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Several properties of Racah polynomials

Racah polinomlarının çeşitli özellikleri

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Abstract

In this paper, bilinear and bilateral generating functions for Racah polynomials are derived, along with a theorem that provides a systematic approach for obtaining these functions. Furthermore, a new recurrence relation and an integral representation for Racah polynomials are established, enhancing their analytical framework. Special attention is given to the limiting cases of Racah polynomials, including Hahn, dual Hahn and Meixner polynomials, for which new recurrence relations are obtained. In particular, a novel integral representation for the dual Hahn polynomial is introduced, offering additional insights into its structural properties. These results contribute to the broader understanding of orthogonal polynomials, enhancing their theoretical significance and potential applications. By expanding the known properties of these polynomials, the findings may provide a basis for further mathematical research and applications in areas such as combinatorics, mathematical physics, and special functions.

Keywords: Generating function, Hypergeometric function, Integral represantations, Racah polynomials, Recurrence relations

Öz

Bu makalede, Racah polinomları için bilinear ve bilateral doğurucu fonksiyonlar türetilmiş ve bu fonksiyonları elde etmek için sistematik bir yaklaşım sağlayan bir teorem sunulmuştur. Ayrıca, Racah polinomları için yeni bir rekürans bağıntısı ve integral gösterim oluşturulmuş, böylece analitik çerçeveleri zenginleştirilmiştir. Racah polinomlarının Hahn, dual Hahn ve Meixner polinomları gibi limit durumlarına özel bir ilgi gösterilmiş ve bu polinomlar için yeni rekürans bağıntıları elde edilmiştir. Özellikle, dual Hahn polinomu için yeni bir integral gösterim sunulmuş ve bu polinomun yapısal özelliklerine dair ek özellikler sağlanmıştır. Bu sonuçlar, ortogonal polinomların daha geniş bir anlayışına katkıda bulunarak teorik anlamlarını ve potansiyel uygulamalarını artırmaktadır. Bu polinomların bilinen özelliklerinin genişletilmesi, kombinatörik, matematiksel fizik ve özel fonksiyonlar gibi alanlarda daha ileri matematiksel araştırmalar ve uygulamalar için bir temel oluşturabilir.

Anahtar kelimeler: *Doğurucu fonksiyon, Hipergeometrik fonksiyon, İntegral gösterim, Racah polinomları, Rekürans bağıntıları*

1. Introduction

Racah polynomials are orthogonal polynomials named after Giulio Racah and Racah polynomials were first described by Wilson (Wilson, 1977). Racah polynomials are a family of discrete orthogonal polynomials in the Askey diagram. They can be reduced to Hahn and Dual Hahn polynomials with some limit operations (Koekoek et al., 2010). Racah polynomials have been the subject of many studies (Rahman, 1980; Rahman, 1981; Area et al., 2004; Geronimo & Iliev, 2009)

In this study, bilinear and bilateral generating function families of Racah polynomials firstly were generated. A relation between the Hahn polynomial, a special case of this family, and the Gegenbauer polynomial is established. Then, with the help of contiguous relations by Wilson (Wilson, 1977), the recurrence relation for Racah polynomials was obtained, and with the help of limit relations, new recurrence relations were formed for Hahn, Dual Hahn and Meixner polynomials. Moreover, integral representation is obtained for Racah polynomials and a new integral representation is obtained for Dual Hahn polynomials using this.

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2. Analytical methods

Let us first give the definition of the Racah polynomial and its generating functions. Racah polynomials are given by (Wilson, 1977; Koekoek et al., 2010)

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

$$= {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, x+\gamma+\delta+1, -x\\ \alpha+1,\ \beta+\delta+1,\ \gamma+1\end{matrix};1\right),\ \alpha+1=-N,\ \beta+\delta+1=-N \text{ or } \gamma+1=-N \tag{2.1}$$

where $n = 0, 1, 2, \ldots, N$, $N_0 \in Z^+$, $\lambda(x) = x(x+\gamma+\delta+1)$ and ${}_4F_3$ denotes the corresponding generalized hypergeometric series.

