



Stancu and Kantorovich-Type Generalizations of a Bernstein Operator:

Approximating Locally Integrable Functions

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Abstract

This study examines the approximation properties of two distinct parameter-dependent generalizations of Stancu-type operators. First, it is established that the operator defined on the interval $[-1,1]$ is a Korovkin-type operator satisfying the corresponding theorem, and its fundamental properties are analyzed. Subsequently, a new class of Kantorovich-type operators is introduced based on this operator, and their approximation properties are investigated in detail. Another significant aspect of the work is the study of the convergence behavior of both classes of operators in L^p - spaces. In this context, the impact of the operators on functions and their convergence characteristics are evaluated, and the advantages of the newly defined operators over classical approximations are demonstrated. Furthermore, graphical illustrations of the approximations are provided, enabling a visual analysis of the operators' effects on functions. By combining theoretical analysis with visual results, the study offers valuable insights into the convergence properties of both operators. The novelty of this study lies in the introduction of a new class of parameter-dependent Kantorovich-type operators, the demonstration of their



Korovkin-type properties, and the detailed investigation of their convergence behavior in L^p -spaces. By providing both theoretical analysis and graphical illustrations, the study highlights the advantages of the newly defined operators over classical approximations.

Keywords: Bernstein stancu operators; Kantorovich operators; Korovkin's Theorem.

Bir Bernstein Operatörünün Stancu ve Kantorovich Tipi Genellemesi: Lokal İntegrallenebilir Fonksiyonlarla Yaklaşım

Öz

Bu çalışma, Stancu tipi operatörlerin iki farklı parametreye bağlı genellemesinin yaklaşım özelliklerini incelemektedir. Öncelikle, $[-1,1]$ aralığında tanımlanan operatörün ilgili teoremi sağlayan Korovkin tipi bir operatör olduğu belirlendi ve temel özellikleri analiz edildi. Ardından, söz konusu operatöre dayalı yeni bir Kantorovich tipi operatörler sınıfı tanıtılmakta ve bu operatörlerin yaklaşım özellikleri ayrıntılı olarak incelenmektedir. Çalışmanın bir diğer önemli yönü, her iki operatör sınıfının L^p uzaylarındaki yakınsama davranışının incelenmesidir. Bu bağlamda, operatörlerin fonksiyonlar üzerindeki etkileri ve yakınsama özellikleri değerlendirilmekte, ayrıca yeni tanımlanan operatörlerin klasik yaklaşımlara göre üstünlükleri ortaya konulmaktadır. Ayrıca, yaklaşım sonuçlarına ilişkin grafiksel gösterimler sunularak operatörlerin fonksiyonlar üzerindeki etkilerinin görsel olarak analiz edilmesi sağlanmaktadır. Kuramsal inceleme ile görsel bulguların bir araya getirilmesi sayesinde, çalışma her iki operatörün yakınsama özelliklerine dair değerli katkılar sunmaktadır. Bu çalışmanın özgünlüğü, parametreye bağlı yeni bir Kantorovich tipi operatörler sınıfının ortaya konulmasında, bu operatörlerin Korovkin tipi özelliklerinin gösterilmesinde ve L^p uzaylarındaki yakınsama davranışlarının ayrıntılı olarak incelenmesinde yatmaktadır. Kuramsal analiz ile grafiksel gösterimleri bir arada sunarak, çalışma yeni tanımlanan operatörlerin klasik yaklaşımlara göre sağladığı üstünlükleri vurgulamaktadır.

Anahtar Kelimeler: Bernstein-Stancu operatörleri; Kantorovich operatörleri; Korovkin Teoremi.

1. Introduction

Approximation theory is a significant branch of mathematics that investigates how complex functions can be approximated by simpler structures such as polynomials or rational functions. One of the fundamental questions addressed by this theory is whether a given family of functions is dense in a broader function space. In 1885, Karl Weierstrass established, through his celebrated

theorems, that algebraic and trigonometric polynomials are dense in $C[a, b]$, thereby laying the foundations of modern approximation theory [1]. However, although Weierstrass's results confirmed the existence of such approximations, they did not provide a constructive method for obtaining them.

To overcome this shortcoming, Sergei Bernstein defined Bernstein operators inspired by the binomial expansion [2]. It provides a constructive proof of Weierstrass's theorem by stating that with these operators, any continuous function $g: [0,1] \rightarrow \mathbb{R}$ can be approximated with polynomials. Furthermore, Bernstein operators have subsequently found applications in a wide range of areas. At present, they are extensively utilized not only in theoretical mathematics but also in applied domains such as computer graphics, numerical analysis, and data approximation. Approximately fifty years after the introduction of the Bernstein operator, D.D. Stancu proposed generalized versions of these operators, which provided a new perspective on the theory [3–10].

Stancu obtained the functions $\check{Y}_{n,\ell,v}(x)$ describe as follows for a $n \in \mathbb{N}$ satisfying the condition

$n > 2v$, and an integer v such that $0 \leq \ell < n$:

$$\check{Y}_{n,\ell,v}(x) = \begin{cases} (\wp_{n-v,\ell}(x))(1-x); & 0 \leq \ell < v \\ (\wp_{n-v,\ell-v}(x))(x+1) + (\wp_{n-v,\ell}(x))(1-x); & v \leq \ell \leq n-v \\ (\wp_{n-v,\ell-v}(x))(x+1); & n-v < \ell \leq n \end{cases}$$

Here $\wp_{n-v,\ell}(x)$ denotes the Bernstein elementary polynomials. Stancu's approach expands the symmetric structure of Bernstein polynomials and defines a more flexible function family for different values of v and ℓ . Moreover, these functions satisfy the following identity:

$$\sum_{\ell=0}^n \check{Y}_{n,\ell,v}(x) = \left(\sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \right) (1-x) + \left(\sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \right) (x+1) = 1.$$

This identity shows that the functions $\check{Y}_{n,\ell,v}(x)$ form a probability distribution and offers a new perspective in approximation theory. Stancu's work has been an essential step in generalising Bernstein operators and extending their applications.

Kantorovich-type generalizations are integral-form extensions of Bernstein operators that have broadened the applicability of Bernstein-type operators, particularly integrable functions. The foundations of these generalizations were established by L.V. Kantorovich in the 1930s [11]. Following his introduction of Bernstein operators in integral form, Kantorovich conducted a

detailed analysis of their effects on integrable functions. In 1953, P.P. Korovkin formulated convergence criteria in $C[a, b]$ and demonstrated their applications [12]. Subsequently, Korovkin examined the convergence properties of Kantorovich-type operators, situated them within the framework of positive linear operators, and strengthened their theoretical underpinnings [13]. In the 1990s, F. Altomare and M. Campiti expanded the scope of Kantorovich-type operators by investigating their behavior in various function spaces, thereby enriching the theoretical framework [14–17]. In the 2010s, A. Kajla and collaborators contributed significantly to the field through their studies on the rates of convergence and the modulus of continuity of Kantorovich-type operators [18]. In 2014, various approximation properties of a Kantorovich-type operator constructed as a generalization of existing Bernstein operators were examined in detail in [19]. Collectively, these contributions have reinforced the central role of Kantorovich-type generalizations in both theoretical and applied mathematics. In 2018, Kajla defined the Kantorovich version of generalized operators of Stancu type and obtained some local and global approximation results in $C[0,1]$ space. This operator defined by Kajla, with some special choices, reduces to the classical form of the Kantorovich operators [20]. This approach provides an integral-based generalization, allowing Bernstein and Stancu type operators to have a broader range of applications. Kajla's work is an important contribution to the theory of approximation, both theoretically and practically.

Generalizations of Stancu and Kantorovich-type operators have significant applications in interpolation, mathematical modeling, numerical analysis, image processing, engineering, integral calculations in physics, and data approximation. They are employed in numerical computations, the evaluation of integrals, and the solution of differential equations. In image processing, these operators are effective in tasks such as image reconstruction and compression [21–24]. Overall, these operators find widespread use in both theoretical studies and practical applications.

Bernstein polynomials have become increasingly crucial in approximation theory, and many different generalizations have been made for both univariate and bivariate functions. These generalizations have been studied and applied by various researchers in different contexts. For example, many authors, such as in [25-34] have extended the theoretical and applied aspects of these structures by dealing with different generalizations of Bernstein polynomials.

Upon reviewing the existing literature, no study was found that simultaneously preserves the advantages of the Stancu operator while incorporating a piecewise-defined function structure suitable for more complex applications in L^p - spaces; therefore, the present study was deemed necessary.

Inspired by these studies, the present work constructs a Stancu-type modification of the operator introduced by Çilo et al. in [25], and then establishes two different Kantorovich-type generalizations of this new operator using the method applied by Bostancı Güney in [35]. Moreover, significant approximation properties of these operators are investigated, and their convergence behavior in the L^p -norm is analyzed. Through the operators defined in this study, it is aimed to make substantial contributions to various fields and approximation theories where integral calculus is required both theoretically and practically.

A review of the literature shows that the operator given in [25], which occupies an essential place in the field, has not yet been generalized by means of a probabilistic approach making use of certain identities of the fundamental functions introduced by Stancu. Therefore, in order to fill this gap in the literature, a new Stancu-type generalization has first been established by means of a piecewise-defined auxiliary function, and subsequently, the Kantorovich generalization, which is the most natural and expected extension in approximation theory, has been constructed in two different forms within the same study. Furthermore, since the L^p -convergence of the operators has also been analyzed, this study not only introduces a new operator to the literature but also provides two distinct generalizations of it simultaneously. Owing to the application of a different approximation approach, the study is quite comprehensive, and since the constructed operator can be reduced to some existing operators in the literature, it also ensures the principle of consistency within the framework of approximation theory.

The organization of the paper is outlined as follows: Section 1 is devoted to the definitions and theorems to be used in the paper concerning approximation theory and the function spaces to be studied. In Section 2, the original hybrid operator is defined, and its various approximation properties are investigated in the space of continuous functions. In Section 3, the Kantorovich version of the operator, which enables approximation within a class of integrable functions, is defined. In Section 4, the approximation properties of both operators are studied on a general L^p -space. In the conclusion section, the study's findings are presented.

Now, recall the basic norm definitions that will be used throughout the paper, which can be found in many functional analysis sources.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function. In this case,

$$\|f\|_{C[a,b]} = \max_{x \in [a,b]} |f(x)|$$

defines a norm on the space $C[a, b]$.

Let $p \geq 1$ and f be a measurable function such that

$$\int_a^b |f(x)|^p dx < \infty.$$

The space of functions whose p -th power is Lebesgue integrable on $[a, b]$ is called the $L^p[a, b]$ space.

