Adıyaman University Journal of Science
https://dergipark.org.tr/en/pub/adyujsci
DergiPark AKADEMİK
ISSN 2147-1630
e-ISSN 2146-586X

On the Solutions of the Diophantine Equations $P_K = R_m R_n$ and $R_K = P_m P_n$

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Received: 26.03.2025 **Accepted:** 12.12.2025 **Published:** 31.12.2025

Abstract

We aim to find all possible solutions (k, m, n) of the Diophantine equations $P_K = R_m R_n$ and $R_K = P_m P_n$. Our proof employs the famous Matveev theorem and Dujella-Pethő reduction lemma.

Keywords: Pell number; Modified Pell number; Diophantine equation; Matveev theorem; Linear forms in logarithms; Dujella-Pethő lemma.

$P_K = R_m R_n$ ve $R_K = P_m P_n$ Diophantine Denklemlerinin Çözümleri Üzerine

Öz

Bu çalışma, $P_K = R_m R_n$ ve $R_K = P_m P_n$ Diophantine denklemleri için tüm olası (k, m, n) çözümlerini belirlemeyi amaçlamaktadır. Bu çözümleri araştırmak için ortaya koyduğumuz teoremin ispatında, Matveev teoremi ve Dujella-Pethő indirgeme lemmasından yararlanılmıştır.

Anahtar Kelimeler: Pell sayısı; Modifiye Pell sayısı; Diophantine denklemi; Matveev teoremi; Logaritmaların lineer formları; Dujella-Pethő lemması.

* Corresponding Author DOI: 10.37094/adyujsci.1666370

1. Introduction

Consider the famous Pell numbers, denoted by $\{P_n\}_{n\geq 0}$, which follows the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, with initial terms $P_0 = 0$ and $P_1 = 1$. Similarly, the Pell-Lucas numbers, represented by $\{Q_n\}_{n\geq 0}$, and the Modified Pell numbers, denoted by $\{R_n\}_{n\geq 0}$, adhere to the same recurrence relation but are distinguished by their initial conditions: $Q_0 = Q_1 = 2$ for Pell-Lucas numbers and $R_0 = R_1 = 1$ for Modified Pell numbers [1]. These sequences, while sharing a common recursive structure, exhibit unique properties due to their distinct starting values, making them fascinating subjects of study in number theory and combinatorics. It is well-known that they can be defined in other ways, which are called Binet's formulas, as follows [1]:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, Q_n = \gamma^n + \delta^n, \text{ and } R_n = \frac{\gamma^n + \delta^n}{\gamma + \delta} \tag{1}$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$. The sequences have beauty and rich applications in just about every branch of mathematics and statistics. To quote from Koshy's monograph [1], the Pell and Pell–Lucas numbers are among the most distinguished sequences in number theory. Herein, we will neglect further details as to their ubiquity. Instead, the current paper will provide all possible solutions to the following open problems:

$$P_K = R_m R_n \tag{2}$$

and

$$R_K = P_m P_n, \tag{3}$$

where k, m, and n are a non-negative integer satisfying that $m \leq n$. This restriction is necessary due to the commutative property of multiplication.

Considering the current literature, several Diophantine equations regarding the well-known Fibonacci, Pell, Jacobsthal, or Mersenne numbers have been analyzed. One can read the papers by Pongsriiam [2], Ddamulira et al. [3], Sahukar and Panda [4], Qu and Zeng [5], Chalebgwa and Ddamulira [6], Gómez et al. [7], and Faye et al. [8]. It should be noted that problems of finding all coincidences in two integer sequences or their variants have a special place in the open literature. To our knowledge, the investigations under consideration have been conducted for two sequences with two different characteristic equations. For example, the Fibonacci numbers vs. the Pell Numbers by Ddamulira et al. [9], the generalized $k-$Fibonacci numbers vs. the generalized $k-$Pell numbers by Bravo et al. [10]. Salim et al. explored the representation of integers by k-generalized Fibonacci sequences, providing important applications in number theory and cryptography [11], the Mersenne numbers vs. the Pell numbers by Alan and Alan [12], the Leonardo numbers vs. the Jacobsthal numbers by Bensella and Behloul [13], and the generalized k −Fibonacci numbers vs. the Padovan numbers by Rihane and Togbé [14]. Bellaouar et al. investigated Diophantine representations involving Padovan and Perrin numbers of the form $7t - 5z - 3y - 2x$, contributing new insights into the arithmetic structure and interrelations of these sequences [15].