Recall that, in general ${}_rF_s$, (r, s $N_0 = N \cup \{0\}$) is defined by

$${}_rF_s\left(\begin{matrix}a_1, a_2, \ldots, a_r\\ b_1, b_2, \ldots, b_s\end{matrix};z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n\, n!} z^n = {}_rF_s\left(a_1, a_2, \ldots, a_r;\ b_1, b_2, \ldots, b_s\ ;z\right).$$

As usual, the Pochammer symbol $(\mu)_s$ is given by

$$(\mu)_s = \frac{\Gamma(\mu+s)}{\Gamma(\mu)} \qquad (\mu+s \in \mathbb{C}\setminus \mathbb{Z}_0^-\ , s \in \mathbb{C}\ ; \mu \in \mathbb{C}\setminus \mathbb{Z}_0^-)$$

$$= \begin{cases} 1 & (s=0)\\ \mu(\mu+1)(\mu+2)\ldots(\mu+k-1) & (s=k\in\mathbb{N}).\end{cases}$$

Provided that the Gamma coefficient exists. It is assumed that $(0)_0 := 1$. Some generating function relations for Racah polynomials are given below (Koekoek et al., 2010):
for $x = 0,1,2,\ldots N$, $\beta+\delta+1 = -N$ or $\gamma+1 = -N$;

$$F\left(\begin{matrix}-x, -x+\alpha-\gamma-\delta\\ \alpha+1\end{matrix};t\right)F\left(\begin{matrix}x+\beta+\delta+1, x+\gamma+1\\ \beta+1\end{matrix};t\right)$$

$$= \sum_{n=0}^{N}\frac{(\beta+\delta+1)_n(\gamma+1)_n}{(\beta+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n = \sum_{n=0}^{\infty}\frac{(-N)_n(\gamma+1)_n}{(\beta+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n, \tag{2.2}$$

and for $\alpha + 1 = -N$ *or* $\gamma + 1 = -N$;

$$F\left(\begin{matrix}-x, -x+\beta-\gamma\\ \beta+\delta+1\end{matrix};t\right)F\left(\begin{matrix}x+\alpha+1, x+\gamma+1\\ \alpha-\delta+1\end{matrix};t\right)$$

$$= \sum_{n=0}^{N}\frac{(\alpha+1)_n(\gamma+1)_n}{(\alpha-\delta+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n = \sum_{n=0}^{\infty}\frac{(\alpha+1)_n(-N)_n}{(\alpha-\delta+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n\ , \tag{2.3}$$

and for $\alpha + 1 = -N$ *or* $\beta + \delta + 1 = -N$;

$$F\left(\begin{matrix}-x, -x-\delta\\ \gamma+1\end{matrix};t\right)F\left(\begin{matrix}x+\alpha+1, x+\beta+\delta+1\\ \alpha+\beta-\gamma+1\end{matrix};t\right)$$

$$= \sum_{n=0}^{N}\frac{(\alpha+1)_n(\beta+\delta+1)_n}{(\alpha+\beta-\gamma+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n = \sum_{n=0}^{\infty}\frac{(-N)_n(\beta+\delta+1)_n}{(\alpha+\beta-\gamma+1)_n n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n, \tag{2.4}$$

and

$$\left[(1-t)^{-\alpha-\beta-1}\ {}_4F_3\left(\begin{matrix}\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2), -x, x+\gamma+\delta+1\\ \alpha+1,\quad \beta+\delta+1,\quad \gamma+1\end{matrix}; -\frac{4t}{(1-t)^2}\right)\right]_N$$

$$= \sum_{n=0}^{N}\frac{(\alpha+\beta+1)_n}{n!}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n\ . \tag{2.5}$$

3. Results

3.1. Bilineer and bilateral generating functions

In this part, we obtain several families of generating functions for the Racah polynomials $R_n(\lambda(x); \alpha, \beta, \gamma, \delta)$ given by (2.1). We should mention that this kind of research has been done in recent years for other polynomials such as Continuous Dual Hahn, Cesaro and Gottlieb and various hypergeometric polynomials (Aktaş & Erkuş-Duman, 2015; Özmen, 2017; Özmen & Erkus-Duman, 2018; Korkmaz-Duzgun & Erkus-Duman, 2018; Erkus-Duman & Choi, 2021; Dumlupinar & Erkus-Duman, 2023;).