Let $f \in L^p[a, b]$. Then

$$\|f\|_{L^p[a,b]} = \left(\int_a^b |f(x)|^p dx \right)^{1/p}.$$

In this case, for the function

$$\|\cdot\|_{L^p[a,b]} : L^p[a, b] \rightarrow \mathbb{R}^+,$$

the following properties hold:

$$\|f\|_{L^p[a,b]} = 0 \Leftrightarrow f = 0 \text{ (h. h. h)}$$

and

$$\|\alpha f\|_{L^p[a,b]} = |\alpha| \|f\|_{L^p[a,b]}.$$

Theorem 1.1 (Korovkin)

If the sequence of positive linear operators $L_n : C[a, b] \rightarrow C[a, b]$ satisfies the conditions

- (a) $\|L_n(1, x) - 1\|_{C[a,b]} \rightarrow 0$, as $n \rightarrow \infty$,
- (b) $\|L_n(t, x) - x\|_{C[a,b]} \rightarrow 0$, as $n \rightarrow \infty$,
- (c) $\|L_n(t^2, x) - x^2\|_{C[a,b]} \rightarrow 0$, as $n \rightarrow \infty$,

then for any function $f \in C[a, b]$, we have

$$\lim_{n \rightarrow \infty} \|L_n(f, x) - f(x)\|_{C[a,b]} = 0, \text{ as } n \rightarrow \infty.$$

Now, some basic information will be given for the operator that is required for the study and whose generalization will be made.

Definition 1.1 ([25], [26])

Let $x \in [-1, 1]$ and $f \in C[-1, 1]$. Then

$$C_n(f; x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} (x+1)^k (1-x)^{n-k} f\left(2\frac{k}{n} - 1\right) \tag{1}$$

is called a Bernstein type operators.

Definition 1.2 ([25], [26], [35])

For every $n \in \mathbb{N}$, $x \in [-1,1]$ and $f \in C[-1,1]$, let

$$\wp_{n,\ell}(x) = \binom{n}{\ell} \left(\frac{x+1}{2}\right)^\ell \left(\frac{1-x}{2}\right)^{n-\ell}.$$

A modified Bernstein operator on the symmetric interval is defined as

$$C_n(f; x) = \sum_{\ell=0}^n \wp_{n,\ell}(x) f\left(2\frac{\ell}{n} - 1\right).$$

Theorem 1.2

For all $f \in C[-1,1]$, $C_n(f; x)$ is a linear and positive operator.

In this paper, significant theorems concerning the approximation properties of the operator defined in the class of p-th power integrable functions are presented. For this purpose, some essential results from the existing literature are first recalled.

Remark 1.1 (Jensen's Inequality)

If I is an open interval on \mathbb{R} and Φ is a convex function, then for every finite family $(x_\zeta)_{1 \leq \zeta \leq m}$ and $\sum_{\zeta=1}^m \lambda_\zeta = 1$ such that $\lambda_\zeta \in [0,1]$, $1 \leq \zeta \leq m$

$$\Phi\left(\sum_{\zeta=1}^m \lambda_\zeta x_\zeta\right) \leq \sum_{\zeta=1}^m \lambda_\zeta \Phi(x_\zeta)$$

inequality is satisfied [16].

Remark 1.2 (Integral Jensen Inequality)

If (\mathfrak{S}, U, μ) is a probability space, I is an open interval on \mathbb{R} , and $f: \mathfrak{S} \rightarrow I$ is a μ -integrable function, then $\int_{\mathfrak{S}} f d\mu \in I$. Moreover, if $\Phi: I \rightarrow \mathbb{R}$ is convex and $\Phi \circ f: \mathfrak{S} \rightarrow \mathbb{R}$ is μ -integrable, then

$$\Phi \left(\int_{\mathfrak{S}} f d\mu \right) \leq \int_{\mathfrak{S}} (\Phi \circ f) d\mu$$

inequality is satisfied [16].

Remark 1.4

For $x \in [-1,1]$ and $0 \leq l \leq n$,

$$\wp_{n,\ell}(x) = \left(\frac{1-x}{2}\right) \wp_{n-1,\ell}(x) + \left(\frac{x+1}{2}\right) \wp_{n-1,\ell-1}(x).$$

Definition 1.3 (Gamma and Beta Functions)

For $Re(\rho), Re(\sigma), Re(\vartheta) > 0$, the gamma and beta functions, respectively,

$$\Gamma(\rho) = \int_0^\infty t^{\rho-1} e^{-t} dt$$

and

$$\beta(\sigma, \vartheta) = \int_0^1 (1-t)^{\vartheta-1} t^{\sigma-1} dt$$

$$\beta(\sigma, \vartheta) = \frac{\Gamma(\sigma)\Gamma(\vartheta)}{\Gamma(\sigma + \vartheta)} \quad [38].$$

Theorem 1.3

Suppose that the next conditions hold for (A_n) transforming from $L^p[c, d]$ to $L^p[c, d]$.

a) Let the norms of A_n be uniformly bounded. That is, let there be a constant H such that $\|A_n\|_{L^p[c,d] \rightarrow L^p[c,d]} \leq H < \infty$, every $n = 1, 2, \dots$

b) For $n \rightarrow \infty$ and $u = 0, 1, 2$,

$$\|A_n(t^u; x) - x^u\|_{L^p[c,d]} \rightarrow 0.$$

Then for all $f \in L^p[c, d]$ as $n \rightarrow \infty$ $\|A_n(f; x) - f(x)\|_{L^p[c,d]} \rightarrow 0$ [39].

Theorem 1.4 [33]

For all $f \in L^p[0,1]$ with $1 \leq p < \infty$,

$$\lim_{m \rightarrow \infty} \|\mathcal{K}_m(f) - f\|_{L^p[0,1]} = 0.$$

2. On New Generalizations of Bernstein Stancu and Kantorovich Operators

Now, a modification of the operator introduced in Definition 1.1 will be constructed by means of piecewise functions, and its essential approximation properties will be discussed.

Definition 2.1

With the help of a non-negative integer v , let $n \in \mathbb{N}$ with $n > 2v$ and let ℓ be an integer such that $\ell \leq n$. For $x \in [-1,1]$, is defined as

$$E_{n,\ell,v}(x) = \begin{cases} \left(\frac{1-x}{2}\right) \wp_{n-v,\ell}(x); & 0 \leq \ell < v \\ \left(\frac{1-x}{2}\right) \wp_{n-v,\ell}(x) + \left(\frac{x+1}{2}\right) \wp_{n-v,\ell-v}(x); & v \leq \ell \leq n-v \\ \left(\frac{x+1}{2}\right) \wp_{n-v,\ell-v}(x); & n-v < \ell \leq n \end{cases}$$

for all n . Here the basic functions $E_{n,\ell,v}(x)$ are expressed in terms of

$$\wp_{n,\ell}(x) = \binom{n}{\ell} \left(\frac{x+1}{2}\right)^\ell \left(\frac{1-x}{2}\right)^{n-\ell}.$$

This auxiliary function will serve to construct the operator defined in Definition 2.2.

Proposition 2.1

$$(i) \sum_{\ell=0}^n E_{n,\ell,v}(x) = 1,$$

$$(ii) \sum_{\ell=0}^n E_{n,\ell,0}(x) = \sum_{\ell=0}^n E_{n,\ell,1}(x) = 1$$

are satisfied.

Definition 2.2

By using the functions $E_{n,\ell,v}(x)$ for each $f \in C[-1,1]$ using equation (1)

$$\tilde{\mathcal{C}}_{n,v}(f; x) = \sum_{\ell=0}^n \left(f \left(2\frac{\ell}{n} - 1 \right) \right) E_{n,\ell,v}(x)$$

is called a modified Stancu type operator.

Theorem 2.2

The operator $\tilde{C}_{n,v}(f, x)$ satisfies the following equality:

$$\tilde{C}_{n,v}(f, x) = \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left[\left(\frac{1-x}{2} \right) f \left(2 \frac{\ell}{n} - 1 \right) + \left(\frac{x+1}{2} \right) f \left(2 \frac{\ell+v}{n} - 1 \right) \right].$$

Remark 2.1

For $v = 0$ and $v = 1$, the operators $\tilde{C}_{n,1}(f, x)$ and $\tilde{C}_{n,0}(f, x)$ are given as follows:

$$\tilde{C}_{n,0}(f, x) = \sum_{\ell=0}^n \wp_{n,\ell}(x) f \left(2 \frac{\ell}{n} - 1 \right),$$

$$\tilde{C}_{n,1}(f, x) = \sum_{\ell=0}^{n-1} \wp_{n-1,\ell}(x) \left[\left(\frac{1-x}{2} \right) f \left(2 \frac{\ell}{n} - 1 \right) + \left(\frac{x+1}{2} \right) f \left(2 \frac{\ell+1}{n} - 1 \right) \right].$$

Theorem 2.3

For a modified Stancu operator $\tilde{C}_{n,v}(f, x)$, the following hold:

(i) $\tilde{C}_{n,v}(1, x) = 1,$

(ii) $\tilde{C}_{n,v}(t, x) = x,$

(iii) $\tilde{C}_{n,v}(t^2, x) = \frac{(n-v)(v-1+n)}{n^2} x^2 + \frac{v^2+n-v}{n^2}.$

Proof

(i) From operator definition

$$\tilde{C}_{n,v}(1; x) = \sum_{\ell=0}^{n-v} \binom{n-v}{\ell} \left(\frac{x+1}{2} \right)^\ell \left(\frac{1-x}{2} \right)^{n-v-\ell} = 1.$$

(ii) If $\tilde{C}_{n,v}(t; x)$ is computed, then

$$\tilde{C}_{n,v}(t; x) = \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left[\left(2 \frac{\ell}{n} - 1 \right) \left(\frac{1-x}{2} \right) + \left(2 \frac{\ell+v}{n} - 1 \right) \left(\frac{x+1}{2} \right) \right] = x.$$

(iii) Finally, for $\tilde{C}_{n,v}(t^2; x)$ the following equality is obtained:

$$\begin{aligned} \tilde{C}_{n,v}(t^2; x) &= \frac{(n-v)(n-v-1)}{n^2} + \frac{(n-v)(n-v-1)2}{n^2}x + \frac{(n-v)(n-v-1)}{n^2}x^2 \\ &+ \frac{2(n-v)}{n^2} - \frac{2(n-v)}{n} + \frac{2v(n-v)}{n^2} + \frac{2v(n-v)}{n^2}x + \left[\frac{2(n-v)}{n^2} - \frac{2(n-v)}{n} \right. \\ &\left. + \frac{2v(n-v)}{n^2} \right]x + \frac{2v(n-v)}{n^2}x^2 + \frac{2v^2}{n^2}x - \frac{2v}{n}x + \frac{2v^2}{n^2} - \frac{2v}{n} + 1 \\ &= \frac{n+v-1}{n^2}(n-v)x^2 + \frac{2n^2-2v^2+2v^2-2n^2}{n^2}x + \frac{v^2+n-v}{n^2}. \end{aligned}$$

Thus, the stated equality is established.

