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Evolution of complex waves via Davey-Stewartson equation

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Abstract

Davey-Stewartson equations (DSEs) have been discussed to examine the features of wave motion in finite depth water which is affected by gravitational force and surface tension. The new versions of the generalized F-expansion method and the generalized -expansion method are suggested to evaluate the analytical solutions of the DSEs. Thus, single, combined and mixed non-degenerative Jacobi elliptic function solutions (JEFSs) and degenerative solutions of the DSEs are obtained to contribute to the literature. These wave solutions are fresh and unexplored. The visual representations of some solutions are also shown in three and two dimensions.

Keywords: New version of the generalized F-expansion method; New version of the generalized $\frac{F'}{F}$-expansion method; Davey-Stewartson equations; Single combined and mixed JEFSs

1. INTRODUCTION

Nonlinear evolution equations (NLEEs) involving time derivative are more common in applied sciences. The progress of the methods to discover the exact solutions of this kind of equations has enhanced the significance of nonlinear models. Sometimes these methods offer similar solutions, and sometimes they can come up with different solutions. These solutions have important benefits in understanding the physical event corresponding to differential equations. The applicability of the solutions of NLEEs can be delineated via wave notion serving to perceive various physical phenomena. Particularly, presenting the motion of a wave by a mathematical function provides us comprehension of physical issues. For example, these wave movements are manifested in a certain order in flexible environments, fluid spaces, optical transmission environments. Therefore, many researches in the field of applied sciences have been carried out to investigate the exact solutions of the NLEEs.

Solitons are one of the vital solution types of the NLEEs having extensive applications in nonlinear optics, biology, fluid mechanics and etc. Thus, many techniques have been proposed and they have been developed over time to discover much more different wave structures for NLEEs. Some of them can be given as, inverse expansion method (Ablowitz 1991), Darboux transformation method (Matveev 1991), sine-cosine technique (Chuntao 1996), tanh function method (Fan 2000), first integral method (Feng 2002), Exp- function method (He 2006), trial equation method (Liu 2005, Liu 2006, Liu 2010, Gurefe 2011, Gurefe 2012), extended trial equation method (Pandir 2012, Pandir 2013, Pandir 2014, Gurefe 2014), Jacobian elliptical function method (Fu 2001, Shen 2003, Chen 2006), Weierstrass elliptical function expansion method (Chen 2006), F-expansion method (Wang 2005, Yang 2004, Abdou 2008, Zhang 2008, Cai 2006, Zhang 2006, Zhang 2007, Zhang 2008) and $\frac{G'}{G}$-expansion method (Wang 2007, Guo 2010, Lu 2010).

Due to the importance of the JEFSs of NLEEs, the Jacobi elliptic F-expansion method (Liu 2001, Yong 2004, Qi 2005, Chunhuan 2011), improved Jacobi elliptic F-expansion method (Yafeng 2012, Alofi 2012, Baojian 2009), generalized Jacobi elliptic F- expansion method (Enam 2010, Inc 2005), extended Jacobi elliptical F- Expansion method (Yafeng 2012) has been developed.

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In the present paper, the new versions of the generalized F-expansion method and the generalized $\frac{F'}{F}$-expansion method have been considered which allow us to reveal combined and mixed JEFSs together. The solutions obtained are fresh in this sense.

The DSEs,

$$\begin{aligned} &i\phi_t + \phi_{xx} - \lambda\phi_{yy} + 2p\left(\varphi + |\phi|^2\right)\phi = 0, \\ &\varphi_{xx} + q\varphi_{yy} + 2\left(|\phi|^2\right)_{xx} = 0, \end{aligned} \tag{1}$$

where λ, p and q are constants have been given in (Liu 2014, Tajiri 2010, Malanyuk 1994, MAlanyuk 1991). In Eq. (1) ϕ and φ correspond to the complex amplitude and the real velocity potential of the wave, respectively. Davey and Stewartson (1974) have propound the DSEs to depict the progression of perturbation in the nonlinear regime of plane Poiseuille flow.

To achieve our objective this research is arranged as follows: In Section 2, the general structure of the new versions of the proposed techniques have been given. In Section 3 and Section 4 the aforementioned analytical methods are applied to DSEs. The results of this search are summed up in Section 5.

2. METHODOLOGY

2.1. General structure of the new version of generalized F-expansion method

In this section, a new version of the generalized F-expansion method is introduced. This proposed method presents more different results than the results obtained with the F-expansion method.

For a given NLEE,

$$\tilde{P}\left(\psi, \psi_x, \psi_y, \psi_z, \ldots, \psi_t, \psi_{xx}, \psi_{xy}, \psi_{xt}, \ldots\right) = 0 \tag{2}$$

where $\psi\left(x, y, z, \ldots, t\right)$ is an unknown function, $x, y, z, \ldots, t$ are independent variables and $\tilde{P}$ is a polynomial of u and its partial derivatives, in which the highest order derivatives and the nonlinear terms are inclusived. When we apply the following traveling wave transformation to Eq. (2)

$$\psi\left(x, y, z, \ldots, t\right) = \psi\left(\xi\right), \qquad \xi = j_1 x + j_2 y + j_3 z + \ldots + j_m t, \tag{3}$$

where $j_h (h = 1, 2, 3, \ldots, m)$ are constants to be determined, we can reduce Eq. (2) into a nonlinear ordinary differential equation (NLODE),

$$O\left(\psi, \psi', \psi'', \psi''', \ldots\right) = 0 \tag{4}$$

The prime denotes the derivation with respect to ξ. Supposing that a solution of Eq. (4) is as,

$$\psi(\xi) = k_0 + \sum_{i=1}^{M}\left(k_i F^i + \frac{l_i}{F^i} + m_i\left(\frac{F'}{F}\right)^i + n_i\left(\frac{F}{F'}\right)^i\right) \tag{5}$$

where $F = F\left(\xi\right)$, $F' = F'\left(\xi\right)$. Also M is a natural number and $k_0, k_i, l_i, m_i, n_i (i = 1, 2, 3, \cdots, M)$ are the coefficients to be evaluated. However, the functions $F\left(\xi\right)$ and $F'\left(\xi\right)$ in Eq. (5) fulfil Eq. (6),

$$\left(F'\right)^2\left(\xi\right) = PF^4\left(\xi\right) + QF^2\left(\xi\right) + R \tag{6}$$

and accordingly, it can be deduced for $F\left(\xi\right)$ and $F'\left(\xi\right)$,

$$\begin{cases} F''(\xi)=2PF^{3}(\xi)+QF(\xi) \\ F'''(\xi)=\left(6PF^{2}(\xi)+Q\right)F'(\xi) \\ F^{(4)}(\xi)=24P^{2}F^{5}(\xi)+20PQF^{3}(\xi)+\left(Q^{2}+12PR\right)F(\xi) \\ F^{(5)}(\xi)=\left(120P^{2}F^{4}(\xi)+60PQF^{2}(\xi)+Q^{2}+12PR\right)F'(\xi) \\ \cdots \end{cases} \tag{7}$$