Theorem 1. Let the function $\Omega_\mu(\xi_1,\dots,\xi_s)$ have complex variables $\xi_1,\dots,\xi_s$ with $s \in \mathrm{N}$:

$$\Lambda_{\mu,\psi}(\xi_1,\dots,\xi_s;\tau) := \sum_{k=0}^{\infty} a_k\, \Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\tau^k \tag{3.1}$$

where $a_k \neq 0$, $\psi \in \mathbb{C}$ and for $n, p \in \mathbb{N}$,

$$\Theta_{n,p}^{\mu,\psi}(\lambda(x);\alpha,\beta,\gamma,\delta;\ \xi_1,\dots,\xi_s;\ \zeta)$$

$$:= \sum_{k=0}^{[n/p]} a_k \frac{(\beta+\delta+1)_{n-pk}(\gamma+1)_{n-pk}}{(\beta+1)_{n-pk}(n-pk)!} R_{n-pk}(\lambda(x);\alpha,\beta,\gamma,\delta)\ \Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\zeta^k \tag{3.2}$$

in this case,

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(\lambda(x);\alpha,\beta,\gamma,\delta;\xi_1,\dots,\xi_s;\frac{\eta}{t^p}\right)t^n$$

$$= F\left(\begin{matrix}-x,-x+\alpha-\gamma-\delta\\ \alpha+1\end{matrix};t\right) F\left(\begin{matrix}x+\beta+\delta+1, x+\gamma+1\\ \beta+1\end{matrix};t\right)\Lambda_{\mu,\psi}(\xi_1,\dots,\xi_s;\eta). \tag{3.3}$$

Proof. The left side of (3.3) is denoted by H. Then using (3.2), we have,

$$H = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k \frac{(\beta+\delta+1)_{n-pk}(\gamma+1)_{n-pk}}{(\beta+1)_{n-pk}(n-pk)!} R_{n-pk}(\lambda(x);\alpha,\beta,\gamma,\delta)\ \Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\frac{\eta^k}{t^{pk}}t^n$$

replacing n by $n+pk$ and using relation (2.2)

$$H = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k \frac{(\beta+\delta+1)_n(\gamma+1)_n}{(\beta+1)_n\, n!} R_n(\lambda(x);\alpha,\beta,\gamma,\delta)\Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\eta^k t^n$$

$$= \sum_{k=0}^{\infty} a_k\, \Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\eta^k \sum_{n=0}^{\infty} \frac{(\beta+\delta+1)_n(\gamma+1)_n}{(\beta+1)_n\, n!} R_n(\lambda(x);\alpha,\beta,\gamma,\delta)t^n$$

$$= F\left(\begin{matrix}-x,-x+\alpha-\gamma-\delta\\ \alpha+1\end{matrix};t\right) F\left(\begin{matrix}x+\beta+\delta+1, x+\gamma+1\\ \beta+1\end{matrix};t\right)\ \Lambda_{\mu,\psi}(\xi_1,\dots,\xi_s;\eta)$$

we get it in the form.

We will now show some results of the Theorem 2 for the multivariate function $\Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)$ ($k \in \mathbb{N}_0$, $s \in \mathrm{N}$) by choosing appropriate functions.

Example 1. In Theorem 1, taking $s = 1$, $\xi_1 = y, \mu = 0,\ \psi = 1,\ a_k = \frac{(\beta+\delta+1)_k(\gamma+1)_k}{(\beta+1)_k\, k!}$ and $\Omega_{\mu+\psi k}(y) = R_{\mu+\psi k}(\lambda(y);\alpha,\beta,\gamma,\delta)$ a class of bilinear genarating functions for Racah polynomials is obtained as follows:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\frac{(\beta+\delta+1)_{n-pk}(\gamma+1)_{n-pk}}{(\beta+1)_{n-pk}(n-pk)!}R_{n-pk}(\lambda(x);\alpha,\beta,\gamma,\delta)\frac{(\beta+\delta+1)_{k}(\gamma+1)_{k}}{(\beta+1)_{k}\,k!}R_{k}(\lambda(y);\alpha,\beta,\gamma,\delta)\eta^{k}t^{n-pk}$$

$$=F\left(\begin{matrix}-x,-x+\alpha-\gamma-\delta\\ \alpha+1\end{matrix};t\right)F\left(\begin{matrix}x+\beta+\delta+1,x+\gamma+1\\ \beta+1\end{matrix};t\right)$$

$$\times F\left(\begin{matrix}-y,-y+\alpha-\gamma-\delta\\ \alpha+1\end{matrix};\eta\right)F\left(\begin{matrix}y+\beta+\delta+1,y+\gamma+1\\ \beta+1\end{matrix};\eta\right).$$

$Q_{\mu+\psi k}(y;\alpha,\beta,N)$ is Hahn polynomials and has a generating function relation as follows (Wilson, 1977; Wilson, 1980; Koekoek et al., 2010):

$${}_1F_1\left(\begin{matrix}-x\\ \alpha+1\end{matrix};-t\right)\;{}_1F_1\left(\begin{matrix}x-N\\ \beta+1\end{matrix};t\right)=\sum_{n=0}^{N}\frac{(-N)_n}{(\beta+1)_n n!}Q_n(x;\alpha,\beta,N)t^n.$$

The following result appears when special values are provided in Theorem 1.