The above review of the literature clearly indicates that Diophantine equations of the types proposed in Eqns. (2) and (3) have not been studied so far, and a mathematical approach to solving these two equations is still lacking. For this reason, in the present work, we aim to reduce these Diophantine equations into suitable logarithmic forms and, by applying Matveev's theorem [18], establish explicit upper bounds. Subsequently, using the Dujella–Pethő lemma, we refine these bounds to obtain more effective limits and, with the aid of the PC algorithm, investigate all possible solutions.

2. Auxiliary Results

Here, we establish a general lower bound for linear forms in logarithms based on Matveev's result and notations [16].

Consider $\eta_1, \eta_2, \dots, \eta_l$ as nonzero elements in a real number field F with degree D, and $b_1, b_2, \dots, b_l$ be integer values. Define

$B := max\{|b_1|, |b_2|, \dots, |b_l|\}$ and $\Lambda := {\eta_1}^{b_1}{\eta_2}^{b_2} \dots {\eta_l}^{b_l}$.

Consider η as a nonzero element in F such that its degree l is a divisor of D. Suppose that the minimal primitive polynomial of η in $\mathbb{Z}[x]$ is given by $P(x) = \sum_{0 \le j \le l} t_j x^j$, where $t_l \neq 0$. The logarithmic height $h(\eta)$ of η is defined as follows:

$$h(\eta) = l^{-1} \log|t_0| + l^{-1} \sum_{s=1}^{l} \log\left(max\left\{\left|\eta^{(s)}\right|, 1\right\}\right)$$

where $\eta^{(s)}, 1 \le s \le l$, are the conjugates of η.

Let $A_1, A_2, \dots, A_l$ be real numbers satisfying the condition

$A_r \geq max\{Dh(\eta_r), |\log \eta_r|, 0.16\}$

for each $r = 1,2,\dots,l$. Utilizing the previous notations, Matveev's main theorem [16] leads to the following bound.

Theorem 1 (Matveev Theorem [16]). Suppose that $\Lambda \neq 0$, and F is a real algebraic number field. Then,

$$\log(|\Lambda|) > -1.4 \times 30^{l+3} \times l^{4.5} \times D^2 \times (1 + \log D) \times (1 + \log B) \times A_1 \times \dots \times A_l.$$

In order to obtain tighter lower bounds using Theorem 1, we rely on certain results from the theory of continued fractions. The next lemma is a modified form of a theorem originally established by Dujella and Pethő [18]. Here, we adopt the version presented by Bravo et al. (see [17], Lemma 1).

Lemma 1 (Dujella-Pethő Lemma [18]). Let τ be an irrational number, and let X, Y and μ be real numbers with $X > 0$ and $Y > 1$. Suppose $M \in \mathbb{Z}^+, p/q$ and is a convergent of the continued fraction expansion of τ, where $q > 6M$. Define $\epsilon \coloneqq \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ represents the distance to the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality

$$0 < |k\tau - n + \mu| < XY^{-\omega}$$

in positive integers k, n and ω such that

$$k \leq M \text{ and } \omega \geq \frac{\log(Xq/\epsilon)}{\log Y}.$$

3. Main Results

The following theorem presents the primary outcomes of the paper.

Theorem 2. Let $k, m,$ and n be a non-negative integer. There is no solution for both Eqns. (2) and (3) except for the ordered triple of $(k, m, n) = (1,1,1)$.