where *P, Q* and *R* are all parameters. Also, getting support from the derivatives in Eq. (7) the value N is determined by the balancing procedure which depends on the highest order derivative and highest power nonlinear terms in Eq. (4). After ascertaining N, Eq. (5) is embed in Eq. (4) by taking account Eq. (6). This leads to a polynomial of $\left(F'\right)^{r}F^{p}$ $\left(r=0,1;p=0,\pm1,\pm2,...\right)$. The related coefficients in this polynomial are matched to zero to reveal an algebraic system of equation. Solving this system, the parameters $j_h(h=1,2,3,...,m)$ and $k_0,k_i,\, l_i,\, m_i,\, n_i(i=1,2,3,\cdots,M)$ are determined which are necessary for exposing the solution. Thus, the new mixed and combined JEFSs are obtained. Considering distinct values of *P, Q* and *R*, the different JEFSs $F(\xi)$ can be attained from Eq. (6).

2.2. General structure of the new version of generalized F-expansion method

In this section, it is aimed to obtain different JEFSs with the newly developed method. Similarly, when Eq. (3) is applied to Eq. (2), a NLODE is obtained as stated in Eq. (4). When the term $\dfrac{F'(\xi)}{F(\xi)}$ is written instead of $F(\xi)$ in the solution function in Eq. (5), a different new solution can be obtained as follows,

$$\kappa(\xi)=k_0+\sum_{i=1}^{M}\left(k_i\left(\frac{F'}{F}\right)^{i}+l_i\left(\frac{F}{F'}\right)^{i}+m_i\left(\left(\frac{F'}{F}\right)'\frac{F}{F'}\right)^{i}+n_i\left(\left(\left(\frac{F'}{F}\right)'\right)^{-1}\frac{F}{F'}\right)^{i}\right) \tag{8}$$

where the $F(\xi)$ is the JEF solving Eq. (6). M can be evaluated from Eq. (4) via balancing the related terms. The related derivatives are obtained with the help of Eq. (7) as follows,

$$\left(\left(\frac{F'}{F}\right)'\right)^{2}=\left(\frac{F'}{F}\right)^{4}-2Q\left(\frac{F'}{F}\right)^{2}-4PR+Q^{2} \tag{9}$$

$$\begin{gathered} \left(\frac{F'}{F}\right)''=2\left(\frac{F'}{F}\right)^{3}-2Q\left(\frac{F'}{F}\right) \\ \left(\frac{F'}{F}\right)'''=6\left(\frac{F'}{F}\right)^{2}\left(\frac{F'}{F}\right)'-2Q\left(\frac{F'}{F}\right)' \\ \cdots \end{gathered} \tag{10}$$

These calculated derivatives are written down in the new solution function in Eq. (8), then a zero polynomial based on the functions $\left(\left(\frac{F'}{F}\right)'\right)^{r}\left(\frac{F'}{F}\right)^{p}$ $\left(r=0,1;p=0,\pm1,\pm2,...\right)$ is constructed. Considering that the coefficients of this polynomial vanish, the algebraic system of equations is attained. When this system is unfastened with the help of Mathematica package, the values $k_0, k_i, l_i, m_i, n_i (i = 1,2,3,...,M)$ and $j_h (j = 1,2,3,\cdots,m)$ are evaluated which are essential for the solution of Eq. (2). These coefficients are replaced by the solution function in Eq. (8) and as aresult fresh, diverse solution forms involving the JEFs can be obtained. Thus, the new mixed and combined JEFSs are occured. According to the diverse values of *P, Q* and *R*, the various JEFSs $F(\xi)$ can be achieved from Eq. (6).

3. Implementation of the new version of generalized F-expansion method to DSEs

In this section, the new version generalized F-Expansion method is applied to the DSEs. When is selected, Eq. (1) can be written as in Eq. (11). The proposed algorithm is employed for the DSEs [42-45],

$$\begin{aligned} & i\phi_t + \phi_{xx} + \phi_{yy} + 2p\left(\varphi + |\phi|^2\right)\phi = 0, \\ & \varphi_{xx} + q\varphi_{yy} + 2\left(|\phi|^2\right)_{xx} = 0. \end{aligned} \tag{11}$$

Generally, the DSE system describes the propagation of weak nonlinear waves propagating in one direction but whose amplitudes occur in two space environments. We introduce traveling wave transformation for this complex equation system

$$\phi(x,y,t) = e^{i\theta}\phi(\omega),\ \varphi(x,y,t) = \varphi(\omega),\ \omega = j_1 x + j_2 y + j_3 t,\ \theta = \vartheta(x + y - \gamma t), \tag{12}$$

where j_1, j_2, j_3, ϑ and γ are constants. By using Eq. (12), Eq. (11) is reduced into NLODEs as follows,

$$\left(\vartheta\gamma - 2\vartheta^2\right)\phi + 2p\left(\phi\varphi + \phi^3\right) + \left(j_1^{\,2} + j_2^{\,2}\right)\phi'' + i\left(j_3 + 2j_1\vartheta + 2j_2\vartheta\right)\phi' = 0, \tag{13}$$

$$\left(j_1^{\,2} + qj_2^{\,2}\right)\varphi'' + 2j_1^{\,2}\left(\phi^2\right)'' = 0. \tag{14}$$

From Eq. (13) $j_3 = -2\vartheta(j_1 + j_2)$ is obtained using the complex numbers equality. Thus, Eq. (13) is rewritten as follows

$$\left(j_1^{\,2} + j_2^{\,2}\right)\phi'' + 2p\left(\phi\varphi + \phi^3\right) + \left(\vartheta\gamma - 2\vartheta^2\right)\phi = 0. \tag{15}$$

The two sides of the Eq. (14) are integrated twice with respect to ω and when the integral constant is chosen to be zero, we get following equation

$$\varphi = -\frac{2j_1^{\,2}\phi^2}{j_1^{\,2} + qj_2^{\,2}}. \tag{16}$$

When the last equation is replaced by Eq. (15), the following NLODE

$$\left(j_1^{\,2} + j_2^{\,2}\right)\phi'' + \left(\vartheta\gamma - 2\vartheta^2\right)\phi + \frac{\left(qj_2^{\,2} - j_1^{\,2}\right)2p}{j_1^{\,2} + qj_2^{\,2}}\phi^3 = 0, \tag{17}$$

is obtained. Balancing ϕ'' and ϕ^3 in Eq. (17), the solution of Eq. (17) is in the form of:

$$\phi(\omega) = k_0 + k_1 F(\omega) + \frac{l_1}{F(\omega)} + m_1\left(\frac{F'(\omega)}{F(\omega)}\right) + n_1\left(\frac{F(\omega)}{F'(\omega)}\right). \tag{18}$$