Example 2. If we take, in Theorem 1, $s=1$, $\xi_1=y$, $\mu=0$, $\psi=1$, $\alpha_k=\frac{(-N)_k}{(\beta+1)_k\,k!}$ and $\Omega_{\mu+\psi k}(y)=Q_{\mu+\psi k}(y;\alpha,\beta,N)$, a class of bilateral genarating functions for Racah polynomials and the Hahn polynomials is obtained as follows:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\frac{(\beta+\delta+1)_{n-pk}(\gamma+1)_{n-pk}}{(\beta+1)_{n-pk}(n-pk)!}R_{n-pk}(\lambda(x);\alpha,\beta,\gamma,\delta)\frac{(-N)_{k}}{(\beta+1)_{k}\,k!}Q_{\mu+\psi k}(y;\alpha,\beta,N)\eta^{k}t^{n-pk}$$

$$=F\left(\begin{matrix}-x,-x+\alpha-\gamma-\delta\\ \alpha+1\end{matrix};t\right)F\left(\begin{matrix}x+\beta+\delta+1,x+\gamma+1\\ \beta+1\end{matrix};t\right)\;{}_1F_1\left(\begin{matrix}-y\\ \alpha+1\end{matrix};-\eta\right)\;{}_1F_1\left(\begin{matrix}y-N\\ \beta+1\end{matrix};\eta\right).$$

The generalized Gottlieb polynomials $l_{\mu+\psi k}(y;\lambda)$ are generated by (Gottlieb, 1938)

$$(1-t)^{x}\left(1-te^{-\lambda}\right)^{-x-1}=\sum_{n=0}^{\infty}l_n(x;\lambda)\,t^n \quad |t|<1. \tag{3.4}$$

Thus the following outcome is achieved when special values are provided in Theorem 1.
Example 3. If we taken as $s=1$, $\xi_1=y$, $\mu=0$, $\psi=1$, $\alpha_k=1$, and $\Omega_{\mu+\psi k}(y)=l_{\mu+\psi k}(y;\lambda)$, using (2.4), it is obtained as follows:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\frac{(\beta+\delta+1)_{n-pk}(\gamma+1)_{n-pk}}{(\beta+1)_{n-pk}(n-pk)!}R_{n-pk}(\lambda(x);\alpha,\beta,\gamma,\delta)l_k(y;\lambda)\eta^{k}t^{n-pk}$$

$$=F\left(\begin{matrix}-x,-x+\alpha-\gamma-\delta\\ \alpha+1\end{matrix};t\right)F\left(\begin{matrix}x+\beta+\delta+1,x+\gamma+1\\ \beta+1\end{matrix};t\right)(1-\eta)^{y}\left(1-\eta e^{-\lambda}\right)^{-y-1}.$$

For each suitable choice of coefficients α_k ($k\in\mathbb{N}_0$) if the multivariate function $\Omega_{\mu+\psi k}(\xi_1,\ldots,\xi_s)$ can be expressed as a convenient product of some simpler functions, then Theorem 1, can be used to obtain various families of multilinear and multilateral generating functions for Racah polynomials defined by (2.1).

3.2. Limit relations

In this section, the limit relation between Racah polynomials and Hahn and Dual Hahn polynomials is given. Moreover, a new limit case between the Hahn polynomial and the Gegenbauer polynomials is obtained.

$Q_n(x;\alpha,\beta,N)$ is Hahn polynomials and is defined as (Koekoek et al., 2010)

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n\ ,n+\alpha+\beta+1\ ,-x\\ \alpha+1\ ,-N\end{matrix}\ ;\ 1\right),\quad n=0,1,2,\ldots,N\,. \tag{3.5}$$

Lemma 1. The following limit cases exist between Racah polynomials and Hahn polynomials (Wilson, 1980; Agarwal & Monocha, 1980; Koekoek et al., 2010):

$$\lim_{\delta\to\infty} R_n(\lambda(x);\alpha,\beta,-N-1,\delta) = Q_n(x;\alpha,\beta,N), \tag{3.6}$$

$$\lim_{\gamma\to\infty} R_n(\lambda(x);\alpha,\beta,\gamma,-\beta-N-1) = Q_n(x;\alpha,\beta,N), \tag{3.7}$$

$$\lim_{\delta\to\infty} R_n(\lambda(x);-N-1,\beta+\gamma+N+1,\gamma,\delta) = Q_n(x;\gamma,\beta,N). \tag{3.8}$$

