ |\theta| \leq C_2 \lambda (x, \partial_x S), & \text{for all } (x, \theta) \in \mathbb{R}^{2n}. \end{cases}$$

(4.3)

(ii) There exists $C_3 > 0$ such that for all $(x, \theta), (x', \theta') \in \mathbb{R}^{2n},$

$$|x - x'| + |\theta - \theta'| \leq C_3 \left[ |(\partial_\theta S) (x, \theta) - (\partial_\theta S) (x', \theta')| + |\theta - \theta'| \right].$$

(4.4)

The proof of the Lemma is similar to that of [12, lemma 3.3]

Remark 4.2. When $\theta = \theta'$ in (4.4), we have for all $(x, x', \theta) \in \mathbb{R}^{3n},$

$$|x - x'| \leq C_3 \left| (\partial_\theta S) (x, \theta) - (\partial_\theta S) (x', \theta) \right|.$$

(4.5)

Lemma 4.2. Assume that $S$ satisfies $(G_1), (G_2), (G_3)$ and $(G_4)$. Then the function $\phi (x, y, \theta) = S (x, \theta) - y\theta$ satisfies $(H_1), (H_2), (H_3), (H_4)$ and $(H_4)$. 

Proof. It is clear that $(H_1)$, $(H_2)$ and $(H_4)$ are satisfied. Let’s prove $(H_3)$.

First observe that the second inequality in $(H_3)$ is a consequence of (4.3). Also from (4.3) we have

$$\lambda (x, \theta, y) \leq \lambda (x, \theta) + \lambda (y) \leq C_4 \left( \lambda (\theta, \partial \theta S) + \lambda (y) \right), \quad C_4 > 0.$$ 

Further $\partial_y \phi = -\theta_j$, and $\partial_y \phi = \partial \theta S - y_j$, so

$$\lambda (\theta, \partial \theta S) = \lambda (\partial_y \phi, \partial_y \phi + y) \leq 2 \lambda (\partial_y \phi, \partial_y \phi, y),$$

which implies for some $C_5 > 0$,

$$\lambda (x, \theta, y) \leq C_4 \left( 2 \lambda (\partial_y \phi, \partial_y \phi, y) \right) \leq \frac{1}{C_5} \lambda (\partial_y \phi, \partial_y \phi, y).$$

The condition $(H_3)$ can be shown in the same way.

**Proposition 4.1.** Assume that $S$ satisfies $(G_1)$, $(G_2)$ and $(G_4)$, so there exists a constant $\varepsilon > 0$ such that the phase function $\phi$ given in (4.3) belongs to $\Gamma^2_\varepsilon (\Omega_{\phi, \varepsilon})$, where

$$\Omega_{\phi, \varepsilon} = \left\{ (x, \theta, y) \in \mathbb{R}^{3n} : |\partial \theta S (x, \theta) - y| < \varepsilon \left( |x|^2 + |y|^2 + |\theta|^2 \right) \right\}.$$

Proof. The matter is to show that:

there exists $\varepsilon > 0$, such that for all $\alpha, \beta, \gamma \in \mathbb{N}^n$, there exist $C_{\alpha, \beta, \gamma} > 0$:

$$|\partial_{\alpha}^\beta \phi (x, \theta, y) | \leq C_{\alpha, \beta, \gamma} \lambda (x, \theta, y)^{(2-|\alpha|-|\beta|-|\gamma|)}, \quad (x, \theta, y) \in \Omega_{\phi, \varepsilon}.$$ (4.6)

For $|\gamma| = 1$, (for some $j \in \{1, \ldots, n\}$, $\gamma_j = 1$) we have

$$\left| \partial_{\alpha}^\beta \partial_{\gamma}^\gamma \phi (x, \theta, y) \right| = \left| \partial_{\alpha}^\beta (-\theta) \right| = \left\{ \begin{array}{ll} 0 & \text{if } |\alpha| \neq 0 \\ \left| \partial_{\theta}^\gamma (-\theta) \right| & \text{if } \alpha = 0 \end{array} \right.;$$

and for $|\gamma| > 1$, we have

$$\left| \partial_{\alpha}^\beta \partial_{\gamma}^\gamma \phi (x, \theta, y) \right| = 0.$$

Then the estimate (4.6) is satisfied. It remains the case $|\gamma| = 0$.

But for all $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha| + |\beta| \leq 1$, and from $(G_2)$ there exists $C_{\alpha, \beta} > 0$ such that

$$|\partial_{\alpha}^\beta \phi (x, \theta, y) | = \left| \partial_{\alpha}^\beta S (x, \theta, y) - \partial_{\alpha}^\beta \theta (y\theta) \right| \leq C_{\alpha, \beta} \lambda (x, \theta, y)^{(2-|\alpha|-|\beta|)}.$$

If $|\alpha| + |\beta| \geq 2$, one has $\partial_{\alpha}^\beta \phi (x, \theta, y) = \partial_{\alpha}^\beta S (x, \theta, y)$, and so in $\Omega_{\phi, \varepsilon}$ we have

$$|y| = |\partial \theta S (x, \theta) - y - \partial \theta S (x, \theta)| \leq \sqrt{\varepsilon} \left( |x|^2 + |y|^2 + |\theta|^2 \right)^{\frac{1}{2}} + C_6 \lambda (x, \theta),$$ (4.7)

with $C_6 > 0$. Choosing $\varepsilon$ small enough, to get a constant $C_7 > 0$ such that

$$|y| \leq C_7 \lambda (x, \theta), \quad \forall (x, \theta, y) \in \Omega_{\phi, \varepsilon}.$$

Which prove the equivalence

$$\lambda (x, \theta, y) \simeq \lambda (x, \theta) \text{ in } \Omega_{\phi, \varepsilon}.$$ (4.8)

Therefore from the last property and $(G_2)$ we obtain (4.6). □

In virtue of the equivalence (4.8), we deduce the following result.

**Proposition 4.2.** If the amplitude $a : (x, \theta) \rightarrow a (x, \theta)$ is in $\Gamma^0_k (\mathbb{R}^n \times \mathbb{R}^n)$, then the amplitude $b : (x, \theta, y) \rightarrow a (x, \theta)$ is in $\Gamma^0_k \left( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \right) \cap \Gamma^0_k (\Omega_{\phi, \varepsilon})$, for $k \in \{0, 1\}$.

As a consequence of the previous calculus we obtain a result of boundedness of $h$-admissible Fourier integral [1, 14, 15] operators with complex phase in $S (\mathbb{R}^n)$ and $S' (\mathbb{R}^n)$.

**Theorem 4.1.** Let $F_h$ be an integral operator of the form

$$(F_h \psi) (x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i \frac{1}{h} (S(x, \theta) - y \theta) a (x, \theta) \psi (y)} dy \theta.$$

where $a \in \Gamma^0_k \left( \mathbb{R}^n_{x, \theta, y} \right), \ k = 0, 1 \ h \in [0, h_0]$ and $S$ satisfies $(G_1), (G_2), (G_3)$ and $(G_4)$.

Then $F_h$ can be extended to a linear continuous operator from $S (\mathbb{R}^n)$ into itself, and from $S' (\mathbb{R}^n)$ into itself.
References


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