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Overdetermined Systems of ODEs with Parameters and Their Applications: The Method of Differential Constraints and the Generalized Separation of Variables in PDEs

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Abstract: Various situations are described where the construction of exact solutions of nonlinear ordinary and partial differential equations leads to overdetermined systems of ODEs with parameters that are not included in the original differential equations. A non-classical problem for ordinary differential equations with parameters is formulated and the concept of the conditional capacity of an exact solution is introduced. The method for investigating overdetermined systems of two ODEs of any order on consistency, which eventually leads to algebraic equations with parameters, is presented. A general description of the method of differential constraints with respect to ordinary differential equations is given and many specific examples of applying this method for obtaining exact solutions are considered. It is shown that the use of the splitting method (and also the method based on the use of invariant subspaces of nonlinear operators) for constructing exact generalized separable solutions of nonlinear PDEs can lead to overdetermined systems of ODEs with parameters. Several nonlinear partial differential equations (including a delay PDE) of higher orders are considered, and their exact solutions are found by analyzing the corresponding overdetermined ODE systems.

1. Introduction

Nonlinear partial differential equations of the second and higher orders (nonlinear equations of mathematical physics) are often encountered in various fields of mathematics, physics, mechanics, chemistry, biology and in numerous applications. The general solution of nonlinear equations of mathematical physics can be obtained only in exceptional cases. Therefore, we usually have to confine ourselves to the search and analysis of particular solutions, which are called exact solutions.

Exact solutions of differential equations of mathematical physics have always played and continue to play a huge role in the formation of a correct understanding of the qualitative features of many phenomena and processes in various fields of natural science. Exact solutions of nonlinear equations clearly demonstrate and allow us to understand the mechanisms of such complex nonlinear effects as the spatial localization of transport processes, the multiplicity or absence of stationary states under certain conditions, the existence of blow-up regimes, etc. Simple solutions are widely used to illustrate the theoretical material and some applications in the courses of universities and technical colleges (theory of heat and mass transfer, hydrodynamics, gas dynamics, wave theory, nonlinear optics, etc.).

Exact traveling wave solutions and self-similar solutions often represent the asymptotics of essentially broader classes of solutions corresponding to other initial and boundary conditions. This property allows us to draw general

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conclusions and predict the dynamics of various phenomena and processes.
Even those particular exact solutions of differential equations, which do not have a clear physical meaning, can be used as a basis for formulating test problems required for verification of correctness and evaluation of the accuracy of various numerical, asymptotic and approximate analytical methods. In addition, model equations and problems that admit exact solutions serve as the basis for the development of new numerical, asymptotic and approximate methods, which, in turn, allow us to study already more complex problems that do not have exact analytic solutions. Exact methods and solutions are also necessary for the development and improvement of the corresponding sections of computer programs intended for symbolic computations (e.g., Maple, Mathematica, etc.)
It is important to note that many equations of applied and theoretical physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow one to plan an experiment to determine these parameters or functions by artificially creating suitable (boundary and initial) conditions.

In this article, by exact solutions of nonlinear equations of mathematical physics we mean the following solutions:

1. Solutions that are expressed in terms of elementary functions.
2. Solutions that are expressed through special functions (definite integrals) or in the form of quadratures.
3. Solutions that are described by ordinary differential equations (systems of ordinary differential equations).

By exact methods of solving nonlinear equations of mathematical physics we mean the methods that allow to obtain exact solutions.
In this paper, the main attention will be paid to methods of searching for exact solutions of the first and second types.

Remark 1.1. By tradition, in the classical theory of ordinary differential equations, it is customary to consider methods[1] that allow one to obtain general solutions of equations (see, for example, [1–11]). At the same time, there is practically no attention to the methods of searching for particular exact solutions of nonlinear equations. This state of affairs has a very negative impact on the development of methods for searching for exact solutions of nonlinear equations of mathematical physics, which can be expressed in terms of elementary or special functions.

2. Traveling Wave Solutions. Two Problems for ODEs with Parameters. Conditional Capacity of Exact Solutions

2.1. Construction of ODEs with the Help of PDEs Having Traveling Wave Solutions

In addition to empirical parameters, ODEs can also depend on additional free parameters that arise with different reductions of the original models (described by PDEs) to ODEs.
Let us illustrate this with the example of constructing the simplest invariant traveling wave solutions of PDEs.
The vast majority of nonlinear equations of mathematical physics are partial differential equations of the form

$$\Phi(w, w_z, w_t, w_{zz}, w_{zt}, w_{tt}, \dots) = 0, \tag{1}$$

which do not depend explicitly on independent variables (here, for simplicity, we consider equations with two independent variables t and z, where t can play the role of time or spatial variable).
In the general case, the equation (1) admits the exact traveling wave solution

$$w = y(x), \quad x = a_1 z + a_2 t, \tag{2}$$

where a_1 and a_2 are arbitrary constants. Substituting (2) into (1), we arrive at the ordinary differential equation

$$\Phi(y, a_1 y'_x, a_2 y'_x, a_1^2 y''_{xx}, a_1 a_2 y''_{xx}, a_2^2 y''_{xx}, \dots) = 0. \tag{3}$$

Thus, the ordinary differential equation (3) describes exact solutions of a special form of the original partial differential equation (1). Since traveling wave solutions (2) are the most common solutions of nonlinear equations of mathematical physics, one must be able to find solutions of the corresponding ordinary differential equations.
In addition to the free parameters a_1 and a_2, the equation (3) often contains other parameters which can also vary. In particular, for equations of the form (1), which can be represented in divergence form (in the form of a conservation law)

$$\begin{aligned} &\frac{\partial}{\partial t}\Phi_1 + \frac{\partial}{\partial z}\Phi_2 = 0, \\ &\Phi_i = \Phi_i(w, w_z, w_t, w_{zz}, w_{zt}, w_{tt}, \dots), \quad i = 1, 2, \end{aligned} \tag{4}$$

[1] Here and in what follows, we are dealing with exact methods for integrating differential equations.

the search for traveling wave solutions (2) leads to an ordinary differential equation

$$\begin{aligned} &a_2\Phi_1 + a_1\Phi_2 + a_3 = 0, \\ &\Phi_i = \Phi_i(y,\, a_1 y'_x, a_2 y'_x, a_1^2 y''_{xx}, a_1 a_2 y''_{xx}, a_2^2 y''_{xx}, \dots), \end{aligned} \tag{5}$$

containing three arbitrary constants a_1, a_2, a_3.