Proof. For $m = 1$, our problem is reduced to $P_k = P_n$, which means that

$$2P_k = P_{n-1} + P_{n+1} = 2P_n + 2P_{n-1} < 4P_n \Rightarrow P_k < 2P_n < P_{n+1}.$$

This means that there are no solutions under consideration since k must be greater than $n + 1$.

Now, assume that $m \geq 2$. So, we conclude that $k > 2n$. Let us start with Eqn. (2). By utilizing Binet's formulas given in Eqn. (1), we can prove the following inequalities:

$$\gamma^{n-2} \leq P_n \leq \gamma^{n-1}, |\delta|^{-n+2} \leq P_n \leq |\delta|^{-n+1}, \gamma^{n-1} \leq R_n \leq \gamma^n, \text{and } |\delta|^{-n+1} \leq R_n \leq |\delta|^{-n}. \quad (4)$$

From Eqns. (2) and (4), we can write

$$\gamma^{k-2} \leq P_k = R_n R_m \leq |\delta|^{-n-m} \quad (5)$$

and

$$(k-2)\log\gamma \leq -(n+m)\log|\delta| \Rightarrow k \leq 2-(n+m)\frac{\log|\delta|}{\log\gamma} \leq 2n+2 \Rightarrow k \leq 4n.$$

By Binet's formula for Pell and Modified Pell numbers, we can write

$$\frac{\gamma^k-\delta^k}{2\sqrt{2}} = \left(\frac{\gamma^m+\delta^m}{2}\right)\left(\frac{\gamma^n+\delta^n}{2}\right) = \frac{\gamma^n\gamma^m+\delta^n\gamma^m+\gamma^n\delta^m+\delta^n\delta^m}{4},$$

which implies that

$$\gamma^k - 2|\delta|^{n+m} = \frac{1}{\sqrt{2}}\gamma^{n+m} + \frac{1}{\sqrt{2}}\delta^n\gamma^m + \frac{1}{\sqrt{2}}\gamma^n\delta^m + \frac{1}{\sqrt{2}}\delta^{n+m} - 2|\delta|^{n+m} + \delta^k .$$

Dividing the last equation by γ^k with mathematical simplifications, we get

$$\left|\gamma^{-k}|\delta|^{n+m}2-1\right| \leq \left|\frac{4+\sqrt{2}}{2\gamma^k} + \frac{3\sqrt{2}}{2\gamma^{k-2n}}\right|.$$

Since $k > 2n$ and $k > m$, we have

$$\left|\gamma^{-k}|\delta|^{n+m}2-1\right| \leq \left|\frac{4+\sqrt{2}}{2\gamma^k} + \frac{3\sqrt{2}}{2\gamma^{n}}\right| \leq \left|\frac{4+\sqrt{2}}{2\gamma^{2n}} + \frac{3\sqrt{2}}{2\gamma^{n}}\right| \leq \left|\frac{2+2\sqrt{2}}{\gamma^n}\right| < \frac{5}{\gamma^m}.$$

As a result, we obtain

$$|\Lambda_1| < \frac{5}{\gamma^m}, \Lambda_1 := \gamma^{-k}|\delta|^{n+m}2-1. \quad (6)$$

In this case, we can write

$$\log(|\Lambda_1|) < \log 5 - m\log\gamma. \quad (7)$$

According to the Matveev theorem, we can consider

$$l = 3, \eta_1 = \gamma, \eta_2 = |\delta|, \eta_3 = 2, b_1 = -k, b_2 = n + m, \text{and } b_3 = 1.$$

Due to $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{2}), F = \mathbb{Q}(\sqrt{2})$ of degree $D = 2$. Assume that $\Lambda_1 = 0$. So, we have $2 = \gamma^k |-\gamma^{-1}|^{-(n+m)} = \gamma^{k+n+m}$, which is absurd. As a result, $\Lambda_1 \neq 0$.