Obtaining the terms ϕ'' and ϕ^3 are from Eq. (18) and substituting them into Eq. (17), gives a zero polynomial depending on the eliptic functions of $F(\omega)$ and $F'(\omega)$. Solving the algebraic equation system, obtained by equating the coefficients of this polynomial to zero, with the help of the Mathematica package program the values $k_0, k_1, l_1, m_1, n_1, \phi$ and e_3 are found. These coefficients and inverse transformations are replaced by the solution function in Eq. (18) and the following solutions are obtained depending on the elliptic functions of $F(\omega)$ and $F'(\omega)$. If we postulate $F(\omega)=sn(\omega)$ from Eq. (6) we have $P=m^2,\ Q=-\left(1+m^2\right),\ R=1$ following situations:

Case (1):

$$k_1=\sqrt{\frac{P\left(j_1^2+qj_2^2\right)\left(j_1^2+j_2^2\right)}{p\left(j_1^2-qj_2^2\right)}},\ k_0=l_1=n_1=m_1=0,$$

$$\vartheta=\frac{\gamma-\sqrt{\gamma^2+8Q\left(j_1^2+j_2^2\right)}}{4},\ j_3=\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2+8Q\left(j_1^2+j_2^2\right)}-\gamma\right). \tag{19}$$

When we use Eq. (19) in Eq. (18), new types of exact solutions of Eq. (11) are evaluated as follows:

$$\begin{aligned}\phi(\omega_1)&=B_1\,m\,e^{i\theta_1}sn(\omega_1),\\ \varphi(\omega_1)&=B_2\,m^2sn^2(\omega_1),\end{aligned} \tag{20}$$

where $\omega_1=j_1x+j_2y+\dfrac{j_1+j_2}{2}\left(\sqrt{\gamma^2-8\left(1+m^2\right)\left(j_1^2+j_2^2\right)}-\gamma\right)t,\ B_1=\sqrt{\dfrac{\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},$

$\theta_1=-\dfrac{\sqrt{\gamma^2-8\left(1+m^2\right)\left(j_1^2+j_2^2\right)}-\gamma}{4}(x+y-\gamma t),\ B_2=-\dfrac{2j_1^2\left(j_1^2+j_2^2\right)}{p\left(j_1^2-qj_2^2\right)}.$

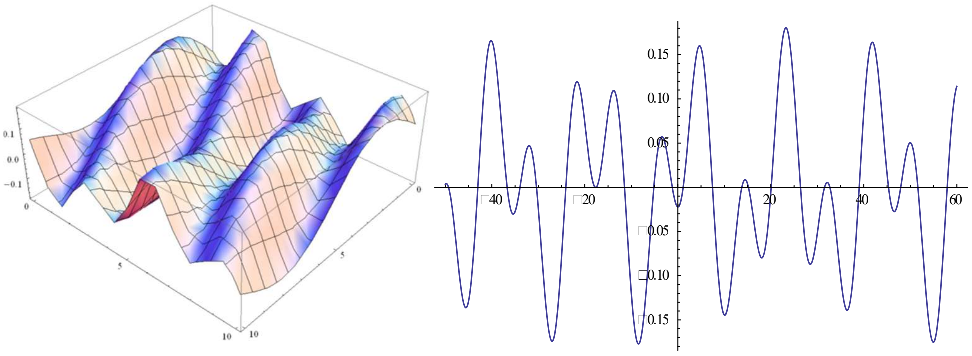

Figure 1: Graph of the solution $\phi(\omega_1)$ corresponding to the values $j_1=1/2,\ j_2=q=1,\ \gamma=7,\ p=-1,\ y=0,\ m=1/8$

Case (2):

$$l_1=-\sqrt{\frac{R\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},\ k_0=k_1=n_1=m_1=0,$$

$$\vartheta=\frac{\gamma+\sqrt{\gamma^2+8Q\left(j_1^2+j_2^2\right)}}{4},\ j_3=-\frac{j_1+j_2}{2}\left(\gamma+\sqrt{\gamma^2+8Q\left(j_1^2+j_2^2\right)}\right). \tag{21}$$

If we substitute Eq. (21) into Eq. (18), we get the JEFSs of Eq. (11)

$$\begin{aligned}\phi\left(\omega_2\right) &= -B_1\, e^{i\theta_2} ns\left(\omega_2\right),\\ \varphi\left(\omega_2\right) &= B_2\, ns^2\left(\omega_2\right),\end{aligned} \tag{22}$$

where $\omega_2 = j_1x + j_2y - \frac{j_1+j_2}{2}\left(\gamma + \sqrt{\gamma^2 - 8\left(1+m^2\right)\left(j_1^2+j_2^2\right)}\right)t,\ \theta_2 = \frac{\gamma + \sqrt{\gamma^2 - 8\left(1+m^2\right)\left(j_1^2+j_2^2\right)}}{4}\left(x+y-\gamma t\right).$

Case (3):

$$m_1 = -\sqrt{\frac{\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2 - pqj_2^2}},\ k_0 = k_1 = l_1 = n_1 = 0,$$

$$\vartheta = \frac{\gamma - \sqrt{\gamma^2 - 16Q\left(j_1^2+j_2^2\right)}}{4},\ j_3 = \frac{j_1+j_2}{2}\left(\sqrt{\gamma^2 - 16Q\left(j_1^2+j_2^2\right)} - \gamma\right). \tag{23}$$

When we put Eq. (23) into Eq. (18), we gain the following combined JEFSs

$$\begin{aligned}\phi\left(\omega_3\right) &= -B_1\, e^{i\theta_3} cs\left(\omega_3\right) dn\left(\omega_3\right),\\ \varphi\left(\omega_3\right) &= B_2\, cs^2\left(\omega_3\right) dn^2\left(\omega_3\right),\end{aligned} \tag{24}$$

where $\omega_3 = j_1x + j_2y - \frac{j_1+j_2}{2}\left(\gamma - \sqrt{\gamma^2 + 16\left(j_1^2+j_2^2\right)\left(1+m^2\right)}\right)t,\ \theta_3 = \frac{\sqrt{\gamma^2 + 16\left(1+m^2\right)\left(j_1^2+j_2^2\right)} - \gamma}{4}\left(x+y-\gamma t\right).$