Theorem 2.4

Let f be any continuous function on $[-1,1]$. Then the sequence of operators $\tilde{C}_{n,v}(f; x)$ converges uniformly to $f(x)$ on $[-1,1]$, that is,

$$\lim_{n \rightarrow \infty} \|\tilde{C}_{n,v}(f; x) - f(x)\|_{C[-1,1]} = 0.$$

Proof It is enough to prove it for the functions $1, t, t^2$, using Korovkin's theorem given in Theorem 1.1.

Example 2.1

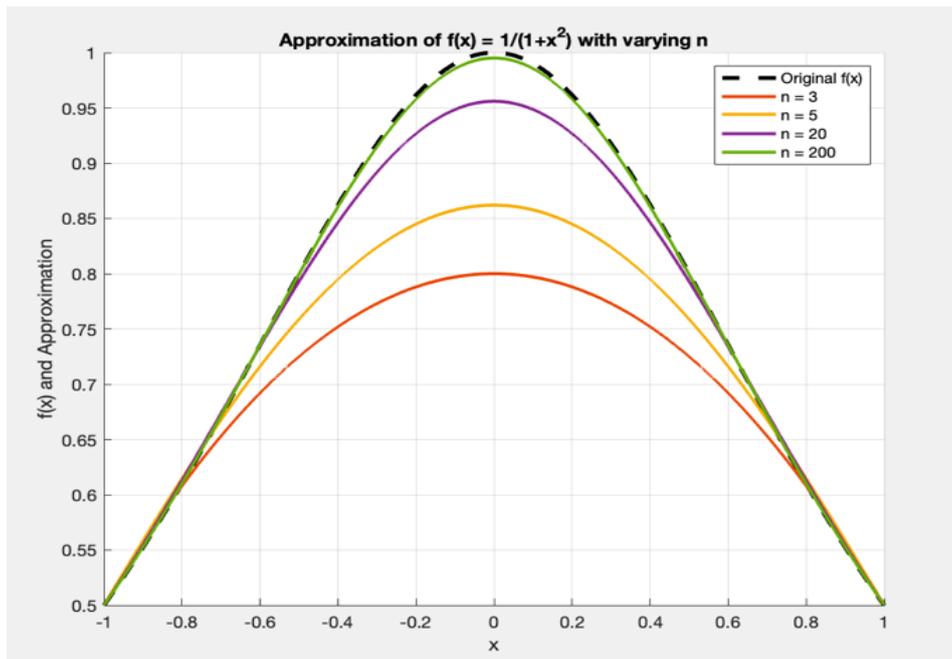


Figure 1: The approximation of $\tilde{C}_{n,\nu}(f; x)$ to $f(x) = \frac{1}{1+x^2}$

The approximation of the operator $\tilde{C}_{n,\nu}(f, x)$ to the function $f(x) = \frac{1}{1+x^2}$ for $n = 3, 5, 20, 200$ is illustrated in Fig. 1. In the figure, the graphs corresponding to $n = 3, 5, 20,$ and 200 are plotted in red, yellow, purple, and green, respectively, while the graph of $f(x)$ is depicted in black.

Example 2.2

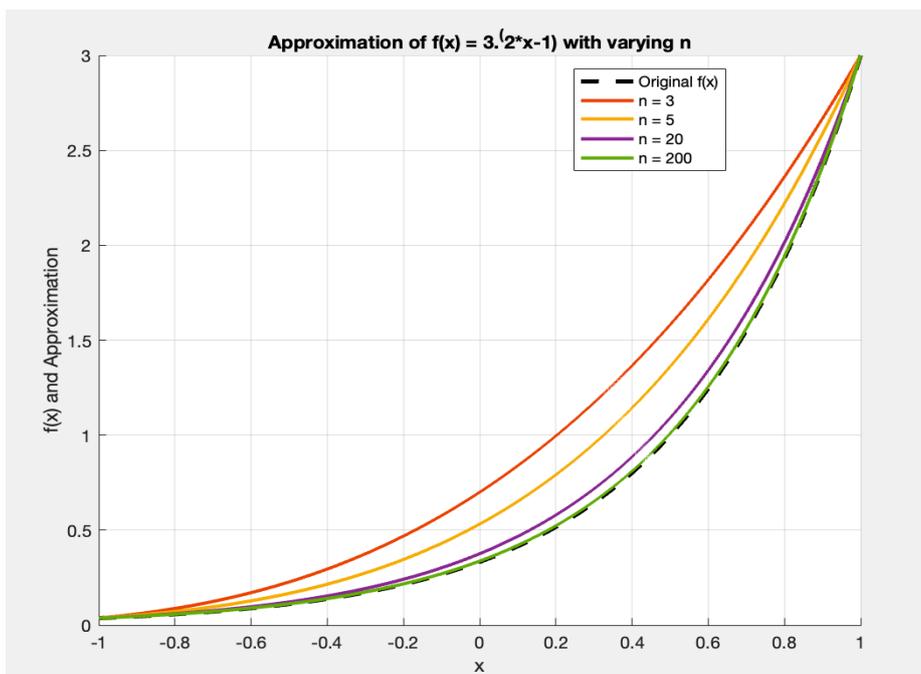


Figure 2: The approximation of $\tilde{C}_{n,\nu}(f; x)$ to $f(x) = 3^{2x-1}$

The convergence of the operator $\tilde{C}_{n,v}(f; x)$ to the function $f(x) = 3^{2x-1}$ for $n = 3, 5, 20, 200$ is illustrated in Fig. 2. In the figure, the graphs corresponding to $n = 3, 5, 20$ and 200 are plotted in red, yellow, purple, and green, respectively, while the graph of $f(x)$ is depicted in black.

3. Kantorovich Modification of the Operators on $C[-1, 1]$

In this section, the theoretical framework concerning the approximation of the two defined operators has been established, and visual findings related to the approximation have been presented.

Definition 3.1

Let $x \in [-1, 1]$ and $f \in L^1[-1, 1]$. For a non-negative integer parameter p and $n \in \mathbb{N}$ with $n > 2v$, the Kantorovich generalization of the modified Stancu operators is defined by

$$\mathcal{K}_{n,v}^{cc}(f, x) := \sum_{\ell=0}^n \mathcal{E}_{n,\ell,v}(x) \left(\frac{1+n}{2} \int_{2\frac{\ell}{n+1}-1}^{2\frac{\ell+1}{n+1}-1} f(t) dt \right).$$

Remark 3.1

The operator $\mathcal{K}_{n,v}^{cc}(f; x)$ can be expressed as

$$\begin{aligned} \mathcal{K}_{n,v}^{cc}(f, x) &= \frac{(1+n)(1-x)}{4} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \int_{2\frac{\ell}{n+1}-1}^{2\frac{\ell+1}{n+1}-1} f(t) dt \\ &+ \frac{(1+n)(1+x)}{4} \sum_{\ell=v}^n \wp_{n-v,\ell-v}(x) \int_{2\frac{\ell}{n+1}-1}^{2\frac{\ell+1}{n+1}-1} f(t) dt. \end{aligned} \tag{2}$$

Theorem 3.1

Let $v \neq 0, 1$, $n \in \mathbb{N}$ and $x \in [-1, 1]$. For the indefinite integral

$$F(x) = F(0) + \int_0^x f(t) dt$$

of a function $f \in L^1[-1, 1]$, the following inequality holds for the operators $\tilde{C}_{n+1,v}(F; x)$ and $\mathcal{K}_{n,v}^{cc}(F'; x)$:

$$\frac{d}{dx} \tilde{C}_{n+1,v}(F; x) \neq \mathcal{K}_{n,v}^{cc}(F'; x).$$

Proof Using the definition of $\tilde{C}_{n+1,v}$,

$$\begin{aligned} &\tilde{C}_{n+1,v}(F, x) \\ &:= \sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) \left[\left(\frac{1-x}{2}\right) F\left(2\frac{\ell}{1+n} - 1\right) + \left(\frac{x+1}{2}\right) F\left(2\frac{\ell+v}{1+n} - 1\right) \right] \end{aligned}$$

it follows that, taking the derivative of both sides with respect to x ,

$$\begin{aligned} \frac{d}{dx} \tilde{C}_{n+1,v}(F, x) &= \frac{d}{dx} \sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) \left[\left(\frac{1-x}{2}\right) F\left(2\frac{\ell}{1+n} - 1\right) \right. \\ &\left. + \left(\frac{x+1}{2}\right) F\left(2\frac{\ell+v}{1+n} - 1\right) \right] \\ &= \left\{ -\frac{(1+n-v)}{2} \sum_{\ell=0}^{n-v} \frac{(n-v)!}{\ell!(n-v-\ell)!} \left(\frac{x+1}{2}\right)^\ell \left(\frac{1-x}{2}\right)^{n-v-\ell} \right. \\ &\left. + \frac{1}{2} \sum_{\ell=1}^{n+1-v} \frac{(1+n-v)!}{\ell!(n+1-v-\ell)!} \left(\frac{x+1}{2}\right)^{\ell-1} \left(\frac{1-x}{2}\right)^{n+1-v-\ell} \right\} \\ &\times \left[\left(\frac{1-x}{2}\right) F\left(2\frac{\ell}{1+n} - 1\right) + \left(\frac{x+1}{2}\right) F\left(2\frac{\ell+v}{1+n} - 1\right) \right] \\ &= \frac{1}{2} \sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) \left[F\left(2\frac{\ell+v}{1+n} - 1\right) - F\left(2\frac{\ell}{1+n} - 1\right) \right] \end{aligned}$$

After computation, this derivative can be written as

$$\begin{aligned} \frac{d}{dx} \tilde{C}_{n+1,v}(F, x) &= \frac{1+n-v}{2} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left\{ \frac{1-x}{2} \int_{2\frac{\ell}{n+1}-1}^{2\frac{\ell+1}{n+1}-1} f(t) dt \right. \\ &\left. + \frac{x+1}{2} \int_{2\frac{\ell+v}{n+1}-1}^{2\frac{\ell+v+1}{n+1}-1} f(t) dt \right\} + \frac{1}{2} \sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) \int_{2\frac{\ell}{n+1}-1}^{2\frac{\ell+v}{n+1}-1} f(t) dt. \end{aligned} \tag{3}$$

is found. Since the right-hand side of (3) contains the Stancu Kantorovich operators $\mathcal{K}_{n,v}^{cc}(f; x)$ given in (2).