$$h(\eta_1) = \frac{1}{2}(\log 1 + \log(max\{|\gamma|, 1\}) + \log(max\{|\delta|, 1\})) = \frac{1}{2}(2\log 1 + \log\gamma) = \frac{1}{2}\log\gamma,$$

$$h(\eta_2) = \frac{1}{2}(\log 1 + \log(max\{|\delta|, 1\}) + \log(max\{|\gamma|, 1\})) = \frac{1}{2}(2\log 1 + \log\gamma) = \frac{1}{2}\log\gamma,$$

$$h(\eta_3) = \frac{1}{2}(\log 1 + \log(max\{|2|, 1\}) + \log(max\{|-2|, 1\})) = \frac{1}{2}(2\log 2) = \log 2,$$

which gives that

$$A_1 = A_2 \geq max\left\{2 \cdot \frac{1}{2}\log\gamma, |\log\gamma|, 0.16\right\} \Rightarrow A_1 = A_2 = \log\gamma,$$

$$A_3 \geq max\{2\log 2, |\log\sqrt{2}|, 0.16\} \Rightarrow A_3 = 2\log 2,$$

$$m < n \Rightarrow m + n < 2n < 4n, k < 4n \Rightarrow B = 4n \geq max\{n + m, |-k|, 1\}.$$

Further, $max\{|-k|, n + m\} \leq B = 4n$. As a result, from the Matveev theorem, we have

$$\log(|\Lambda_1|) > -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1 + \log 2)(1 + \log 4n) \times (\log\gamma)^2 \times 2\log 2,$$

or, more slightly

$$\log(|\Lambda_1|) > -1.05 \times 10^{12} \times (1 + \log 4n). \tag{8}$$

By comparing Eqn. (7) with Eqn. (8), we derive

$$m < 1.21 \times 10^{12} \times (1 + \log 4n). \tag{9}$$

Considering Eqn. (2) once again, we can write

$$|\Lambda_2| < \frac{6}{\gamma^n}, \Lambda_2 := \gamma^{-k}|\delta|^n(\sqrt{2}R_m) - 1. \tag{10}$$

Similarly, repeating the above approach to Eqn. (10), we have

$$l = 3, \eta_1 = \gamma, \eta_2 = |\delta|, \eta_3 = \sqrt{2}R_m, b_1 = -k, b_2 = n, b_3 = 1, F = \mathbb{Q}(\sqrt{2}), D = 2,$$

$$h(\eta_1) = \frac{1}{2}\log\gamma, h(\eta_2) = \frac{1}{2}\log\gamma, h(\eta_3) = \log(\sqrt{2}R_m), A_1 = \log\gamma, A_2 = \log\gamma,$$

$$A_3 = 4m\log\gamma, \text{and } B = 4n.$$

Thus, by Theorem 1, we obtain

$$\log(|\Lambda_2|) > -2.66 \times 10^{12} \times (1 + \log 4n) \times m.$$

From the right-hand side of Eqn. (10), we have

$$\log(|\Lambda_2|) < \log 6 - n\log\gamma.$$

Considering these two inequalities together, we obtain

$$n < 3.03 \times 10^{12} \times (1 + \log 4n) \times m. \quad (11)$$

In this case, we conclude from Eqns. (9) and (11) that

$$n < 1.67 \times 10^{28}. \quad (12)$$

This time, we shall consider Eqn. (3) without details. So, we can obtain the following statements similarly:

$$|\Lambda_3| < \frac{6}{\gamma^m}, \Lambda_3 := \gamma^{-k}|\delta|^{n+m}\sqrt{2} - 1;\ |\Lambda_4| < \frac{3}{\gamma^n}, \Lambda_4 := \gamma^{-k}|\delta|^n R_m - 1; n < 3.96 \times 10^{27}. \quad (13)$$

According to the above, we have the following inference.

Corollary 1. All the possible solutions of both Eqns. (2) and (3) are in the range $k < 4n$ and $0 \leq m \leq n < 1.67 \times 10^{28}$.

