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On the Refined Hermite-Hadamard Inequalities

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Abstract
In this paper, we give some new refinements of Hermite-Hadamard inequality for co-ordinated convex function. These refinements provide us better estimation as compare to the earlier established refinements of Hadamard's inequality.

Keywords: *Convex function; Co-ordinated convex function; Hermite-Hadamard's inequality.*

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1. Introduction

Let $\mathtt{I}$ be an interval in $\mathbb{R}$ and $f : \mathtt{I} \subseteq \mathbb{R} \to \mathbb{R}$ such that $a, b \in \mathtt{I}$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hadamard's inequality for convex mappings. Note that some of the classical inequalities for means can be derived from Hadamard's for appropriate particular selections of the mapping f. The inequality (1.1) which appeared for the first time in ([11], 1893) gives us an estimate of the mean value of a convex function f. Since then, many important refinements of Hermite-Hadamard inequality for convex functions have been investigated extensively. For example see [1]-[18].

In ([6], 1992) S. S. Dragomir defined a mapping $\mathtt{H}$ in the following way:

$$\mathtt{H}(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx, \tag{1.2}$$

where $\mathtt{H} : [0,1] \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is a convex function defined on $[a,b]$, and proved a new refinement of Hadamard's inequality:

Theorem 1.1 ([6])**.** *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $\mathtt{H} : [0,1] \to \mathbb{R}$ be defined as above, then we have:*

$$f\left(\frac{a+b}{2}\right) \leq \mathtt{H}(t) \leq \frac{1}{b-a}\int_a^b f(x)dx \tag{1.3}$$

In ([3], 2010), A. E. Farissi established a simple proof and a new generalization of the inequality (1.1) as follows:

Theorem 1.2 ([3])**.** *Assume that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$, then for all $t \in [0,1]$ we have:*

$$f\left(\frac{a+b}{2}\right) \leq \mathtt{l}(t) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \mathtt{L}(t) \leq \frac{f(a)+f(b)}{2}, \tag{1.4}$$

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where

$$\mathit{l}(t) =: tf\left(\frac{tb+(2-t)a}{2}\right)+(1-t)f\left(\frac{(1+t)b+(1-t)a}{2}\right)$$

and

$$\mathit{L}(t) =: \frac{1}{2}\left[f(tb+(1-t)a)+tf(a)+(1-t)f(b)\right].$$

To complete this section we state definition of co-ordinated convex function and its related Hermite-Hadamard type inequalities:

Let us consider the bi-dimensional interval $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le\lambda f(x,y)+(1-\lambda)f(z,w) \tag{1.5}$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex for all $y\in[c,d]$ and $x\in[a,b]$.
A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1.1. A function $f:\Delta\to\mathbb{R}$ is said to be convex on co-ordinates on Δ if the inequality

$$\begin{aligned} f(\lambda x+(1-\lambda)z,ty+(1-t)w) &\le \lambda tf(x,y)+\lambda(1-t)f(x,w)\\ &+ (1-\lambda)tf(z,y)+(1-t)(1-\lambda)f(z,w)\end{aligned}$$

holds for all $(x,y),\ (x,w),\ (z,y),\ (z,w)\in\Delta$ and $t\in[0,1],\ \lambda\in[0,1]$.

S. S. Dragomir in [7] established the following Hadamard-type inequalities for co-ordinated convex functions:

Theorem 1.3 ([7])**.** *Suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ \le\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\ \le\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\ \le\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ +\ & \left.\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\ \le\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned} \tag{1.6}$$

The above inequalities are sharp.

In ([1], 2008), M. Alomari and Darus defined co-ordinated s-convex functions and proved some inequalities. In ([16], 2009), analogous results for h-convex functions on the co-ordinates were proved by M. A. Latif and M. Alomari.

In ([2], 2009), Alomari et al. established some Hadamard type inequalities for co-ordinated log-convex functions. In ([17]), M. A. Latif and S. S. Dragomir obtained some new Hadamard type inequalities for differentiable co-ordinated convex and concave functions.

For recent results and generalizations concerning Hermite-Hadamard type inequality for co-ordinated convex functions see ([18], 2012) and some of the references given therein.

In this paper, we obtain some new Hermite-Hadamard type inequalities for convex functions on the co-ordinates. These results refine the Hermite-Hadamard type inequalities given in Theorem 1.3 as well as in [3].

2. Refinements

Now, we give first refinement of Hermite-Hadamard inequality for co-ordinated convex function.