Now, another Kantorovich-type generalization of the Stancu operators can be described in Definition 3.2 as follows:

Definition 3.2

Let $x \in [-1,1]$ and $f \in L^1[-1,1]$. For a non-negative integer parameter v and $n \in \mathbb{N}$ with $n > 2v$, the Kantorovich-type Stancu operator is defined by

$$\begin{aligned} \mathcal{K}_{n,v}(f, x) := & \frac{(1+n-v)}{2} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left\{ \left(\frac{1-x}{2} \right) \int_{\frac{\ell}{n+1}-1}^{\frac{\ell+1}{n+1}-1} f(t) dt \right. \\ & \left. + \left(\frac{x+1}{2} \right) \int_{\frac{\ell+v}{n+1}-1}^{\frac{\ell+v+1}{n+1}-1} f(t) dt \right\} + \frac{1}{2} \sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) \int_{\frac{\ell}{n+1}-1}^{\frac{\ell+v}{n+1}-1} f(t) dt. \end{aligned} \tag{4}$$

The operators $\mathcal{K}_{n,v}(f; x)$ is called another modified Stancu-Kantorovich type. Then,

$$\frac{d}{dx} \tilde{C}_{n+1,v}(F; x) = \mathcal{K}_{n,v}(F'; x).$$

Remark 3.2

The operators $\mathcal{K}_{n,v}$ and $\mathcal{K}_{n,v}^{cc}$ are linear and positive.

Proposition 3.1

For $v = 0, 1$ these operators yield a modified Kantorovich operator given by

$$\mathcal{K}_n(f, x) = \frac{1+n}{2} \sum_{\ell=0}^n \wp_{n,\ell}(x) \int_{\frac{\ell}{n+1}-1}^{\frac{\ell+1}{n+1}-1} f(t) dt. \tag{5}$$

Hence,

$$\mathcal{K}_n(f, x) = \mathcal{K}_{n,0}(f; x) = \mathcal{K}_{n,1}(f, x) \text{ and } \mathcal{K}_n(f; x) = \mathcal{K}_{n,0}^{cc}(f, x) = \mathcal{K}_{n,1}^{cc}(f, x).$$

Both cases represent Kantorovich-type generalizations of the operator introduced by Cilo *et al.* (2012) in [25]. It is clearly seen that the operator defined here is a generalization of the operator given in [19]. In other words, for $v = 0,1$, the defined operator reduces to the one in [19].

Lemma 3.1

For the operator $\mathcal{K}_{n,v}^{cc}(f, x)$, the following properties are satisfied:

(i) $\mathcal{K}_{n,v}^{cc}(1, x) = 1,$

$$(ii) \mathcal{K}_{n,v}^{cc}(t, x) = \frac{n}{1+n} x,$$

$$(iii) \mathcal{K}_{n,v}^{cc}(t^2, x) = x + \frac{(n+v+3)(n-v)}{(1+n)^2} + \frac{12v^2 + 12v + 8}{6(1+n)^2} - 1 + \frac{2v}{(1+n)} + \frac{(n+3v-1)(n-v)}{(1+n)^2} x^2.$$

Proof (i) If we first compute $\mathcal{K}_{n,v}^{cc}(1, t)$, then we obtained

$$\mathcal{K}_{n,v}^{cc}(1, t) = \frac{1+n}{2} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left[\left(\frac{1-x}{2}\right) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} dt + \left(\frac{x+1}{2}\right) \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} dt \right] = 1.$$

(ii) For $\mathcal{K}_{n,v}^{cc}(t; x)$

$$\begin{aligned} \mathcal{K}_{n,v}^{cc}(t, x) &= \frac{1+n}{2} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left[\left(\frac{1-x}{2}\right) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} t dt + \left(\frac{x+1}{2}\right) \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} t dt \right] \\ &= \frac{2}{1+n} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \tau + \left[\frac{v}{1+n} x - 1 + \frac{v+1}{1+n} \right] \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x), \\ &= \frac{1+n}{1+n} - 1 + \frac{n}{1+n} x \end{aligned}$$

is obtained.

(iii) $\mathcal{K}_{n,v}^{cc}(t^2; x)$

$$= \frac{1+n}{2} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left[\left(\frac{1-x}{2}\right) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} t^2 dt + \left(\frac{x+1}{2}\right) \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} t^2 dt \right]$$

which simplifies to

$$\begin{aligned} \mathcal{K}_{n,v}^{cc}(t^2; x) &= \frac{1+n}{6} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left[\left(\frac{1-x}{2}\right) \left\{ \left(2 \frac{\ell+1}{1+n}\right)^3 - 3 \left(2 \frac{\ell+1}{1+n}\right)^2 \right. \right. \\ &\quad \left. \left. + 3 \left(2 \frac{\ell+1}{1+n}\right) - \left(2 \frac{\ell}{1+n}\right)^3 + 3 \left(2 \frac{\ell}{1+n}\right)^2 - 3 \left(2 \frac{\ell}{1+n}\right) \right\} \right] \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{x+1}{2}\right) \left\{ \left(2\frac{\ell+v+1}{1+n}\right)^3 - 3\left(2\frac{\ell+v+1}{1+n}\right)^2 + 3\left(2\frac{\ell+v+1}{1+n}\right) \right. \\
 & \left. - \left(2\frac{\ell+v}{1+n}\right)^3 + 3\left(2\frac{\ell+v}{1+n}\right)^2 - 3\left(2\frac{\ell+v}{1+n}\right) \right\}.
 \end{aligned}$$

If the brackets are further simplified, we obtain

$$\begin{aligned}
 & \frac{1+n}{6} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left[\frac{24\ell^2}{(1+n)^3} + \left\{ \frac{24v+24xv+24}{(1+n)^3} - \frac{24}{(1+n)^2} \right\} \ell \right. \\
 & \left. + \frac{(12v^2+12v)(1+x)+8}{(1+n)^3} - \frac{12v+12vx+12}{(1+n)^2} + \frac{6}{n+1} \right]
 \end{aligned}$$

After simplification, this yields

$$\begin{aligned}
 \mathcal{K}_{n,v}^{cc}(t^2; x) & = \frac{(n-v)(3v-1+n)}{(1+n)^2} x^2 + \left[\frac{2n(1+n)}{(1+n)^2} - \frac{2(n-v)}{1+n} \right] x \\
 & + \frac{(n+v+3)(n-v)}{(1+n)^2} + \frac{12v^2+12v+8}{6(1+n)^2} - \frac{2n+2}{n+1} + 1.
 \end{aligned}$$

Thus, the expression for $\mathcal{K}_{n,v}^{cc}(t^2; x)$ is obtained.

Theorem 3.3

For every $f \in C[-1,1]$, the following holds:

$$\lim_{n \rightarrow \infty} \left\| \mathcal{K}_{n,v}^{cc}(f, x) - f(x) \right\|_{C[-1,1]} = 0.$$

Proof The convergence can be directly deduced from Korovkin’s Theorem.

In the following example, a graph illustrating the approximation of the previously defined operator to the selected function for increasing values of n is presented.

Example 3.1

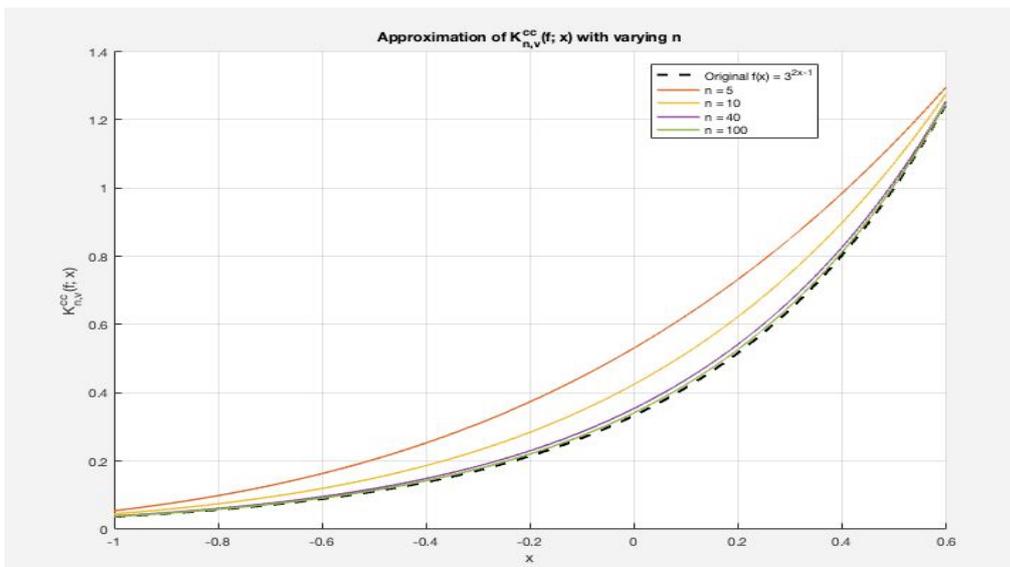


Figure 3: The approximation of $\mathcal{K}_{n,v}^{cc}(f; x)$ to $f(x) = 3^{2x-1}$

The approximation of $\mathcal{K}_{n,v}^{cc}(f; x)$ to $f(x) = 3^{2x-1}$ for $n = 5, 10, 40, 100$ is shown in Fig. 3. In the figure, the graphs correspond to $n = 5$ (red), $n = 10$ (yellow), $n = 40$ (purple), and $n = 100$ (green), while the black curve represents the graph of $f(x)$.

Example 3.2

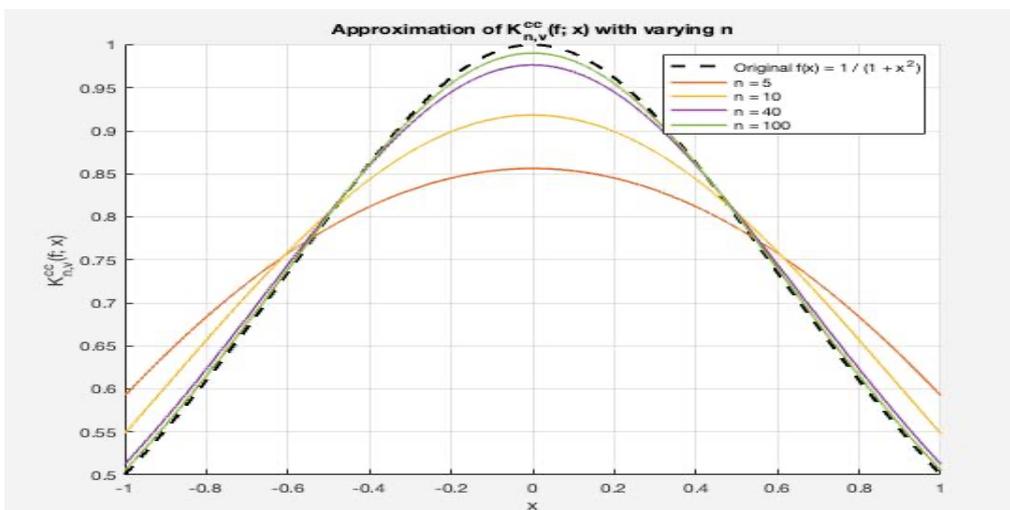


Figure 4: The approximation of $\mathcal{K}_{n,v}^{cc}(f; x)$ to $f(x) = \frac{1}{1+x^2}$.