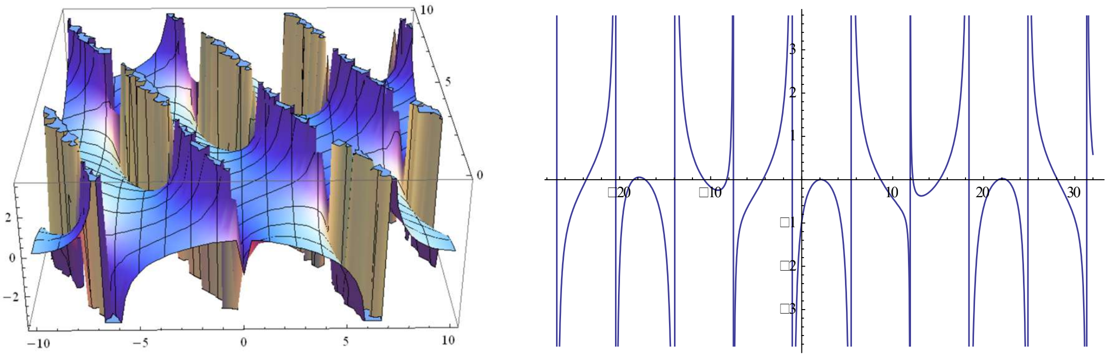

Figure 2: Graph of the solution $\phi\left(\omega_3\right)$ corresponding to the values $j_1 = 1/2,\ j_2 = m = 1/3,\ \gamma = 2,\ p = -1,\ q = 1,\ y = 0.$

Case (4):

$$n_1 = \sqrt{\frac{\left(Q^2 - 4PR\right)\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2 - pqj_2^2}},\ k_1 = l_1 = k_0 = m_1 = 0,$$

$$\vartheta = \frac{\gamma + \sqrt{\gamma^2 - 16Q\left(j_1^2+j_2^2\right)}}{4},\ j_3 = -\frac{j_1+j_2}{2}\left(\gamma + \sqrt{\gamma^2 - 16Q\left(j_1^2+j_2^2\right)}\right).. \tag{25}$$

When we put Eq. (25) into Eq. (18), we find new exact solution named as combined JEFSs of Eq. (11)

$$\begin{aligned}\phi\left(\omega_4\right) &= \left(1-m^2\right) B_1\, e^{i\theta_4} sc\left(\omega_4\right) nd\left(\omega_4\right),\\ \varphi(\omega_4) &= \left(1-m^2\right)^2 B_2\, sc^2(\omega_4) nd^2(\omega_4),\end{aligned} \tag{26}$$

where $\omega_4 = j_1x + j_2y - \frac{j_1+j_2}{2}\left(\gamma+\sqrt{\gamma^2+16\left(1+m^2\right)\left(j_1^2+j_2^2\right)}\right)t,\ \theta_4 = \frac{\gamma+\sqrt{\gamma^2+16\left(1+m^2\right)\left(j_1^2+j_2^2\right)}}{4}\left(x+y-\gamma t\right).$

Case (5):

$$k_1 = \sqrt{\frac{P\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},\ m_1 = n_1 = k_0 = 0,\ \ l_1 = \sqrt{\frac{R\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},$$

$$\vartheta = \frac{\gamma-\sqrt{\gamma^2+8\left(Q-6\sqrt{PR}\right)\left(j_1^2+j_2^2\right)}}{4},$$

$$j_3 = \frac{-\left(j_1+j_2\right)}{2}\left(\gamma-\sqrt{\gamma^2+8\left(Q-6\sqrt{PR}\right)\left(j_1^2+j_2^2\right)}\right), \tag{27}$$

Putting Eq. (27) into Eq. (18), we achieve the following new combined JEFSs of Eq. (11)

$$\phi\left(\omega_5\right) = B_1\frac{e^{i\theta_5}\left(1+msn^2\left(\omega_5\right)\right)}{sn\left(\omega_5\right)},$$

$$\varphi\left(\omega_5\right) = B_2\frac{\left(1+msn^2\left(\omega_5\right)\right)^2}{sn^2\left(\omega_5\right)}, \tag{28}$$

where $\omega_5 = j_1x + j_2y + \frac{j_1+j_2}{2}\left(\sqrt{\gamma^2-8\left(1+m^2+6m\right)\left(j_1^2+j_2^2\right)}-\gamma\right)t,$

$$\theta_5 = \frac{\gamma-\sqrt{\gamma^2-8\left(1+m^2+6m\right)\left(j_1^2+j_2^2\right)}}{4}\left(x+y-\gamma t\right).$$

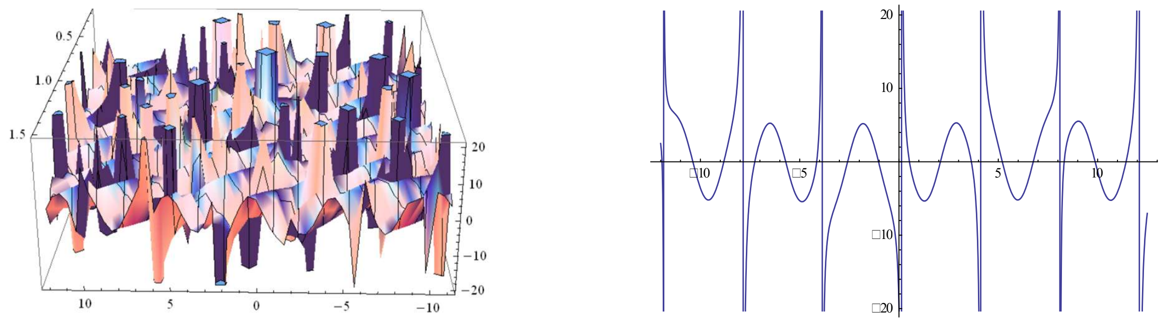

Figure 3: Graph of the solution $\phi\left(\omega_5\right)$ corresponding to the values $j_1 = q = 1,\ j_2 = 2,\ \gamma = 20,\ p = -1,\ y = 0,\ m = 4/5.$

Case (6):

$$m_1 = \sqrt{\frac{\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},\ n_1 = \sqrt{\frac{\left(Q^2-4PR\right)\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},\ k_0 = k_1 = l_1 = 0,$$

$$\vartheta = \frac{\gamma-\sqrt{\gamma^2-16\left(Q+3\sqrt{Q^2-4PR}\right)\left(j_1^2+j_2^2\right)}}{4},$$

$$j_3=-\frac{j_1+j_2}{2}\left(\gamma-\sqrt{\gamma^2-16\left(Q+3\sqrt{Q^2-4PR}\right)\left(j_1^2+j_2^2\right)}\right). \tag{29}$$

Writing Eq. (29) into Eq. (18), we attain the following new mixed JEFSs of Eq. (11)

$$\begin{aligned}\phi(\omega_6)&=B_1\frac{e^{i\theta_6}\left(1-m^2+m^2cn^4(\omega_6)\right)}{cn(\omega_6)dn(\omega_6)sn(\omega_6)},\\ \varphi(\omega_6)&=B_2\frac{\left(\left(1-m^2+m^2cn^4(\omega_6)\right)\right)^2}{cn^2(\omega_6)dn^2(\omega_6)sn^2(\omega_6)},\end{aligned} \tag{30}$$

where $\omega_6=j_1x+j_2y+\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2-16\left(2-4m^2\right)\left(j_1^2+j_2^2\right)}-\gamma\right)t,$

$$\theta_6=\frac{\gamma-\sqrt{\gamma^2-16\left(2-4m^2\right)\left(j_1^2+j_2^2\right)}}{4}\left(x+y-\gamma t\right).$$