On the other hand, $R_n(\lambda(x);\alpha,\delta,N)$ is Dual Hahn polynomials and is defined as (Koekoek et al., 2010)

$$R_n(\lambda(x);\alpha,\delta,N) = {}_3F_2\left(\begin{matrix}-n\,,-x\,,x+\alpha+\delta+1\\ \alpha+1\ ,-N\end{matrix}\ ;\ 1\right),\quad n=0,1,2,\ldots,N.$$

Lemma 2. The following limit cases exist between Racah polynomials and Dual Hahn polynomials (Wilson, 1980; Agarwal & Monocha, 1980; Koekoek et al., 2010):

$$\lim_{\beta\to\infty} R_n(\lambda(x);-N-1,\beta,\gamma,\delta) = R_n(\lambda(x);\gamma,\delta,N), \tag{3.9}$$

$$\lim_{\alpha\to\infty} R_n(\lambda(x);\alpha,-\delta-N-1,\gamma,\delta) = R_n(\lambda(x);\gamma,\delta,N), \tag{3.10}$$

$$\lim_{\beta\to\infty} R_n(\lambda(x);\alpha,\beta,-N-1,\alpha+\delta+N+1) = R_n(\lambda(x);\alpha,\delta,N). \tag{3.11}$$

$C_n^{\lambda}(x)$ are Gegenbauer polynomials, defined as follows (Rainville, 1960; Szegö, 1975):

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k\,(2\lambda+n)_k}{\left(\lambda+\frac{1}{2}\right)_k\,k!}\left(\frac{1-x}{2}\right)^k;\quad \lambda>-\frac{1}{2},\quad \lambda\neq 0.$$

Theorem 2. The following limit relation between the Hahn polynomial and the Gegenbauer polynomials are provided:

$$\lim_{N\to\infty}\frac{(2\lambda)_n}{n!}Q_n\left(\frac{N(1-x)}{2};\lambda-\frac{1}{2},\lambda-\frac{1}{2},N\right) = C_n^{(\lambda)}(x). \tag{3.12}$$

Proof. In the definition of Hahn polynomials given in (3.5) taking $\alpha=\beta=\lambda-\frac{1}{2}$ and $x\to\frac{N(1-x)}{2}$ multiplying both sides of the resulting equation by $\frac{(2\lambda)_n}{n!}$ and taking the limit for $N\to\infty$ gives the Gegenbauer polynomials.

3.3. Reccurrence relations

In this section, a new recurrence relation for the Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ will be derived. Furthermore, by employing the limit relations between Racah polynomials and the Dual Hahn, Hahn, and Meixner polynomials, additional new recurrence relations for these related families will also be obtained.

Lemma 3. For seven complex numbers a,b,c,d,e,f and g there holds the three term relation (Wilson, 1977);

$$fg\ {}_4F_3\left(\begin{matrix}a\,,b\,,c\,,d\\ e\,,f\,,g\end{matrix};\ 1\right) - (f-a)(g-a)\ {}_4F_3\left(\begin{matrix}a\,,b+1\,,c+1\,,d+1\\ e+1\,,f+1\,,g+1\end{matrix};\ 1\right) + \frac{a(e-b)(e-c)(e-d)}{e(e+1)}$$

$$\times\ {}_4F_3\left(\begin{matrix} a+1\,,b+1\,,c+1\,,d+1 \\ e+2\,,f+1\,,g+1 \end{matrix};\quad 1\right)=0. \tag{3.13}$$

Theorem 3. Racah polynomials have the following recurrence relation:

$$(\beta+\delta+1)(\gamma+1)R_n(\lambda(x);\alpha,\beta,\gamma,\delta)-(\beta+\delta+n+1)(\gamma+n+1)R_n(\lambda(x-1);\alpha+1,\beta,\gamma+1,\delta+1)$$

$$+\frac{n(n+\beta)(\alpha+x+1)(\alpha-x-\gamma-\delta)}{(\alpha+1)(\alpha+2)}R_{n-1}(\lambda(x-1);\alpha+2,\beta,\gamma+1,\delta+1)=0,\quad n\geq 1. \tag{3.14}$$