The convergence of $\mathcal{K}_{n,v}^{cc}(f, x)$ to $f(x) = \frac{1}{1+x^2}$ for $n = 5, 10, 40, 100$ is illustrated in Fig. 4. In the figure, the graphs correspond to $n = 5$ (red), $n = 10$ (yellow), $n = 40$ (purple), and $n = 100$ (green), while the black curve represents the graph of $f(x)$.

Lemma 3.2

For the operator $\mathcal{K}_{n,v}(f, x)$, the following equations hold:

(i) $\mathcal{K}_{n,v}(1, x) = 1,$

(ii)
$$\mathcal{K}_{n,v}(t; x) = \frac{(n-v+1)(2n-v+2)}{2(1+n)^2}x + \frac{(n+1-v)(2n-v+4)}{2(1+n)^2} + \frac{(v+1)}{(1+n)^2} - \frac{(n-v+2)}{(1+n)},$$

(iii)
$$\begin{aligned} \mathcal{K}_{n,v}(t^2; x) &= \left\{ \frac{(n+2v-1)(n-v+1)(n-v)}{(n+1)^3} \right\} x^2 \\ &+ \left\{ \frac{(2n-2v^2+2n^2+2vn+2)(n-v+1)}{(n+1)^3} + \frac{(n+1-v)}{2(1+n)} \left(\frac{2v}{1+n} - 1 \right)^2 \right. \\ &\left. - \frac{(n+1-v)}{2(1+n)} - \frac{2n(n+1-v)}{(1+n)^2} \right\} x + \frac{(3n+v+n^2+vn)(n-v+1)}{(n+1)^3} \\ &+ \left(\frac{2v}{1+n} - 1 \right)^2 \frac{(n+1-v)}{2(1+n)} - \frac{(n-v+1)(2n+2)+2v^2}{(1+n)^2} + \frac{4v^3+4(n-v+1)}{3(1+n)^3} \\ &+ \frac{(n+1+v)}{2(1+n)}. \end{aligned}$$

Proof By applying Lemma 3.1, the proof is established.

Theorem 3.4 For every $f \in C[-1,1],$

$$\lim_{n \rightarrow \infty} \|\mathcal{K}_{n,v}(f, x) - f(x)\|_{C[-1,1]} = 0.$$

Proof Applying Korovkin’s Theorem, it suffices to prove the statement for 1, t and t^2 .

Example 3.3

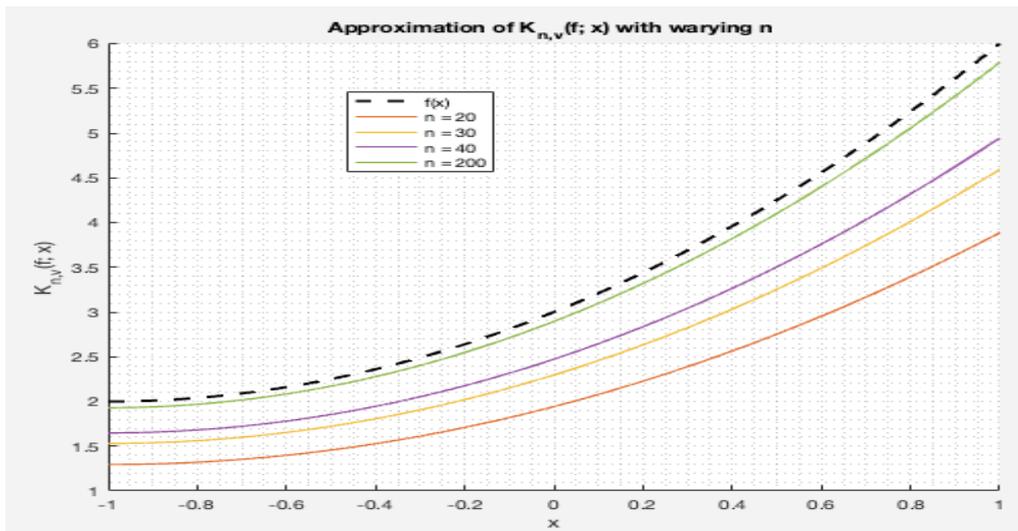


Figure 5: The approximation of $\mathcal{K}_{n,v}(f, x)$ to $f(x) = x^2 + 2x + 3$

Figure 5 depicts the convergence of $\mathcal{K}_{n,v}(f, x)$ to $f(x) = x^2 + 2x + 3$ for $n = 20, 30, 40, 200$. The graphs are plotted as follows: $n = 20$ in red, $n = 30$ in yellow, $n = 40$ in purple, and $n = 200$ in green, while the black curve represents $f(x)$.

Example 3.4

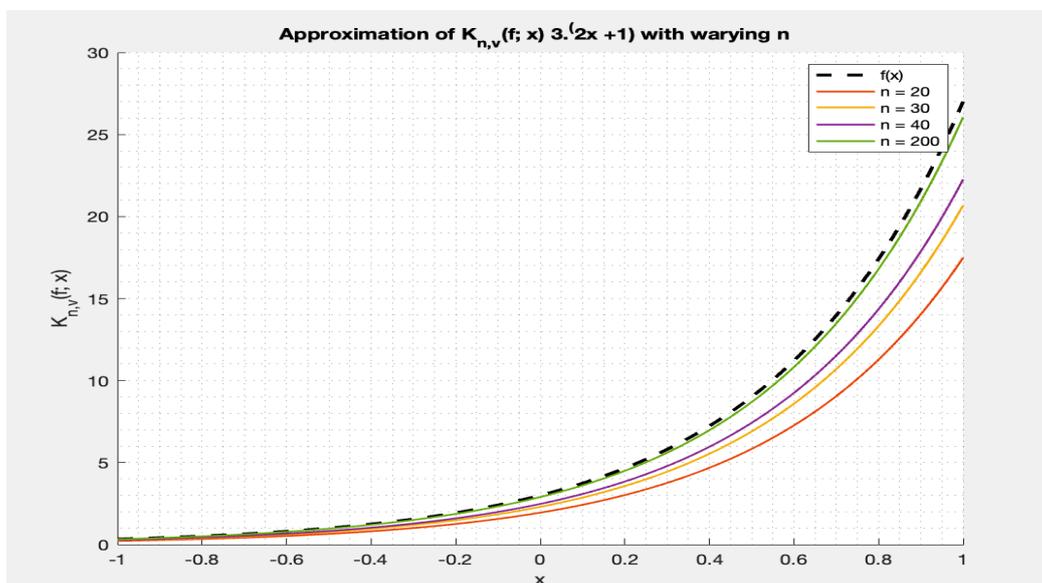


Figure 6: The approximation of $\mathcal{K}_{n,v}(f, x)$ to $f(x) = 3^{2x+1}$

Figure 6 presents the approximation of $\mathcal{K}_{n,v}(f; x)$ to the function $f(x) = 3^{2x+1}$ for $n = 20, 30, 40, 200$. The graphs are plotted as follows: $n = 20$ in red, $n = 30$ in yellow, $n = 40$ in purple, and $n = 200$ in green, while the black curve represents $f(x)$.

Corollary 3.1

By utilizing the fact that the operator given in Definition 3.2 reduces to the operator presented in [19] when $v = 1$, the approximation of the same function f by the operator in [19] is shown in Fig. 3.5, while the approximation by the operator defined in Definition 3.2 is illustrated in Fig. 3.6. When the maximum error in Fig. 3.6 is examined, it can be observed that the operator defined herein provides a better approximation to the function.

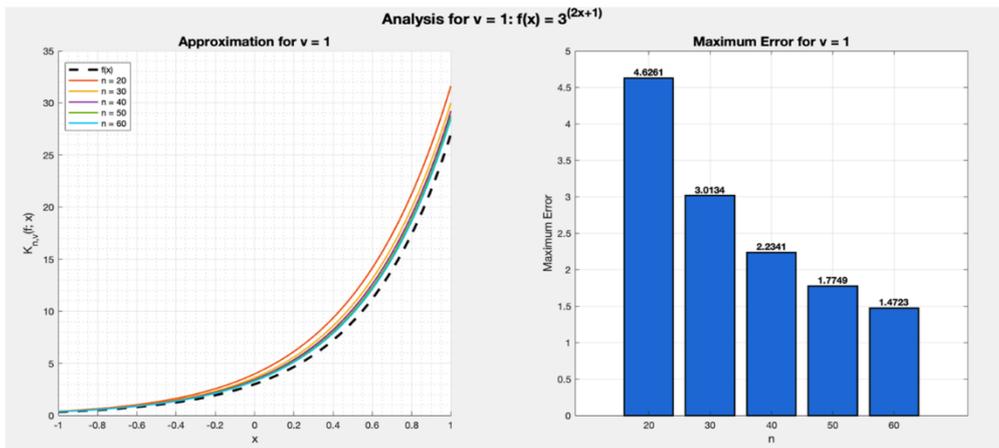


Figure 7: The approximation of $\mathcal{K}_{n,v}(f, x)$ to $f(x) = 3^{2x+1}$

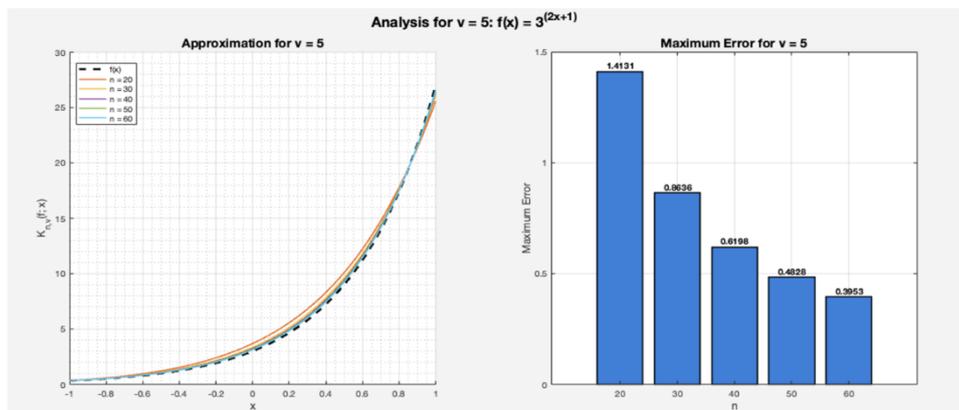


Figure 8: The approximation of $\mathcal{K}_{n,v}(f, x)$ to $f(x) = 3^{2x+1}$

4. Approximation by New Operators in $L^p[-1,1]$ Space

In this section, we present important theorems that provide the approximation of the operator within the class of p -th power integrable functions.