Theorem 2.1. *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ, then we have the following refinement:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \Omega_1(t,s) \\ \leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ \leq\ & \Omega_2(t,s) \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \end{aligned} \tag{2.1}$$

where

$$\Omega_1(t,s) = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)dy\right]$$

and

$$\Omega_2(t,s) = \frac{1}{2}\left[\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \left[f\left(x, sy+(1-s)\frac{c+d}{2}\right) + f\left(tx+(1-t)\frac{a+b}{2}, y\right)\right]dydx\right].$$

Proof. Since $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ, it follows that the mapping $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ is convex on $[c,d]$ for all $x \in [a,b]$. Then by making use of Theorem 1.2 one has

$$f_x\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f_x(sy+(1-s)\frac{c+d}{2})dy \leq \frac{1}{d-c}\int_c^d f_x(y)dy.$$

Integrating this inequality on $[a,b]$, we have

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(x, sy+(1-s)\frac{c+d}{2}\right)dydx \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx. \end{aligned} \tag{2.2}$$

By a similar argument applied for the mapping $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ is convex on $[a,b]$ for all $y \in [a,b]$, we get

$$\begin{aligned} & \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2}, y\right)dydx \\ \leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx. \end{aligned} \tag{2.3}$$

Summing the inequalities (2.5) and (2.6), we get the third and the fourth inequalities in (2.1). By Hadamard's inequality, we also have

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right)dx \leq \frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)dy \le \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy$$

which give, by addition, the first and the second inequalities in (2.1). □

Theorem 2.2. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is co-ordinated convex on Δ. Then for all $t\in[0,1]$, $s\in[0,1]$, one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le A \\ &\le \gamma(t,s) \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\le \Gamma(t,s) \\ &\le B \\ &\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}, \end{aligned} \tag{2.4}$$

where

$$A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right],$$

$$B = \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right],$$

$$\begin{aligned} \gamma(t,s) &= \frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[sf\left(x,\frac{sd+(2-s)c}{2}\right)+(1-s)f\left(x,\frac{(1+s)d+(1-s)c}{2}\right)\right]dx\right. \\ &+ \left.\frac{1}{d-c}\int_c^d\left[tf\left(\frac{tb+(2-t)a}{2},y\right)+(1-t)f\left(\frac{(1+t)b+(1-t)a}{2},y\right)\right]dy\right] \end{aligned}$$

and

$$\begin{aligned} \Gamma(t,s) &= \frac{1}{4}\left[\frac{1}{b-a}\int_a^b[f(x,sd+(1-s)c)+sf(x,c)+(1-s)f(x,d)]\,dx\right. \\ &+ \left.\frac{1}{d-c}\int_c^d[f(tb+(1-t)a,y)+tf(a,y)+(1-t)f(b,y)]\,dy\right]. \end{aligned}$$

Proof. Since $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ, it follows that the mapping $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ is convex on $[c,d]$ for all $x\in[a,b]$. Then under the utility of Theorem 1.2 one has

$$f_x\left(\frac{c+d}{2}\right) \le sf_x\left(\frac{sd+(2-s)c}{2}\right)+(1-s)f_x\left(\frac{(1+s)d+(1-s)c}{2}\right) \le \frac{1}{d-c}\int_c^d f_x(y)dy$$

and integrating this inequality over $[a,b]$, we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\ \le\ &\frac{1}{b-a}\int_a^b\left[sf\left(x,\frac{sd+(2-s)c}{2}\right)+(1-s)f\left(x,\frac{(1+s)d+(1-s)c}{2}\right)\right]dx \\ \le\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx. \end{aligned} \tag{2.5}$$

Similarly applying Hadamard's inequality for the convex mapping $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ on $[a,b]$ for all $y \in [c,d]$, we get

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy \\
\leq\ & \frac{1}{d-c}\int_c^d \left[tf\left(\frac{tb+(2-t)a}{2}, y\right) + (1-t)f\left(\frac{(1+t)b+(1-t)a}{2}, y\right)\right] dy \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx.
\end{aligned} \tag{2.6}$$

Summing the inequalities (2.5) and (2.6), we get the second and the third inequalities in (2.4). Again by convexity of the mapping $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ on $[c,d]$ for all $x \in [a,b]$ and using the right double inequality in (1.4) one has

$$\frac{1}{d-c}\int_c^d f(x,y)dy \leq \frac{1}{2}\left[f(x, sd+(1-s)c) + sf(x,c) + (1-s)f(x,c)\right] \leq \frac{f(x,c)+f(x,d)}{2}$$

and integrating this over the interval $[a,b]$, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
\leq\ & \frac{1}{2(b-a)}\int_a^b \left[f(x, sd+(1-s)c) + sf(x,c) + (1-s)f(x,d)\right] dx \\
\leq\ & \frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]dx
\end{aligned} \tag{2.7}$$