Case (7):

$$k_1=\sqrt{\frac{P\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{4pj_1^2-4pqj_2^2}},\ l_1=\sqrt{\frac{R\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{4pj_1^2-4pqj_2^2}},\ m_1=\sqrt{\frac{\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{4pj_1^2-4pqj_2^2}},\ k_0=0,$$

$$n_1=0,\ \vartheta=\frac{\gamma-\sqrt{\gamma^2-4\left(Q+6\sqrt{PR}\right)\left(j_1^2+j_2^2\right)}}{4},$$

$$j_3=\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2-4\left(Q+6\sqrt{PR}\right)\left(j_1^2+j_2^2\right)}-\gamma\right). \tag{31}$$

When we put Eq. (31) into Eq. (18), we find the combined JEFSs of Eq. (11)

$$\begin{aligned}\phi(\omega_7)&=B_1e^{i\theta_7}\frac{\left(1+cn(\omega_7)dn(\omega_7)+msn^2(\omega_7)\right)}{2sn(\omega_7)},\\ \varphi(\omega_7)&=B_2\frac{\left(1+cn(\omega_7)dn(\omega_7)+msn^2(\omega_7)\right)^2}{4sn^2(\omega_7)},\end{aligned} \tag{32}$$

where $\omega_7=j_1x+j_2y+\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2-4\left(-1-m^2+6m\right)\left(j_1^2+j_2^2\right)}-\gamma\right)t,$

$$\theta_7=\frac{\gamma-\sqrt{\gamma^2-4\left(-1-m^2+6m\right)\left(j_1^2+j_2^2\right)}}{4}\left(x+y-\gamma t\right).$$

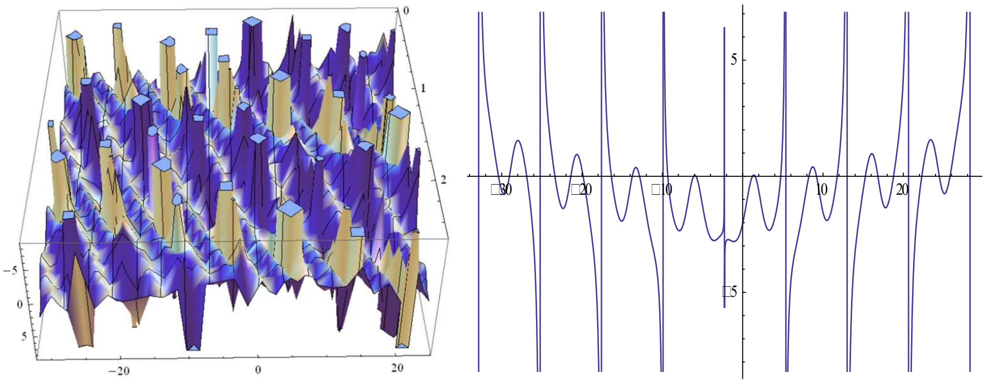

Figure 4: Graph of the solution $\phi(\omega_7)$ corresponding to the values $j_1=q=1,\ j_2=2,\ \gamma=10,\ p=-1,\ y=0, m=3/4$

Remark 1: According to our findings, all the JEFSs of Eq. (11) obtained with the new version of generalized F-expansion method are uncharted and are not proved before in researches. When the results obtained in reference [43] are examined, it is seen that trigonometric and hyperbolic function solutions are obtained. The results obtained here are completely different from these and are new Jacobian elliptic function solutions. In addition, 2D and 3D graphics of the obtained solution functions are shown in Figs. 1-4 which are established with suitable parametric selections.

4. Implementation of the new version of the generalized -expansion method to DSEs

In this section, the application of the new version of the generalized -expansion method to DSEs is given. The traveling wave transformation as in Section 3 has been used for DSEs,

$$\phi(x,y,t)=e^{i\sigma}\phi(\mu),\ \varphi(x,y,t)=\varphi(\mu),\ \mu=j_1x+j_2y+j_3t,\ \sigma=\zeta(x+y-\gamma t), \tag{33}$$

where j_1, j_2, j_3, ζ and γ are constants. When the transformation and necessary operations are performed, the NLODE below

$$\left(j_1^2+j_2^2\right)\phi''+\left(\zeta\gamma-2\zeta^2\right)\phi+\frac{2p\left(qj_2^2-j_1^2\right)}{j_1^2+qj_2^2}\phi^3=0, \tag{34}$$

is obtained. Considering the new solution function Eq. (8), the balancing procedure is applied between the terms ϕ'' and ϕ^3 in Eq. (34). Since it can be written as $F'=\pm\sqrt{PF^4+QF^2+R}\cong\pm\sqrt{P}F^2+\dots$ using Eq. (6), the solution function in Eq. (8) is briefly taken as $\phi=k_M^M\left(\frac{F'}{F}\right)^M+\dots\cong\pm k_M^M P^{M/2}F^M+\dots$ Accordingly, the related terms that provide balancing are calculated as

$$\phi^3=k_M^{3M}\left(\frac{F'}{F}\right)^{3M}+\dots\cong\pm k_M^{3M}P^{3M/2}F^{3M}+\dots$$

$$\phi''\cong M(M+1)k_M^M P^{(M/2)+1}F^{M+2}+\dots$$

Then, the balancing term is obtained as $M=1$. Thus, the general form of the new solution function is written as follows

$$\phi(\mu)=k_0+k_1\frac{F'(\mu)}{F(\mu)}+l_1\frac{F(\mu)}{F'(\mu)}+m_1\left(\frac{F'(\mu)}{F(\mu)}\right)'\frac{F(\mu)}{F'(\mu)}+n_1\frac{F'(\mu)}{F(\mu)}\left(\left(\frac{F'(\mu)}{F(\mu)}\right)'\right)^{-1}. \tag{35}$$

When the terms ϕ'' and ϕ^3 obtained from Eq. (35) are subsituted into Eq. (34), a zero polynomial is obtained depending on the eliptic functions of the $\left(\left(\frac{F'(\mu)}{F(\mu)}\right)'\right)^r\left(\frac{F'(\mu)}{F(\mu)}\right)^p$ $(r=0,1;p=0,\pm1,\pm2,...)$. When the algebraic equation system, which is obtained by equalizing the coefficients of this polynomial with zero, is solved with the help of the Mathematica package program, $k_0,k_1,l_1,m_1,n_1,\zeta$ and e_3 are obtained. These coefficients and inverse transformations are replaced by Eq. (35) and the following solutions are obtained depending on the JEFs of the $\frac{F'(\mu)}{F(\mu)}$ and $\left(\frac{F'(\mu)}{F(\mu)}\right)'$. Then, unlike the solutions in Section 3, $\phi(\mu)$ and $\varphi(\mu)$ solutions for $F(\mu)=sn(\mu)$ were obtained respectively.