Theorem 4.1

Let p be a non-negative constant integer with $1 \leq p < \infty$, and let $f \in L^p[-1,1]$. Then,

$$\lim_{n \rightarrow \infty} \|\mathcal{K}_{n,v}^{cc}(f, x) - f(x)\|_{L^p[-1,1]} = 0 \quad (\forall f \in L^p[-1,1]).$$

Proof

$$|\mathcal{K}_{n,0}^{cc}(f; x)|^p \leq \sum_{\ell=0}^n \wp_{n,\ell}(x) \left| \frac{1+n}{2} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} f(\hbar) d\hbar \right|^p.$$

Thus,

$$\begin{aligned} |\mathcal{K}_{n,1}^{cc}(f, x)|^p &\leq \frac{(1+n)(1-x)}{4} \sum_{\ell=0}^{n-1} \wp_{n-1,\ell}(x) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \\ &+ \frac{(1+n)(1+x)}{4} \sum_{\ell=p}^{n-1} \wp_{n-1,\ell}(x) \int_{2^{\frac{\ell+1}{n+1}-1}}^{2^{\frac{\ell+2}{n+1}-1}} |f(\hbar)|^p d\hbar. \end{aligned}$$

is found. In the last inequality, integrating the two sides over $[-1,1]$

Integrating both sides over $[-1,1]$,

$$\begin{aligned} \int_{-1}^1 |\mathcal{K}_{n,1}^{cc}(f, x)|^p dx &\leq \frac{1+n}{2} \int_{-1}^1 \left(\frac{1-x}{2}\right) \sum_{\ell=0}^{n-1} \wp_{n-1,\ell}(x) dx \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \\ &+ \frac{1+n}{2} \int_{-1}^1 \left(\frac{x+1}{2}\right) \sum_{\ell=p}^{n-1} \wp_{n-1,\ell}(x) dx \int_{2^{\frac{\ell+1}{n+1}-1}}^{2^{\frac{\ell+2}{n+1}-1}} |f(\hbar)|^p d\hbar \end{aligned}$$

is obtained. By the reasoning given in Theorem [31] the cases $v = 0$ and $v = 1$ are clear.

A detailed proof for the case $v > 1$ will now be presented. It is sufficient to show that the norm of the operator is uniformly bounded by a Korovkin-type theorem in the space $L^p[-1,1]$.

In this case, for all $n \in \mathbb{N}, v > 1$, and $n > 2v$, let $\|\mathcal{K}_{n,v}^{cc}\|$ denote the operator norm. To ensure that the operator norm is uniformly bounded, it must be shown that there exists a constant $\mathfrak{S} > 0$ such that

$$\|\mathcal{K}_{n,v}^{cc}(f, x)\|_{L^p[-1,1]} \leq \|f\|_{L^p[-1,1]} \mathfrak{S} \text{ [25].}$$

In the operator setting with $1 \leq p < \infty$ and $\hbar \in [0,1]$, let us consider the convex function $\psi(\hbar) = |\hbar|^p$. Since $\mathcal{E}_{n,\ell,v}(x) \geq 0$ and $\sum_{\ell=0}^n \mathcal{E}_{n,\ell,v}(x) = 1$, an application of Jensen’s inequality, together with the definition of $\mathcal{E}_{n,\ell,v}(x)$,

$$|\mathcal{K}_{n,v}^{cc}(f; x)|^p \leq \sum_{\ell=0}^n E_{n,\ell,v}(x) \frac{1+n}{2} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar.$$

Moreover, this inequality can equivalently be expressed in the form

$$\left\{ \left(\frac{1-x}{2}\right) \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) + \left(\frac{x+1}{2}\right) \sum_{\ell=v}^n \wp_{n-v,\ell-v}(x) \right\} \frac{1+n}{2} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar.$$

Thus, the operator $\mathcal{K}_{n,v}^{cc}(f; x)$ admits a representation that is consistent with Jensen’s framework, which will be instrumental for establishing its norm estimates in the subsequent analysis.

In the last inequality, integrating the two sides over $[-1,1]$

$$\int_{-1}^1 |\mathcal{K}_{n,v}^{cc}(f; x)|^p dx \leq \frac{1+n}{2} \left\{ \sum_{\ell=0}^{n-v} \binom{n-v}{\ell} \int_{-1}^1 \left(\frac{1+x}{2}\right)^\ell \left(\frac{1-x}{2}\right)^{1+n-\ell-v} dx + \sum_{\ell=v}^n \binom{n-v}{\ell-v} \int_{-1}^1 \left(\frac{1-x}{2}\right)^{n-\ell} \left(\frac{x+1}{2}\right)^{\ell+1-v} dx \right\} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar$$

if parentheses are taken in the expression,

$$\sum_{\ell=0}^{n-v} \binom{n-v}{\ell} \int_{-1}^1 \left(\frac{x+1}{2}\right)^\ell \left(\frac{1-x}{2}\right)^{1+n-\ell-v} dx =: S_1$$

$$\sum_{\ell=v}^n \binom{n-v}{\ell-v} \int_{-1}^1 \left(\frac{x+1}{2}\right)^{\ell-v+1} \left(\frac{1-x}{2}\right)^{n-\ell} dx =: S_2$$

and if $\frac{1+x}{2} = u$ variable is changed, integrals

$$2 \sum_{\ell=0}^{n-v} \binom{n-v}{\ell} \int_0^1 u^\ell (1-u)^{n-v-\ell+1} du = S_1,$$

$$2 \sum_{\ell=v}^n \binom{n-v}{\ell-v} \int_0^1 u^{\ell-v+1} (1-u)^{n-\ell} du = S_2$$

is obtained in the form. It follows that

$$\int_0^1 u^\ell (1-u)^{n-v-\ell+1} du = \beta(\ell+1, 2-v-\ell+n),$$

$$\int_0^1 u^{\ell-v+1}(1-u)^{n-\ell} = \beta(\ell-v+2, n-\ell+1).$$

Since the following inequality is established:

$$\begin{aligned} |\mathcal{K}_{n,v}^{cc}(f; x)|^p &\leq \frac{1+n}{2} \left\{ 2 \sum_{\ell=0}^{n-v} \binom{n-v}{\ell} \beta(\ell+1, 2-v-\ell+n) \right. \\ &\left. + 2 \sum_{\ell=v}^n \binom{n-v}{\ell-v} \beta(\ell-v+2, n-\ell+1) \right\} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar. \end{aligned}$$

Here the notations β, Γ denotes the Beta functions and the Gamma function; respectively.

Using the Gamma function,

$$\begin{aligned} \binom{n-v}{\ell} \beta(\ell+1, 2-v-\ell+n) &= \binom{n-v}{\ell} \frac{\Gamma(\ell+1)\Gamma(n-v-\ell+2)}{\Gamma(n-v+3)} \\ &= \frac{n-v-\ell+1}{(n-v+2)(n-v+1)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \binom{n-v}{\ell-p} \beta(\ell-v+2, n-\ell+1) &= \binom{n-v}{\ell-p} \frac{\Gamma(\ell-v+2)\Gamma(1-\ell+n)}{\Gamma(n-v+3)} \\ &= \frac{\ell-v+1}{(n-v+2)(n-v+1)} \end{aligned}$$

is obtained. In this case, the following inequality is obtained:

$$\begin{aligned} |\mathcal{K}_{n,p}^{cc}(f; x)|^p &\leq \sum_{\ell=0}^{n-v} \frac{(1+n)(n-v-\ell+1)}{(n-v+1)(n-v+2)} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \\ &+ \sum_{\ell=v}^n \frac{(1+n)(\ell-v+1)}{(1-v+n)(n-v+2)} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar. \end{aligned}$$

Since $n > 2v$, it follows that $n-v > v$. Therefore, the sums in the previous expression can be partitioned as follows:

$$|\mathcal{K}_{n,p}^{cc}(f; x)|^p \leq \frac{1+n}{n-v+2} \left\{ \sum_{\ell=0}^{v-1} \frac{(1-v-\ell+n)}{(1-v+n)} + \sum_{\ell=v}^{n-v} \frac{(n-2v+2)}{(n-v+1)} \right\}$$

$$+ \sum_{\ell=n-v+1}^n \frac{(\ell - v + 1)}{(n - v + 1)} \left\} \times \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar$$

Here, since $n - 2v + 1 < n - v + 1$ in the second sum, we have $n - 2v + 2 \leq n - v + 1$.

On the other hand, since

$$\frac{1-v-\ell+n}{n+1-v} \leq 1 \text{ for } 0 \leq \ell \leq v - 1$$

in the first sum and

$$\frac{\ell-v+1}{n-v+1} \leq 1 \text{ for } n - v + 1 \leq \ell \leq n$$

in the third sum, we obtain

$$\int_{-1}^1 |\mathcal{K}_{n,v}^{cc}(f, x)|^p dx \leq \frac{1+n}{n-v+2} \left\{ \sum_{\ell=0}^{v-1} 1 + \sum_{\ell=v}^{n-v} 1 + \sum_{\ell=n-v+1}^n 1 \right\} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar.$$

Also

$$\frac{(1+n)}{n-v+2} (1+n) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \leq \frac{2(1+n)}{(n-v+2)} \int_{-1}^1 |f(\hbar)|^p d\hbar$$

is obtained. For fixed $v > 1$, take $\sup_{n>2v} \frac{2(1+n)}{(n-v+2)} =: M_v$. Here M_v is a constant and $M_v < 2$,

$$\int_{-1}^1 |\mathcal{K}_{n,v}^{cc}(f; x)|^p dx \leq M_v \int_{-1}^1 |f(\hbar)|^p d\hbar$$

inequality can be written. Using the definition of $L^p[-1,1]$ norm

$$\|\mathcal{K}_{n,v}^{cc}(f; x)\|_{L^p[-1,1]} \leq \|\mathcal{K}_{n,v}^{cc}\|_{L^p[-1,1]} \|f\|_{L^p[-1,1]} \leq M_v^{\frac{1}{p}} \|f\|_{L^p[-1,1]}.$$

Then the norms of the operators $\mathcal{K}_{n,v}^{cc}$ are bounded by $\|\mathcal{K}_{n,v}^{cc}\|_{L^p[-1,1]} \leq M_v^{\frac{1}{p}}$. On the other hand, let $\varepsilon > 0$ be arbitrary. Since $C[-1,1]$ is dense in the space $L^p[-1,1]$, there exists a $g \in C[-1,1]$ where $\|f - g\|_{L^p[-1,1]} < \varepsilon$. $\mathcal{K}_{n,v}^{cc}$ from Korovkin's theorem, for a $g(x)$, there exists n_0 where for all $n > n_0$, $\|\mathcal{K}_{n,v}^{cc}(g; x) - g\|_{L^p[-1,1]} \leq \varepsilon$. Hence, for $\mathfrak{S} := 3M_v^{\frac{1}{p}}$,

$$\begin{aligned} \|\mathcal{K}_{n,v}^{cc}(f, x) - f(x)\|_{L^p[-1,1]} &\leq M_v^{\frac{1}{p}} \|f - g\|_{L^p[-1,1]} + \|\mathcal{K}_{n,v}^{cc}(g; x) - g\|_{L^p[-1,1]} \\ &+ \|g - f\|_{L^p[-1,1]} < \varepsilon \mathfrak{S}. \end{aligned}$$

The proof is accomplished from the inequality.