Proof. If we take $a\to -n$, $b\to n+\alpha+\beta+1$, $c\to -x$, $d\to x+\gamma+\delta+1$, $e\to\alpha+1$, $f\to\beta+\delta+1$ and $g\to\gamma+1$ in (3.13), we have

$$(\beta+\delta+1)(\gamma+1)\ {}_4F_3\left(\begin{matrix} -n,n+\alpha+\beta+1\,,-x\,,x+\gamma+\delta+1 \\ \alpha+1\,,\quad \beta+\delta+1\,,\quad \gamma+1 \end{matrix};1\right)-(\beta+\delta+n+1)(\gamma+n+1)$$

$$\times\ {}_4F_3\left(\begin{matrix} -n,n+\alpha+\beta+2\,,-x+1\,,x+\gamma+\delta+2 \\ \alpha+2\,,\quad \beta+\delta+2\,,\quad \gamma+2 \end{matrix};1\right)+\frac{n(n+\beta)(\alpha+1+x)(\alpha-x-\gamma-\delta)}{(\alpha+1)(\alpha+2)}$$

$$\times\ {}_4F_3\left(\begin{matrix} -n+1,n+\alpha+\beta+2\,,-x+1\,,x+\gamma+\delta+2 \\ \alpha+3\,,\quad \beta+\delta+2\,,\quad \gamma+2 \end{matrix};1\right)=0$$

using the defition of the Racah polynomials given by (2.1), the desired result is obtained that completes the proof.

Using (3.14), a new recurrence relation for Dual Hahn polynomials is obtained.

Corollary 1. Dual Hahn polynomials have the following recurrence relation,

$$(-N)(\gamma+1)R_n(\lambda(x);\gamma,\delta,N)+(N-n)(\gamma+n+1)R_n(\lambda(x-1);\gamma+1,\delta,N-1)$$

$$+n(n-\delta-N-1)R_{n-1}(\lambda(x-1);\gamma+1,\delta,N-1)=0,\quad n\geq 1. \tag{3.15}$$

Proof. If $\beta=-\delta-N-1$ is written in the recurrence relation given in (3.14), we get

$$(-N)(\gamma+1)\ {}_4F_3\left(\begin{matrix} -n,n+\alpha-\delta-N\,,-x\,,x+\gamma+\delta+1 \\ \alpha+1\,,\quad -N\,,\quad \gamma+1 \end{matrix};1\right)+(N-n)(\gamma+n+1)$$

$$\times{}_4F_3\left(\begin{matrix} -n,n+\alpha-\delta-N+1,-x+1\,,x+\gamma+\delta+2 \\ \alpha+2\,,\quad -N+1\,,\quad \gamma+2 \end{matrix};1\right)$$

$$+\frac{n(n-\delta-N-1)(\alpha+x+1)(\alpha-x-\gamma-\delta)}{(\alpha+1)(\alpha+2)}$$

$$\times\ {}_4F_3\left(\begin{matrix} -n+1,n+\alpha-N-\delta+1\,,-x+1\,,x+\gamma+\delta+2 \\ \alpha+3\,,\quad -N+1,\quad \gamma+2 \end{matrix};1\right)=0.$$

The limit is taken for the $\alpha\to\infty$, the desired result is easily obtained as follows:

$$(-N)(\gamma+1)\lim_{\alpha\to\infty}\sum_{k=0}^{n}\frac{(-n)_k(-x)_k(x+\gamma+\delta+1)_k(n+\alpha-\delta-N)_k}{(-N)_k(\gamma+1)_k(\alpha+1)_k k!}+(N-n)(\gamma+n+1)$$

$$\times\lim_{\alpha\to\infty}\sum_{k=0}^{n}\frac{(-n)_k(-x+1)_k(x+\gamma+\delta+2)_k(n+\alpha-\delta-N+1)_k}{(-N+1)_k(\gamma+2)_k(\alpha+2)_k k!}+n(n-\delta-N-1)$$

$$\times \lim_{\alpha\to\infty} \frac{(\alpha+x+1)(\alpha-x-\gamma-\delta)}{(\alpha+1)(\alpha+2)} \sum_{k=0}^{n} \frac{(-n+1)_k(-x+1)_k(x+\gamma+\delta+2)_k(n+\alpha-\delta-N+1)_k}{(-N+1)_k(\gamma+2)_k(\alpha+3)_k k!} = 0$$

$$(-N)(\gamma+1)\sum_{k=0}^{n} \frac{(-n)_k(-x)_k(x+\gamma+\delta+1)_k}{(-N)_k(\gamma+1)_k k!}$$