Case 1:

$$k_1=\sqrt{\frac{(j_1^2+qj_2^2)(j_1^2+j_2^2)}{pj_1^2-pqj_2^2}},\ \zeta=\frac{\gamma+\sqrt{\gamma^2-16Q(j_1^2+j_2^2)}}{4},\ k_0=l_1=n_1=m_1=0,$$

$$j_3=-\frac{j_1+j_2}{2}\left(\gamma+\sqrt{\gamma^2-16Q(j_1^2+j_2^2)}\right). \tag{36}$$

When we substitute Eq. (36) into Eq. (35), the following new mixed exact solutions of Eq. (11) are gained as follows:

$$\begin{aligned}\phi(\mu_1)&=D_1\frac{e^{i\sigma_1}cn(\mu_1)dn(\mu_1)}{sn(\mu_1)},\\ \varphi(\mu_1)&=D_2\frac{cn^2(\mu_1)dn^2(\mu_1)}{sn^2(\mu_1)},\end{aligned} \tag{37}$$

where $\mu_1=j_1x+j_2y-\frac{j_1+j_2}{2}\left(\gamma+\sqrt{\gamma^2+16(1+m^2)(j_1^2+j_2^2)}\right)t,$ $D_1=\sqrt{\frac{(j_1^2+j_2^2)(j_1^2+qj_2^2)}{pj_1^2-pqj_2^2}},$

$\sigma_1=\frac{\gamma+\sqrt{\gamma^2+16(j_1^2+j_2^2)(1+m^2)}}{4}(x+y-\gamma t),\ D_2=-\frac{2(j_1^2+j_2^2)j_1^2}{p(j_1^2-qj_2^2)}.$

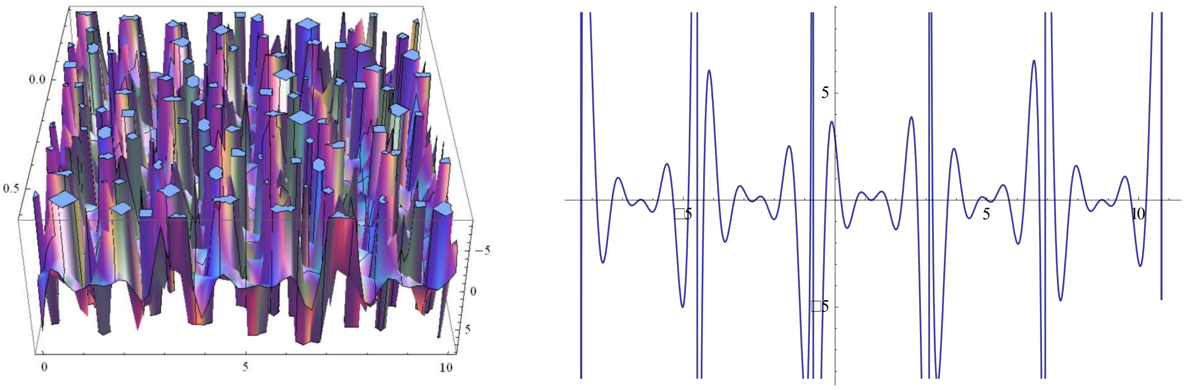

Figure 5: Graph of the solution $\phi(\mu_1)$ corresponding to the values $j_1=q=1,\ j_2=2,\ \gamma=10,\ p=-1,$ $y=0,\ m=3/4$

Case (2):

$$k_0 = m_1 = k_1 = n_1 = 0,\ l_1 = -\sqrt{\frac{\left(j_1^2 + qj_2^2\right)\left(j_1^2 + j_2^2\right)\left(Q^2 - 4PR\right)}{pj_1^2 - pqj_2^2}},$$

$$\zeta = \frac{\gamma - \sqrt{\gamma^2 - 16Q\left(j_1^2 + j_2^2\right)}}{4},\ j_3 = \frac{j_1 + j_2}{2}\left(\sqrt{\gamma^2 - 16Q\left(j_1^2 + j_2^2\right)} - \gamma\right). \tag{38}$$

If we put Eq. (38) into Eq. (35), we get the following new mixed JEFSs of Eq. (11)

$$\begin{gathered}\phi\left(\mu_2\right) = D_3 \frac{e^{i\sigma_2} sn\left(\mu_2\right)}{cn\left(\mu_2\right) dn\left(\mu_2\right)},\\ \varphi\left(\mu_2\right) = D_4 \frac{sn^2\left(\mu_2\right)}{dn^2\left(\mu_2\right) cn^2\left(\mu_2\right)},\end{gathered} \tag{39}$$

where $\mu_2 = j_1 x + j_2 y + \frac{j_1 + j_2}{2}\left(\sqrt{\gamma^2 + 16\left(1 + m^2\right)\left(j_1^2 + j_2^2\right)} - \gamma\right)t,\ D_4 = -\frac{2j_1^2\left(j_1^2 + j_2^2\right)\left(1 - m^2\right)^2}{p\left(j_1^2 - qj_2^2\right)},$

$D_3 = \left(1 - m^2\right)\sqrt{\frac{\left(j_1^2 + qj_2^2\right)\left(j_1^2 + j_2^2\right)}{pj_1^2 - pqj_2^2}},\ \sigma_2 = \frac{\sqrt{\gamma^2 - 8\left(1 + m^2\right)\left(j_1^2 + j_2^2\right)} + \gamma}{4}\left(x + y - \gamma t\right).$

Case (3):

$$m_1 = -\sqrt{\frac{\left(j_1^2 + qj_2^2\right)\left(j_1^2 + j_2^2\right)}{pj_1^2 - pqj_2^2}},\ k_0 = n_1 = k_1 = l_1 = 0,$$

$$\zeta = \frac{\gamma + \sqrt{\gamma^2 + 32Q\left(j_1^2 + j_2^2\right)}}{4},\ j_3 = -\frac{j_1 + j_2}{2}\left(\sqrt{\gamma^2 + 32Q\left(j_1^2 + j_2^2\right)} + \gamma\right). \tag{40}$$

When we substitute Eq. (40) into Eq. (35), we obtain the following combined JEFSs,

$$\begin{gathered}\phi\left(\mu_3\right) = D_1 \frac{e^{i\sigma_3}\left(1 - m^2 sn^4\left(\mu_3\right)\right)}{cn\left(\mu_3\right) dn\left(\mu_3\right) sn\left(\mu_3\right)},\\ \varphi\left(\mu_3\right) = D_2 \frac{\left(1 - m^2 sn^4\left(\mu_3\right)\right)^2}{cn^2\left(\mu_3\right) dn^2\left(\mu_3\right) sn^2\left(\mu_3\right)},\end{gathered} \tag{41}$$

where $\mu_3 = j_1 x + j_2 y - \frac{j_1 + j_2}{2}\left(\sqrt{\gamma^2 - 32\left(1 + m^2\right)\left(j_1^2 + j_2^2\right)} + \gamma\right)t,\ \sigma_3 = \frac{\gamma + \sqrt{\gamma^2 - 32\left(1 + m^2\right)\left(j_1^2 + j_2^2\right)}}{4}\left(x + y - \gamma t\right).$