Like the theorem given in [38], the following theorem can be given for the operator we define.

Theorem 4.2

Let p be a nonnegative fixed integer and $1 \leq p < \infty$. For every $f \in L^p[-1,1]$,

$$\lim_{n \rightarrow \infty} \|\mathcal{K}_{n,v}(f, x) - f(x)\|_{L^p[-1,1]} = 0.$$

Proof Since the cases $v = 0$ and $v = 1$ are given in [31], only the case $v > 1$ will be proved in detail as same as [35]. It suffices to show that the norm of the operator is uniformly bounded with a Korovkin-type theorem on the space L^p .

So, for every $v > 1$, $n \in \mathbb{N}$ and $n > 2v$, $\|\mathcal{K}_{n,v}\|$ is the operator norm, and for the operator norm to be uniformly bounded, it must be shown that there exists a $\mathfrak{S} > 0$ where $\|\mathcal{K}_{n,v}\| \leq \mathfrak{S}$. For modified Stancu Kantorovich type operator

$$\begin{aligned} \mathcal{K}_{n,v}(f, x) &= \frac{(1+n-v)}{2} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left\{ \left(\frac{1-x}{2}\right) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} f(\hbar) d\hbar \right. \\ &+ \left. \left(1 - \frac{1-x}{2}\right) \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} f(\hbar) d\hbar \right\} + \frac{1}{2} \sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+v}{n+1}-1}} f(\hbar) d\hbar. \end{aligned}$$

Then if we divide these equality $\mathcal{K}_{n,v}(f; x) = G_{n,v}(f; x) + G_{n+1,v}(f; x)$. Here,

$$\begin{aligned} G_{n,v}(f, x) &= \frac{(1-v+n)}{2} \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left\{ \left(\frac{1-x}{2}\right) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} f(\hbar) d\hbar \right. \\ &+ \left. \left(1 - \frac{1-x}{2}\right) \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} f(\hbar) d\hbar \right\} \end{aligned}$$

$$G_{n+1,v}(f, x) = \frac{1}{2} \sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+v}{n+1}-1}} f(\hbar) d\hbar.$$

Then, using

$$|\mathcal{K}_{n,v}(f; x)|^p \leq 2^p \left(|G_{n,v}(f, x)|^p + |G_{n+1,v}(f, x)|^p \right) \tag{6}$$

so, the $L^p[-1,1]$ -norm of $\mathcal{K}_{n,v}(f; x)$ will be found. For the first term of the right-hand side of (6), for $t \in [0,1]$ using the convexity of the function $\psi(t) = |t|^p$ and $1 \leq p < \infty$, $\lambda_1 = \frac{1-x}{2}$ and $\lambda_2 = \left(1 - \frac{1-x}{2}\right)$ for $x \in [-1,1]$, $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$. Thus, from the Jensen inequality given in Remark 1.1

$$\begin{aligned} |G_{n,v}(f; x)|^p &\leq \left(\frac{1-x}{2}\right) \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left| \frac{1+n-v}{2} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} f(\hbar) d\hbar \right|^p \\ &+ \left(\frac{x+1}{2}\right) \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \left| \frac{1+n-v}{2} \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} f(\hbar) d\hbar \right|^p \end{aligned}$$

is found. In the last expression, for the function $\psi(t) = |t|^p$ and the measure $(n+1-v)$, where $\hbar \in [0,1]$ and $1 \leq p < \infty$, from Remark 1.2 applying the integral form of Jensen's inequality

$$\begin{aligned} |G_{n,v}(f; x)|^p &\leq \sum_{\ell=0}^{n-v} \wp_{n-v,\ell}(x) \frac{1-v+n}{2} \left\{ \left(\frac{1-x}{2}\right) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \right. \\ &\left. + \left(\frac{x+1}{2}\right) \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} |f(\hbar)|^p d\hbar \right\} \tag{7} \end{aligned}$$

is obtained. For the function $\psi(\hbar) = |\hbar|^p$ with $t \in [0,1]$ and $1 \leq p < \infty$, in the second term of the right-hand side of (6); $x \in [-1,1], \ell = 0, \dots, n-v$ while $\wp_{n+1-v,\ell}(x) \geq 0$ and $\sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) = 1$

$$|G_{n+1,v}(f; x)|^p = \frac{1}{2} \sum_{\ell=0}^{n+1-v} \wp_{n+1-v,\ell}(x) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+v}{n+1}-1}} |f(\hbar)|^p d\hbar \tag{8}$$

is obtained. If taking integration of (7) and the beta function is used

$$\int_{-1}^1 |G_{n,v}(f; x)|^p dx \leq \frac{1+n-v}{2} \sum_{\ell=0}^{n-v} \binom{n-v}{\ell}$$

$$\times \left\{ \int_{-1}^1 \left(\frac{x+1}{2}\right)^\ell \left(\frac{1-x}{2}\right)^{1-v-\ell+n} dx \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \right. \\ \left. + \int_{-1}^1 \left(\frac{x+1}{2}\right)^{\ell+1} \left(\frac{1-x}{2}\right)^{n-v-\ell} dx \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} |f(\hbar)|^p d\hbar \right\}.$$

For $\frac{1+x}{2} = u$ on the right side,

$$\int_{-1}^1 |G_{n,v}(f; x)|^p dx \leq \frac{1}{2(n-v+2)} M_{n,v} \tag{9}$$

is found. Since $n > 2v, n - v > v$, thus, for $v > 1$,

$$M_{n,v} = \sum_{\ell=0}^{v-1} (n+1-v-\ell) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \\ + \sum_{\ell=v}^{n-v} (n+1-v-\ell) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar + \sum_{\ell=0}^{n-2v} (\ell+1) \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} |f(\hbar)|^p d\hbar \\ + \sum_{\ell=n-2v+1}^{n-v} (\ell+1) \int_{2^{\frac{\ell+v}{n+1}-1}}^{2^{\frac{\ell+v+1}{n+1}-1}} |f(\hbar)|^p d\hbar$$

can be written. If $\ell - v$ is taken instead of ℓ in the second summation,

$$M_{n,v} = \sum_{\ell=0}^{v-1} (n+1-v-\ell) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \\ + \sum_{\ell=v}^{n-v} (n+2-2v) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar + \sum_{\ell=n-v+1}^n (\ell-v+1) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar$$

is obtained. On the other hand, similarly, taking integration of (8) over $[-1,1]$,

$$\int_{-1}^1 |G_{n+1,v}(f; x)|^p dx \leq \frac{1}{2(n-v+2)} \sum_{\ell=0}^{n+1-v} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \tag{10}$$

is found. By combining similar terms, the sum in formula (4.19) can be divided into three parts as follows:

$$\begin{aligned}
 \sum_{\ell=0}^{n+1-v} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+v}{n+1}-1}} |f(\hbar)|^p d\hbar &= \sum_{\ell=0}^{n+1-v} \sum_{r=0}^{v-1} \int_{2^{\frac{\ell+r}{n+1}-1}}^{2^{\frac{\ell+r+1}{n+1}-1}} |f(\hbar)|^p d\hbar \\
 &\leq \sum_{\ell=0}^{v-1} (\ell + 1) \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar + v \sum_{\ell=v}^{n-v} \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar \\
 &+ (n + 1 - v) \sum_{\ell=n-v+1}^n \int_{2^{\frac{\ell}{n+1}-1}}^{2^{\frac{\ell+1}{n+1}-1}} |f(\hbar)|^p d\hbar. \tag{11}
 \end{aligned}$$

Using (8) - (11); in (6)

$$\int_{-1}^1 |K_{n,v}(f; x)|^p dx \leq 2^p \int_{-1}^1 |f(\hbar)|^p d\hbar.$$

Therefore, moving to the L^p - norm, for every $f \in L^p[-1,1]$ we get $\|\mathcal{K}_{n,v}(f)\|_{L^p[-1,1]} \leq 2\|f\|_{L^p[-1,1]}$.

That is, for every $n \in \mathbb{N}$ and $n > 2v$, since $\|\mathcal{K}_{n,v}(f)\| \leq 2$, operator is bounded. On the other hand, let $\varepsilon > 0$ be arbitrarily given. Since $C[-1,1]$ is dense in the space $L^p[-1,1]$, so there exists a $f \in C[-1,1]$ where $\|f - g\|_{L^p[-1,1]} < \varepsilon$. For $\mathcal{K}_{n,v}^{cc}$ from Korovkin's theorem, for every $n \geq n_0$, there exists $n_0 \in \mathbb{N} : \|\mathcal{K}_{n,v}^{cc}(g; x) - g\|_{L^p[-1,1]} \leq \varepsilon$. Hence the proof follows from the inequality

$$\begin{aligned}
 \|\mathcal{K}_{n,v}^{cc}(f; x) - f\|_{L^p[-1,1]} &\leq 2\|f - g\|_{L^p[-1,1]} + \|\mathcal{K}_{n,v}^{cc}(g; x) - g\|_{L^p[-1,1]} \\
 &+ \|g - f\|_{L^p[-1,1]} < 4\varepsilon.
 \end{aligned}$$

5. Conclusion

In this study, two distinct parameter-dependent generalizations of Stancu-type operators have been investigated, and a new class of Kantorovich-type operators has been introduced based on these generalizations. First, the generalization of Stancu operators with the requirements arising from the interval change is defined and then the generalization of this new operator in the Kantorovich type is defined. The operators defined on the interval $[-1,1]$ were shown to possess Korovkin-type properties, and their fundamental approximation characteristics were rigorously analyzed. The convergence behavior of both classes of operators in L^p - spaces was examined in detail, highlighting their effectiveness compared to classical approximations.

Graphical illustrations complemented the theoretical analysis, providing a visual understanding of the operators' effects on functions. The novelty of the work lies in the introduction of the new Kantorovich-type operators, the demonstration of their Korovkin-type properties, and the comprehensive study of their convergence in L^p -spaces.