$$+(N-n)(\gamma+n+1)\sum_{k=0}^{n} \frac{(-n)_k(-x+1)_k(x+\gamma+\delta+2)_k}{(-N+1)_k(\gamma+2)_k k!}$$

$$+n(n-\delta-N-1)\sum_{k=0}^{n-1} \frac{(-n+1)_k(-x+1)_k(x+\gamma+\delta+2)_k}{(-N+1)_k(\gamma+2)_k k!} = 0$$

$$(-N)(\gamma+1)\,{}_3F_2\left(\begin{matrix}-n, -x\,, x+\gamma+\delta+1\\ -N\,, \quad \gamma+1\end{matrix};1\right) + (N-n)(\gamma+n+1)\,{}_3F_2\left(\begin{matrix}-n, -x+1\,, x+\gamma+\delta+2\\ -N+1\,, \quad \gamma+2\end{matrix};1\right)$$

$$+n(n-\delta-N-1)\,{}_3F_2\left(\begin{matrix}-n+1\,, -x+1\,, x+\gamma+\delta+2\\ -N+1, \quad \gamma+2\end{matrix};1\right) = 0,$$

which completes the proof.

After that if we write $\delta = -\beta - N - 1$ in (3.14), we reached the following recurrence relation fort he Hahn polynomials.

Corollary 2. Hahn polynomials have the following recurrence relation,

$$(-N)Q_n(x;\alpha,\beta,N) + (N-n)Q_n(x-1;\alpha+1,\beta,N-1) - \frac{n(n+\beta)(\alpha+x+1)}{(\alpha+1)(\alpha+2)}$$

$$\times Q_{n-1}(x-1;\alpha+1,\beta,N-1) = 0, \quad n > 0. \tag{3.16}$$

Finally, if we write $\gamma = \beta - 1$ and $\delta = \frac{N(1-c)}{c}$ in (3.14) and limit for $N \to \infty$, a new recurrence relation for Meixner polynomials is obtained:

Corollary 3. Meixner polynomials have the following recurrence relation,

$$-N\beta R_n\left(\lambda(x);\beta-1,\frac{N(1-c)}{c},N\right) + (N-n)(\beta+n)R_n\left(\lambda(x-1);\beta,\frac{N(1-c)}{c},N-1\right)$$

$$+n\left(n-1-\tfrac{N}{c}\right)R_{n-1}\left(\lambda(x-1);\beta,\tfrac{N(1-c)}{c},N-1\right) = 0. \tag{3.17}$$

If the left-hand side of equation (3.17) is multiplied by $\frac{1}{N}$ and the limit is taken as $N \to \infty$, the following result is obtained.

Remark 1. Meixner polynomials have the following recurrence relation,

$$-\beta M_n(x;\beta,c) + (\beta+n)M_n(x-1;\beta+1,c) - \tfrac{n}{c}M_{n-1}(x-1;\beta+1,c) = 0, \; n \geq 0,$$

$M_n(x;\beta,c)$ is Meixner polynomials and is defined as follows (Koekoek et al., 2010):

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix}-n, -x\\ \beta\end{matrix};1-\tfrac{1}{c}\right).$$

On the other hand, another recurrence relation for Meixner polynomials can beobtained using the generating function as follows.

Theorem 4. Meixner polynomials have the following recurrence relation,

$$M_n(x;\beta,c)-\frac{x(1-c)}{\beta c}M_n(x-1;\beta+1,c)-M_{n+1}(x;\beta,c)=0,\quad n\geq 0. \tag{3.18}$$

Proof. A generating function relation of Meixner polynomials is as follows (Koekoek et al., 2010)

$$e^t\,{}_1F_1\left(\begin{matrix}-x\\ \beta\end{matrix};\left(\frac{1-c}{c}\right)t\right)=\sum_{n=0}^{\infty}\frac{M_n(x;\beta,c)}{n!}t^n. \tag{3.19}$$

If we take the derivative of both sides of for (3.19) with respect to t and equalize the coefficients of t in the resulting equation, the desired recurrence relation is obtained.

3.4. Integral representations

Theorem 5. Racah polynomials have an integral representation as follows:

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)=\frac{\Gamma(\gamma+1)}{\Gamma(x+\gamma+\delta+1)\Gamma(-x-\delta)}\int_0^1\left[t^{x+\gamma+\delta}(1-t)^{-x-\delta-1}\times{}_3F_2\left(\begin{matrix}-n,n+\alpha+\beta+1,-x\\ \alpha+1,\ \beta+\delta+1\end{matrix};t\right)\right]dt \tag{3.20}$$

where $n=0,1,,\dots.,N$, $\lambda(x)=x(x+\gamma+\delta+1)$, $Re(\gamma+1)>Re(x+\gamma+\delta+1)>0$.