Case (4):

$$n_1 = 4\sqrt{\frac{PR\left(j_1^2 + qj_2^2\right)\left(j_1^2 + j_2^2\right)}{pj_1^2 - pqj_2^2}},\ k_0 = m_1 = k_1 = l_1 = 0,$$

$$\zeta=\frac{\gamma-\sqrt{\gamma^2+32Q\left(j_1^2+j_2^2\right)}}{4},\ j_3=\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2+32Q\left(j_1^2+j_2^2\right)}-\gamma\right).. \tag{42}$$

When we put Eq. (42) into Eq. (35), we achieve new exact solution name as combined JEFSs of Eq. (11)

$$\begin{aligned}\phi\left(\mu_4\right)&=D_5\frac{e^{i\sigma_4}cn\left(\mu_4\right)dn\left(\mu_4\right)sn\left(\mu_4\right)}{\left(1-m^2sn^4\left(\mu_4\right)\right)},\\ \varphi\left(\mu_4\right)&=D_6\frac{cn^2\left(\mu_4\right)dn^2\left(\mu_4\right)sn^2\left(\mu_4\right)}{\left(m^2sn^4\left(\mu_4\right)-1\right)^2},\end{aligned} \tag{43}$$

where $\mu_4=j_1x+j_2y+\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2-32\left(1+m^2\right)\left(j_1^2+j_2^2\right)}-\gamma\right)t,\ D_6=-\frac{32j_1^2\left(j_1^2+j_2^2\right)m^2}{p\left(j_1^2-qj_2^2\right)},$

$$D_5=-4m\sqrt{\frac{\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{p\left(j_1^2-qj_2^2\right)}},\ \sigma_4=-\frac{\sqrt{\gamma^2-32\left(1+m^2\right)\left(j_1^2+j_2^2\right)}-\gamma}{4}\left(x+y-\gamma t\right).$$

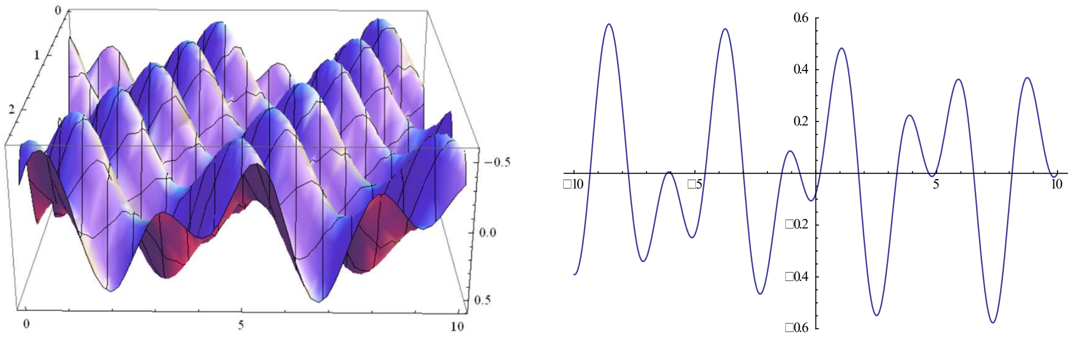

Figure 6: Graph of the solution $\phi\left(\mu_4\right)$ corresponding to the values $j_1=q=1,\ j_2=1/2,\ \gamma=10,\ p=-1,\ y=0,\ m=1/5.$

Case (5):

$$k_1=\sqrt{\frac{\left(j_1^2+e_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},\ l_1=\sqrt{\frac{\left(Q^2-4PR\right)\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},\ k_0=0,$$

$$m_1=n_1=0,\quad \zeta=\frac{\gamma-\sqrt{\gamma^2-16\left(Q+3\sqrt{Q^2-4PR}\right)\left(j_1^2+j_2^2\right)}}{4},$$

$$j_3=\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2-16\left(Q+3\sqrt{Q^2-4PR}\right)\left(j_1^2+j_2^2\right)}-\gamma\right) \tag{44}$$

When we substitute Eq. (44) into Eq. (35), we find following new combined JEFSs of Eq. (11),

$$\phi(\mu_5)=D_1\frac{e^{i\sigma_5}\left(cn^2(\mu_5)dn^2(\mu_5)+\left(1-m^2\right)sn^2(\mu_5)\right)}{cn(\mu_5)dn(\mu_5)sn(\mu_5)},$$
$$\varphi(\mu_5)=D_2\frac{\left(cn^2(\mu_5)dn^2(\mu_5)+\left(1-m^2\right)sn^2(\mu_5)\right)^2}{cn^2(\mu_5)dn^2(\mu_5)sn^2(\mu_5)}, \tag{45}$$

where $\mu_5=j_1x+j_2y+\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2-16\left(2-4m^2\right)\left(j_1^2+j_2^2\right)}-\gamma\right)t,$

$\sigma_5=\frac{\gamma-\sqrt{\gamma^2-16\left(2-4m^2\right)\left(j_1^2+j_2^2\right)}}{4}\left(x+y-\gamma t\right).$

Case (6):

$$m_1=-\sqrt{\frac{\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},\ n_1=4\sqrt{\frac{PR\left(j_1^2+j_2^2\right)\left(j_1^2+qj_2^2\right)}{pj_1^2-pqj_2^2}},\ k_0=l_1=k_1=0,$$

$$\zeta=\frac{\gamma-\sqrt{\gamma^2+32\left(Q+6\sqrt{PR}\right)\left(j_1^2+j_2^2\right)}}{4},$$

$$j_3=\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2+32\left(Q+6\sqrt{PR}\right)\left(j_1^2+j_2^2\right)}-\gamma\right). \tag{46}$$

When we replace Eq. (46) into Eq. (35), we attain following new combined and mixed type of JEFSs of Eq. (11),

$$\phi(\mu_6)=D_1e^{i\sigma_6}\left(\frac{\left(1-m^2sn^4(\mu_6)\right)}{cn(\mu_6)dn(\mu_6)sn(\mu_6)}-\frac{4m\,cn(\mu_6)dn(\mu_6)sn(\mu_6)}{\left(1-m^2sn^4(\mu_6)\right)}\right),$$
$$\varphi(\mu_6)=D_2\left(\frac{\left(1-m^2sn^4(\mu_6)\right)}{cn(\mu_6)dn(\mu_6)sn(\mu_6)}-\frac{4m\ cn(\mu_6)dn(\mu_6)sn(\mu_6)}{\left(1-m^2sn^4(\mu_6)\right)}\right)^2, \tag{47}$$

where $\mu_6=j_1x+j_2y+\frac{j_1+j_2}{2}\left(\sqrt{\gamma^2+32\left(-1-m^2+6m\right)\left(j_1^2+j_2^2\right)}-\gamma\right)t,$