Motivated by the studies of Çilo et al. [25], Stancu [10] and Bostancı Güney [35], the present work extended the approximation framework by generalizing the Stancu operator with a focus on L^p -norm convergence. There exist several generalizations of the operator presented in [25] in the literature. Among these, the operator given in [19] provides a Kantorovich-type generalization. However, the present study demonstrates that this generalization is in fact a special case of the operator we have obtained using a more recent and refined method. Although various classical generalizations of the operator in [25] can be found in the literature, no version based on Stancu's work [10] has been encountered.

The operator defined in [35], based on the classical Bernstein operator, provided approximation properties for a Stancu-type operator. Subsequently, they studied the Kantorovich-type generalizations of the operator they introduced. In parallel with this work, which fills the gap in the Stancu-type generalization of the Bernstein operator, the present study addresses a similar gap in the literature for the operator given in [25].

In addition to the fact that the defined operator reduces to the related operator given in [19] for certain special choices, as presented in Corollary 3.1, its yielding a better approximation result than the operator in [19] can be considered another contribution of the defined operator to the literature.

In conclusion, the results of this study demonstrate that the newly constructed operators provide notable advantages in terms of approximation accuracy and convergence behavior. Overall, the present work contributes both theoretically and practically to the field of approximation theory and related disciplines in which integral operators play a key role. The findings are expected to serve as a solid foundation for future research and applications in mathematical analysis, computational methods, and the applied sciences. Moreover, this work offers valuable insights, particularly for studies on Bernstein-type operators and integral equations, which have recently attracted considerable attention. The proposed operator can also be introduced to different frameworks through the definitions provided in [39–40]. In this context, the generalizations presented in [41–47] may be employed to evaluate the operators from different perspectives and to extend their range of applications.

As a recommendation for new researchers, this study can be extended to new applications by utilizing various generalizations of the operator given in [35], as examined in [48-50].

References

- [1] Weierstrass, K., *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 2, 633-639, 1885.
- [2] Bernstein, S., *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*. Сообщения Харьковского математического общества, 13(1), 1-2, 1912.
- [3] Stancu, D.D., *On Some Polynomials of Bernstein Type*, *Studii și Cercetări Științifice, Seria Matematică* (Iași), 11, 221-233, 1960.
- [4] Stancu, D.D., *Approximation of functions by a new class of polynomial operators*, *Revue Roumaine de Mathématiques Pures et Appliquées*, 13(8), 1173-1194, 1968.
- [5] Stancu, D.D., *Quadrature formulas constructed by using certain linear positive operators*, In *Numerical Integration: Proceedings of the Conference Held at the Mathematisches Forschungsinstitut Oberwolfach*, October 4–10, 1981, 241-25, Birkhäuser, Basel, 1982.
- [6] Stancu, D.D., Occorsio, M.R., *On approximation by binomial operators of Tiberiu Popoviciu type*, *Revue d'analyse numérique et de théorie de l'approximation*, 27(1), 167-181, 1998.
- [7] Stancu, D.D., *A note on a multiparameter Bernstein-type approximating operator*, *Mathematica (Cluj)*, 26(49), 153-157, 1984.
- [8] Stancu, D.D., *Approximation of functions by means of a new generalized Bernstein operator*, *Calcolo*, 20, 211-229, 1983.
- [9] Stancu, D.D., *Asupra unei generalizări a polinoamelor lui Bernstein*, *Studia Universitatis Babeș-Bolyai, Mathematica*, 14(2), 31-45, 1969.
- [10] Stancu, D.D., *The remainder in the approximation by a generalized Bernstein Operator: a representation by a convex combination of second-order divided differences*, *Calcolo*, 35(1), 53-62, 1998.
- [11] Kantorovich, L.V., *Sur certains développements suivant les polynômes de la forme S. Bernstein*, *Comptes Rendus de l'Académie des Sciences de l'URSS*, 20, 563–568, 1930.
- [12] Korovkin, P.P., *On convergence of linear positive operators in the space of continuous functions*, *Doklady Akademii Nauk SSSR*, 90, 961–964, 1953.
- [13] Korovkin, P.P., *Linear operators and approximation theory*, Hindustan Publishing Corporation, 1960.
- [14] Campiti, M., Metafune, G., *L^p -convergence of Bernstein-Kantorovich-type operators*, *Annales Polonici Mathematici*, 63(3), 273-280, 1996.
- [15] Altomare, F., Campiti, M., *Korovkin-type approximation theory and its applications*, Walter de Gruyter, 1994.
- [16] Altomare, F., *Korovkin-type theorems and approximation by positive linear operators*, arXiv:1009.2601, 2010.
- [17] Altomare, F., Cappelletti Montano, M., Leonessa, V., Raşa, I., *A generalization of Kantorovich operators for convex compact subsets*, *Banach Journal of Mathematical Analysis*, 11(3), 591-614, 2017.

- [18] Kajla, A., Mursaleen, M., Acar, T., *Durrmeyer-type generalization of parametric Bernstein operators*, *Symmetry*, 12(7), 1141, 2020.
- [19] Kahvecibaşı, İ., *Approximation properties of Bernstein–Kantorovich operators on the interval $[-1,1]$* , Harran University, Şanlıurfa, Türkiye, 2014.
- [20] Kajla, A., *The Kantorovich variant of an operator defined by D.D. Stancu*, *Applied Mathematics and Computation*, 316, 400-408, 2018.
- [21] Stancu, D.D., *Approximation of functions by a new class of linear polynomial operators*, *Revue Roumaine de Mathématiques Pures et Appliquées*, 13(8), 1173–1194, 1968.
- [22] Kantorovich, L.V., Akilov, G.P., *Functional analysis*, Pergamon Press, 1st ed., 1982.
- [23] Sasi, S., Jyothi, L.S., *A heuristic approach for secured transmission of image based on Bernstein polynomial*. In International Conference on Circuits, Communication, Control and Computing, 312-315, 2014.
- [24] Bilgin, N.G., Kaya, Y., and Eren, M., *Security of image transfer and innovative results for (p, q) -Bernstein-Schurer operators*, *AIMS Mathematics*, 9(9), 23812-23836, 2024.
- [25] Çilo, A., Ural, A., İzgi, A., *Comparison of Bernstein polynomials and some modifications*, In XXV. National Mathematics Symposium - Niğde University, 5-8 September, 2012.
- [26] Çilo, A., *Approximation properties and rates of convergence of Bernstein polynomials on the interval $[-1,1]$* , Harran University, Şanlıurfa, Türkiye, 2012.
- [27] İzgi, A. ve Büyükyazıcı, İ., *Approximation in boundedness interval and order of approximation*, *Kastamonu Education Journal*, 11(2), 451-460, 2003.
- [28] İzgi, A., *Approximation by a class of new type Bernstein polynomials of one and two variables*, *Global Journal of Pure and Applied Mathematics*, 8(5), 55-71, 2012.
- [29] Büyükyazıcı, İ., *Bernstein polynomials of functions of two variables*, Ankara University, Ankara, Türkiye, 1999.
- [30] Bilgin, N.G., Eren, M., *A generalization of two dimensional Bernstein-Stancu operators*, *Sinop University Journal of Natural Sciences* <https://dergipark.org.tr/en/pub/sinopfbid>, 6(2), 130-142, 2021.
- [31] Lorentz, G. G., *Bernstein Polynomials*. University of Toronto Press, 1953.
- [32] Tunç, T. and Alhazzori, G., *On conic equations under Bernstein operators*, *Bitlis Eren University Journal of Science*, 13(1), 161-169, 2024.
- [33] Aral, A., Ökten, O., Acar, T., *A note on Bernstein-Stancu-Chlodowsky operators*, *Kırıkkale Üniversitesi Bilimde Gelişmeler Dergisi*, 11, 2012.
- [34] Aral, A., Cardenas-Morales, D., Garrancho, P., *Bernstein-type operators that reproduce exponential functions*, *Journal of Mathematical Inequalities*, 12(3), 861-872, 2018.
- [35] Güney, B.T., *Generalized Stancu Operators*. Ankara University, Ankara, Türkiye, 2022.
- [36] Abramowitz, M., Stegun I.A., *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, US Government Printing Office, 1965.
- [37] Hacısalihoglu, H., Hacıyev, A., *Convergence of sequences of linear positive operators*, Ankara University, Ankara, Türkiye, 1-30, 1995.
- [38] Bostancı, T., Başcanbaz-Tunca, G., *On Stancu operators depending on a non-negative integer*, *Filomat*, 36(18), 6129-6138, 2022.

- [39] Kaytmaz, Ö., *The problem of determining source term in a kinetic equation in an unbounded domain*, *AIMS Mathematics*, 9(4), 9184-9194, 2024.
- [40] Bilgin, N.G., Coşkun, N., *Comparison result of some Gadjiev-Ibragimov type operators*, *Karaelmas Fen ve Mühendislik Dergisi*, 8(1), 188-196, 2018.
- [41] Aslan, R., *Some approximation results on λ Szász-Mirakjan-Kantorovich operators*, *Fundamental Journal of Mathematics and Applications*, 4(3), 150-158, 2021.
- [42] Çiçek, H., İzgi, A., *Approximation by modified bivariate Bernstein-Durrmeyer and GBS bivariate Bernstein-Durrmeyer operators on a triangular region*, *Fundamental Journal of Mathematics and Applications*, 5(2), 135-144, 2022.
- [43] Baytunç, E., Aktuğlu, H., Mahmudov, N., *A new generalization of Szász-Mirakjan Kantorovich operators for better error estimation*, *Fundamental Journal of Mathematics and Applications*, 6(4), 194-210, 2023.
- [44] Arı, D.A., Yılmaz, G.U., *A note on Kantorovich type operators which preserve affine functions*, *Fundamental Journal of Mathematics and Applications*, 7(1), 53-58, 2024.
- [45] Yılmaz, Ö.G., *On the eigenstructure of the q -Stancu operator*, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 73(3), 820-832, 2024.
- [46] Bodur, M., Bostancı, T., Başcanbaz-Tunca, G., *On Kantorovich variant of Brass-Stancu operators*, *Demonstratio Mathematica*, 57(1), 20240007, 2024.
- [47] Kumar, A., *A new kind of variant of the Kantorovich type modification operators introduced by D.D. Stancu*, *Results in Applied Mathematics*, 11, 100158, 2021.
- [48] Yeter, S., Çetin, N., *Approximation by generalized Stancu-Kantorovich operators*, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 74(3), 503-512, 2025.
- [49] Başcanbaz-Tunca, G., *A note on Stancu operators with three parameters*, *Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science*, 5(67), 55-70, 2025.
- [50] Çetin, N., Manav Mutlu, N., *Approximation by Generalization of Bernstein-Schurer Operators*. In *Advanced mathematics for the modeling and solution of challenging problems in engineering*, 582, 375-391, 2025.