By a similar argument applied for the mapping $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ is convex on $[a,b]$ for all $y \in [a,b]$, we get

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
\leq\ & \frac{1}{2(d-c)}\int_c^d \left[f(tb+(1-t)a, y) + tf(a,y) + (1-t)f(a,y)\right] dy \\
\leq\ & \frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]dy.
\end{aligned} \tag{2.8}$$

Adding the inequalities (2.7) and (2.8), we get the fourth and fifth inequalities in (2.4) and hence completing the proof. □

Corollary 2.1. *Under the assumptions of Theorem 2.2, we have the following inequalities:*

$$\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \sup_{t,\, s\in[0,1]} \gamma(t,s) \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
&\leq \inf_{t,\, s\in[0,1]} \Gamma(t,s) \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4},
\end{aligned} \tag{2.9}$$

where $\gamma(t,s)$ and $\Gamma(t,s)$ are as defined in Theorem 2.2.

Corollary 2.2. *With notations above, we have the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \max\left\{\sup_{0\leq t,\, s\leq 1} \gamma(t,s),\ \sup_{0\leq t,\, s\leq 1} \Omega_2(t,s)\right\} \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\ &\leq \min\left\{\inf_{0\leq t,\, s\leq 1} \Gamma(t,s),\ \mathtt{B}\right\} \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{2.10}$$

Where $\mathtt{B}$, $\gamma(t,s)$ *and* $\Gamma(t,s)$ *are as defined above.*

Now, we present the importance of the refinement (2.4) with the help of example.

Example 2.1. Let $\Delta = [0,1] \times [0,1]$ and $f(x,y) = x^3y^3$. Clearly the function $f(x,y) = x^3y^3$ is convex on the co-ordinates on Δ.

Now,

$$\begin{aligned} \int_0^1 \int_0^1 f(x,y)dydx &= \frac{1}{16}, \\ \mathtt{A} &= \frac{1}{2}\left[\int_0^1 f\left(x, \frac{1}{2}\right)dx + \int_0^1 f\left(\frac{1}{2}, y\right)dy\right] = \frac{1}{32}, \\ \Omega_2\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{2}\left[\int_0^1 \int_0^1 \left[f\left(x, \frac{2y+1}{4}\right) + f\left(\frac{2x+1}{4}, y\right)\right]dxdy\right] = \frac{5}{128}, \\ \gamma\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{4}\left[\int_0^1 \left[f\left(x, \frac{1}{4}\right) + f\left(x, \frac{3}{4}\right)\right]dx + \int_0^1 \left[f\left(\frac{1}{4}, y\right) + f\left(\frac{3}{4}, y\right)\right]dy\right] \\ &= \frac{56}{1024}, \\ \Gamma\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{4}\left[\int_0^1 \left[f\left(x, \frac{1}{2}\right) + \frac{1}{2}f(x,1)\right]dx + \int_0^1 \left[f\left(\frac{1}{2}, y\right) + \frac{1}{2}f(1,y)\right]dy\right] \\ &= \frac{5}{64}, \\ \mathtt{B} &= \frac{1}{2}\left[\int_0^1 [f(x,0) + f(x,1)]dx + \int_0^1 [f(0,y) + f(1,y)]dy\right] \\ &= \frac{1}{8}. \end{aligned}$$

Therefore,

$$\begin{aligned} f\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{64} \\ &\leq \mathtt{A} = \frac{1}{32} \\ &\leq \gamma\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{56}{1024} \\ &\leq \sup_{0\leq t,\, s\leq 1} \gamma(t,s) \\ &\leq \int_0^1 \int_0^1 f(x,y)dydx = \frac{1}{16} \end{aligned}$$

$$\begin{aligned}
&\leq \inf_{0\leq t,\, s\leq 1} \Gamma(t,s) \\
&\leq \Gamma\left(\frac{1}{2},\frac{1}{2}\right) = \frac{5}{64} \\
&\leq \mathrm{B} = \frac{1}{8} \\
&\leq \frac{f(0,0)+f(0,1)+f(1,0)+f(1,1)}{4} = \frac{1}{4},
\end{aligned}$$

we get an estimation better than the estimation obtained in *[3, Example 1]* as well as in *Theorem 1.3*.

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