$\sigma_6=\frac{\gamma-\sqrt{\gamma^2+32\left(-1-m^2+6m\right)\left(j_1^2+j_2^2\right)}}{4}\left(x+y-\gamma t\right).$

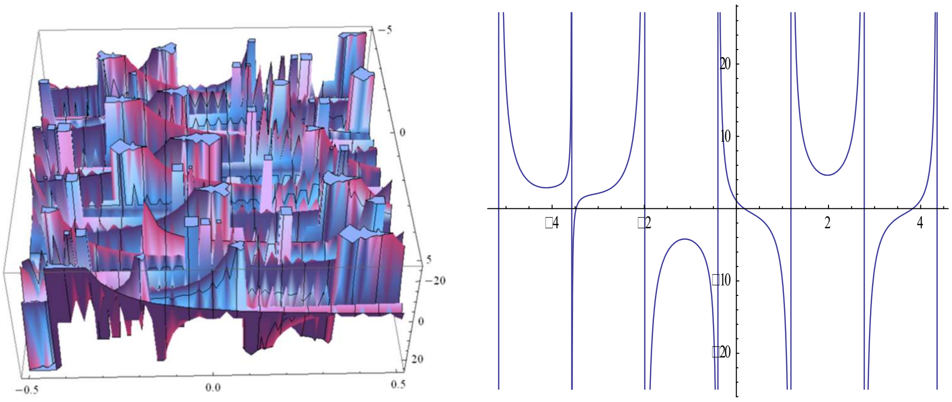

Figure 7: Graph of the solution $\phi(\mu_6)$ corresponding to the values $j_1 = q = 1,\ j_2 = \gamma = 2,\ p = -1,\ y = 0,\ m = 1/5.$

Case (7):

$$l_1 = \frac{1}{2}\sqrt{\frac{(Q^2 - 4PR)(j_1^2 + j_2^2)(j_1^2 + qj_2^2)}{pj_1^2 - pqj_2^2}},\ k_1 = \frac{1}{2}\sqrt{\frac{(j_1^2 + j_2^2)(j_1^2 + qj_2^2)}{pj_1^2 - pqj_2^2}},\ k_0 = 0,$$

$$m_1 = \frac{1}{2}\sqrt{\frac{(j_1^2 + j_2^2)(j_1^2 + qj_2^2)}{pj_1^2 - pqj_2^2}},\ n_1 = 0,\ \zeta = \frac{\gamma - \sqrt{\gamma^2 - 8(j_1^2 + j_2^2)\left(-Q + 3\sqrt{Q^2 - 4PR}\right)}}{4},$$

$$j_3 = \frac{j_1 + j_2}{2}\left(\sqrt{\gamma^2 - 8\left(-Q + 3\sqrt{Q^2 - 4PR}\right)(j_1^2 + j_2^2)} - \gamma\right). \tag{48}$$

When we put Eq. (48) into Eq. (35), we obtain the combined JEFSs of Eq. (11),

$$\begin{aligned} \phi(\mu_7) &= D_1 \mathrm{e}^{\mathrm{i}\sigma_7} \frac{\left(1 - m^2 - dn^2(\mu_7)\right) sn(\mu_7)}{cn(\mu_7)\, dn(\mu_7)}, \\ \varphi(\mu_7) &= D_2 \frac{\left(1 - m^2 - dn^2(\mu_7)\right)^2 sn^2(\mu_7)}{cn^2(\mu_7)\, dn^2(\mu_7)}, \end{aligned} \tag{49}$$

where $\mu_7 = j_1 x + j_2 y + \frac{j_1 + j_2}{2}\left(\sqrt{\gamma^2 - 8(4 - 2m^2)(j_1^2 + j_2^2)} - \gamma\right)t,\ \sigma_7 = \frac{\gamma - \sqrt{\gamma^2 - 8(4 - 3m^2)(j_1^2 + j_2^2)}}{4}(x + y - \gamma t).$

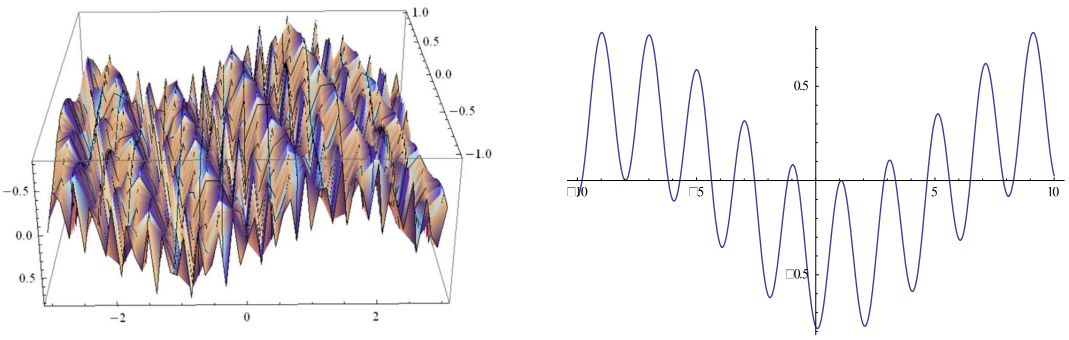

Figure 8: Graph of the solution $\phi(\mu_7)$ corresponding to the values $j_1 = q = 1,\ j_2 = 2,\ \gamma = 10,\ p = -1,\ y = 0, m = 3/4$

Remark 2: According to the results obtained, single, combined and mixed JEFSs of Eq. (11) that we acquired with the new version of the generalized F'/F-expansion method is new and are not observed in the literature. Also, 2D and 3D graphics of the obtained solutions are shown in Figs. 5-8 which represent with appropriate parametric choices.

5. CONCLUSION

In this paper, the new versions of the generalized F-expansion method and the generalized F'/F-expansion method have been successfully implemented to achieve new exact solutions of the DSEs. These methods make it possible to obtain new function solutions. The new obtained results for DSEs seem to be very different and interesting. When the results obtained in reference [43] are examined, it is seen that trigonometric and hyperbolic function solutions are obtained. The results obtained here are completely different from these and include different types of Jacobi elliptic function solutions. These solutions include single, combined and multiple combined JEFSs. Consequently, the current study is assumed an enterprise to fill the gap in the related literature. If we set the parameters in the attained JEFSs as special values, a variety of special solutions like different solitary wave solutions are obtained. We propose that new solutions of other nonlinear complex models and nonlinear complex systems can be obtained with these fertile algorithms.

CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

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