Proof. If the beta function is used in (2.1), we have

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

$$={}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\ \beta+\delta+1,\ \gamma+1\end{matrix};1\right)$$

$$=\sum_{k=0}^{n}\frac{(-n)_k\ (n+\alpha+\beta+1)_k\,(-x)_k(\,x+\gamma+\delta+1)_k}{(\alpha+1)_k(\beta+\delta+1)_k\,(\gamma+1)_k\ \ k!}$$

$$=\sum_{k=0}^{n}\frac{(-n)_k\ (n+\alpha+\beta+1)_k\,(-x)_k}{(\alpha+1)_k(\beta+\delta+1)_k k!}\frac{\Gamma(x+\gamma+\delta+1+k)}{\Gamma(x+\gamma+\delta+1)}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+k)}\frac{\Gamma(-x-\delta)}{\Gamma(-x-\delta)}$$

$$=\sum_{k=0}^{n}\frac{(-n)_k\ (n+\alpha+\beta+1)_k\,(-x)_k\Gamma(\gamma+1)}{(\alpha+1)_k(\beta+\delta+1)_k k!\,\Gamma(x+\gamma+\delta+1)\Gamma(-x-\delta)}B(x+\gamma+\delta+1+k,-x-\delta)$$

$$=\frac{\Gamma(\gamma+1)}{\Gamma(x+\gamma+\delta+1)\Gamma(-x-\delta)}\int_0^1 t^{x+\gamma+\delta}(1-t)^{-x-\delta-1}\sum_{k=0}^{n}\frac{(-n)_k\ (n+\alpha+\beta+1)_k\,(-x)_k}{(\alpha+1)_k(\beta+\delta+1)_k\ k!}t^k dt$$

$$=\frac{\Gamma(\gamma+1)}{\Gamma(x+\gamma+\delta+1)\Gamma(-x-\delta)}\int_0^1\left[t^{x+\gamma+\delta}(1-t)^{-x-\delta-1}\ {}_3F_2\left(\begin{matrix}-n,n+\alpha+\beta+1,-x\\ \alpha+1,\ \beta+\delta+1\end{matrix};t\right)\right]dt,$$

which completes the proof.

In Theorem 5, if we write $\gamma=-N-1$, $\delta\to\alpha+\delta+N+1$ and take the limit of both sides of the equality for $\beta\to\infty$, a new integral represantations of Dual Hahn polynomial is obtained.

Corollary 4. Dual Hahn polnomials have an integral represantion as follows:

$$R_n(\lambda(x);\alpha,\delta,N)$$

$$= \frac{\Gamma(-N)}{\Gamma(x+\alpha+\delta+1)\Gamma(-x-\alpha-\delta-N-1)} \int_0^1 t^{x+\alpha+\delta}(1-t)^{-x-\alpha-\delta-N-2} F\left(\begin{matrix}-n\ ,-x\\ \alpha+1\ ,\end{matrix};t\right) dt$$

where $n = 0,1,\dots,N$, $\lambda(x) = x(x+\gamma+\delta+1)$, $Re(x+\alpha+\delta+1) > 0$.

4. Discussion and conclusions

In this paper, bilinear and bilateral generating functions for Racah polynomials have been presented. Furthermore, a novel recurrence relation and an integral representation for Racah polynomials have been obtained. Additionally, new recurrence relations for the Hahn, dual Hahn, and Meixner polynomials, have been derived. A new integral representation for the Dual Hahn polynomial has also been provided.

The results of this investigation have expanded the theoretical framework of Racah polynomials and offered new perspectives on their features. Significant findings are obtained for Racah polynomials as well as related families of orthogonal polynomials, including Dual Hahn, Hahn, and Meixner polynomials, using the recurrence relationships and representations. These findings may find utilize in coding theory, quantum mechanics, and other branches of mathematical physics where orthogonal polynomials are important. Furthermore, the techniques and representations offered in this paper might stimulate novel concepts for researching related families of hypergeometric orthogonal polynomials. As a result, this study lays a strong basis for future, more thorough research on integral representations, generating functions, and recurrence relations.

Declaration of ethical code

The author of this article declares that the materials and methods used in this study do not require ethics committee approval and/or legal-special permission.

Conflicts of interest

The author declares that there is no conflict of interest.

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