As a result, we have proved that there are a finite number of solutions to our problems. However, since the boundaries are quite large, examining the solution will take a long time. So now, we will focus on reducing the upper bounds using the Dujella-Pethő lemma, which is Lemma 1. For this purpose, introduce

$$\Gamma_1 \coloneqq -k\log\gamma + (n+m)\log|\delta| + \log 2.$$

According to Eqn. (2), we can write

$$\Lambda_1 = |exp(\Gamma_1) - 1| < \frac{5}{\gamma^m}.$$

It is known from Ddamulira et al. [9, p.16] that $|x| < 2|e^x - 1|$ whenever $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. From this, we get

$$\Gamma_1 \coloneqq \left|k\frac{\log\gamma}{\log|\delta|} - (n+m) + \frac{\log\frac{1}{2}}{\log|\delta|}\right| < \frac{10}{\gamma^m|-\log|\delta||} < \frac{12}{\gamma^m}.$$

To be able to apply the Dujella-Pethő lemma, let us consider

$$\tau \coloneqq \frac{\log\gamma}{\log|\delta|}, \mu \coloneqq \frac{\log\frac{1}{2}}{\log|\delta|}, X = 12, Y = \gamma, w = m, \text{ and } M = 6.69 \times 10^{28} (M > 4n > k).$$

To be clear, all the criteria of the Dujella-Pethő lemma are provided in the 63rd convergent of the continued fraction expansion of τ, namely,

$$\frac{p_{63}}{q_{63}} = \frac{38328591297114539835198029343 3}{14189421881349479414793903193 49}, \qquad 6M < q_{63}, \text{and}$$

$$\epsilon \coloneqq \|\mu q_{63}\| - M\|\tau q_{63}\| = 0.0075 > 0.$$

As a result, we obtain that $m \leq 87$.

For the case $m \leq 87$, let us consider

$$\Gamma_2 \coloneqq -k\log\gamma + n\log|\delta| + \log(\sqrt{2}R_m).$$

Then, Eqn. (10) can be reorganized as follows:

$$\Lambda_2 = |exp(\Gamma_2) - 1| < \frac{4}{\gamma^n}.$$

From this, we can write

$$\Gamma_2 = \left|k\frac{\log\gamma}{\log|\delta|} - n + \frac{\log(\sqrt{2}R_m)}{\log|\delta|}\right| < \left|\frac{4}{\gamma^n(-\log|\delta|)}\right| < \frac{2}{\gamma^n}.$$

So, we can consider the following in the Dujella-Pethő lemma:

$$\tau := \frac{\log\gamma}{\log|\delta|}, \mu := \frac{\log(\sqrt{2}R_m)}{\log|\delta|}, X = 2, Y = \gamma, w = n, \text{and}$$

$$M := 6.69 \times 10^{28}\ (M > 4n > k).$$

Similarly, we get

$$\frac{p_{66}}{q_{66}} = \frac{27961057263325020868222902 16885}{1035131280151582676595599 0569711}, \qquad 6M < q_{66}, \text{and}$$

$$\epsilon := \|\mu q_{66}\| - M\|\tau q_{66}\| = 0.0019 > 0.$$

From this, we conclude that $m \leq 87$ and $n \leq 88$.

Repeating the same procedure above with those in Eqn. (13), we conclude that $m \leq 83$ and $n \leq 86$. Summing up, all the solutions in question are in the range $m \leq 87$, $n \leq 88$, and $k < 352$. Implementing a looping in Wolfram Mathematica© proves our results.

Further, the results from Theorem 2 show that the Pell and the Modified Pell numbers have only one overlapping term, i.e., $P_1 = R_1 = 1$. At the same time, it is also the only term that can be written as the square of the Pell numbers or Modified Pell numbers, i.e., $P_1 = 1 = R_1^2$ and $R_1 = 1 = P_1^2$.

4. Conclusion

In this study, we investigated the Diophantine equations $P_K = R_m R_n$ and $R_K = P_m P_n$ where P_n denotes the Pell numbers and R_n represents the modified Pell numbers, aiming to determine all possible integer solutions (k, m, n). Utilizing Matveev's theorem and the Dujella-Pethő lemma, we established a complete characterization of the solutions. Our results provide a deeper understanding of the structural properties of Pell and modified Pell numbers in the context of multiplicative relations.

A natural direction for future research is to explore whether similar techniques can be applied to other classes of recurrence sequences, such as balancing or associated Pell numbers. Additionally, studying variants of these equations under different arithmetic constraints may lead to further insights into the interplay between linear forms in logarithms and integer sequences.

Acknowledgements

The authors wish to express profound gratitude to the anonymous reviewers for their insightful critiques and thoughtful recommendations. Their discerning observations and expert counsel have been instrumental in refining the rigor, clarity, and scholarly depth of this work.

References

[1] Koshy, T., *Pell and Pell-Lucas numbers with applications*, Springer New York, 431p, 2014.

[2] Pongsriiam, P., *Fibonacci and Lucas numbers which are one away from their products*, Fibonacci Quarterly, 55(1), 29-40, 2017.

[3] Ddamulira, M., Gómez, C.A., Luca, F., *On a problem of Pillai with* $k-$ *generalized Fibonacci numbers and powers of 2*, Monatshefte für Mathematik, 187(4), 635–664, 2018.

[4] Sahukar, M.K., Panda, G.K., *Diophantine equations with balancing-like sequences associated to Brocard-Ramanujan-type problem*, Glasnik Matematički, 54(2), 255–270, 2019.

[5] Qu, Y., Zeng, J., *Lucas numbers which are concatenations of two repdigits*, Mathematics, 8(8), 1360, 2020.

[6] Chalebgwa, T.P., Ddamulira, M., *Padovan numbers which are palindromic concatenations of two distinct repdigits*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 115(3), 108, 2021.

[7] Gómez, C.A., Gómez, J.C., Luca, F., *On the Exponential Diophantine Equation* $F_{n+1}^x - F_{n-1}^x = F_m^y$, Taiwanese Journal of Mathematics, 26(4), 685–712, 2022.

[8] Faye, B., Luca, F., Rihane, S.E, Togbé, A., *The Diophantine equations* $P_n^x + P_{n+1}^y = P_m^x$ *or* $P_n^y + P_{n+1}^x = P_m^x$, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 117(2), 89, 2023.

[9] Ddamulira, M., Luca, F., Rakotomalala, M., *Fibonacci Numbers which are products of two Pell Numbers*, Fibonacci Quarterly, 54(1), 11–18, 2016.

[10] Bravo, J.J., Herrera, J.L., Luca, F., *Common values of generalized Fibonacci and Pell sequences*, Journal of Number Theory, 226, 51–71, 2021.

[11] Badidja, S., Mokhtar, A.A., Özer, Ö., *Representation of integers by k-generalized Fibonacci sequences and applications in cryptography*, Asian-European Journal of Mathematics, 14, 9, 2150157, 2021.

[12] Alan, M., Alan, K.S., *Mersenne numbers which are products of two Pell numbers*, Boletín de la Sociedad Matemática Mexicana, 28(2), 38, 2022.

[13] Bensella, H., Behloul, D., *Common terms of Leonardo and Jacobsthal numbers*, Rendiconti del Circolo Matematico di Palermo Series II, 73(1), 259–265, 2024.

[14] Rihane, S.E., Togbé, A., k*-Fibonacci numbers which are Padovan or Perrin numbers*, Indian Journal of Pure and Applied Mathematics, 54(2), 568–582, 2023.

[15] Bellaouar, D., Özer, Ö., Azzouza, N., *Padovan and Perrin numbers of the form* $7^t - 5^z - 3^y - 2^x$, Notes on Number Theory and Discrete Mathematics, 31(1), 191–200, 2025.

[16] Matveev, E.M., *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II*, Izvestiya: Mathematics, 64(6), 1217–1269, 2000.

[17] Bravo, J.J., Gomez, C.A., Luca, F., *Powers of two as sums of two k–Fibonacci numbers,* Miskolc Mathematical Notes, 17(1), 85–100, 2016.

[18] Dujella, A., Pethő, A., *A generalization of a theorem of Baker and Davenport,* Quarterly Journal of Mathematics, 49(195), 291–306, 1998.