2.2. Construction of PDEs, Having Two Types of Exact Solutions, with the Help of ODEs

Below, some simple ways are described, that allow one, by using ODEs, to construct nonlinear PDEs that admit exact solutions.
Suppose that we know a particular solution $y = y_{\rm p}(x)$ of an autonomous ordinary differential equation

$$F(y, y'_x, \dots, y_x^{(n)}) = 0, \tag{6}$$

which can depend on the parameters $a_1, \dots, a_k$. Then the following statements are valid:
Statement 1. Nonlinear evolution PDE

$$w_t + c w_z + F(w, w_z, \dots, w_z^{(n)}) = 0$$

admits the exact traveling wave solution $w = y_{\rm p}(z - ct)$.
Statement 2. Nonlinear PDE

$$w_{tt} - c^2 w_{zz} + F(w, w_z, \dots, w_z^{(n)}) = 0$$

admits the exact traveling wave solutions $w_1 = y_{\rm p}(z - ct)$ and $w_2 = y_{\rm p}(z + ct)$.
Statement 3. If the function F in (6) does not depend on the first argument, then the nonlinear evolution PDE

$$w_t - b + F(w_z, \dots, w_z^{(n)}) = 0$$

admits an exact solution in the form of a sum of functions of different arguments $w = bt + y_{\rm p}(z) + A$, where A is an arbitrary constant.
Statement 4. If the left-hand side of (6) has the form $F = by + c + F_1(y'_x, \dots, y_x^{(n)})$, then the nonlinear evolution PDE

$$w_t + bw - c + F_1(w_z, \dots, w_z^{(n)}) = 0$$

admits an exact solution in the form of a sum of functions of different arguments $w = \varphi(t) + y_{\rm p}(z)$, where

$$\varphi(t) = \begin{cases} Ae^{-bt} + (c/b) & \text{if } b \neq 0, \\ ct + A & \text{if } b = 0, \end{cases}$$

and A is an arbitrary constant.
Statement 5. If the left-hand side of (6) has the form $F = by + c + F_1(y'_x, \dots, y_x^{(n)})$, then the nonlinear PDE

$$w_{tt} + bw - c + F_1(w_z, \dots, w_z^{(n)}) = 0$$

admits an exact solution in the form of a sum of functions of different arguments $w = \varphi(t) + y_{\rm p}(z)$, where

$$\varphi(t) = \begin{cases} A\cos(kt) + B\sin(kt) + (c/b) & \text{if } b = k^2 > 0, \\ A\exp(-kt) + B\exp(kt) + (c/b) & \text{if } b = -k^2 < 0, \\ \frac{1}{2}ct^2 + At + B & \text{if } b = 0, \end{cases}$$

and A and B are arbitrary constants.

2.3. Two Problems for ODEs with Parameters. Conditional Capacity of Exact Solutions

There are many nonlinear equations of mathematical physics (see, for example, [10, 12–18]) whose solutions can be expressed in terms of ordinary differential equations[2]

$$F(x, y, y'_x, \dots, y_x^{(n)}; a_1, \dots, a_k) = 0, \tag{7}$$

[2]The form of these solutions may differ from (2).

containing a set of free parameters a_i $(i = 1, \dots, k)$ that do not include into the original partial differential equation. In this case, two fundamentally different problems arise, which are described below.

Problem 1. It is required to find the values of the parameters a_i, for which we can find the general solution of the equation (7) (here and in what follows, we are dealing with solutions that can be expressed in terms of elementary or special functions).
Problem 2. It is required to find the values of the parameters a_i, for which we can find a particular (exact) solution of the equation (7).

For a comparative analysis of the results of solving problems 1 and 2, it is useful to introduce the following definition.

Definition. The *conditional capacity* of an exact solution of a nonlinear PDE is equal to the number of arbitrary constants involved in the solution but not in the original equation. The conditional capacity of a solution will be denoted by "cc."

The practical meaning of this definition is obvious: the more arbitrary constants the solution contains, the more important and interesting this solution is (since the generality of the solution is determined by the number of arbitrary constants).
In problem 1, the general solution of the corresponding ordinary differential equation (7) can usually be obtained in a closed form only for specific numerical values of the parameters a_i or for imposing a certain number of constraints on them (in this case, there will be less free parameters $a_1,\dots, a_p$, and the other parameters $a_{p+1},\dots, a_k$ will be expressed through them). The conditional capacity of such a solution for the original PDE is determined by the formula

$$\mathrm{cc}_1 = p + n, \tag{8}$$

where n is the order of the equation (7).
In problem 2, it is usually possible to obtain an exact solution of the ordinary differential equation (7) with a smaller number of constraints on the parameters a_i, i.e., there remain more free parameters $a_1,\dots, a_q$, than in problem 1 $(q \geq p)$. In this case, the exact solution itself may additionally depend on m integration constants, where $m \leq n$. The conditional capacity of such a solution for the original PDE is determined by the formula

$$\mathrm{cc}_2 = q + m. \tag{9}$$

A comparison of the formulas (8) and (9) shows that the conditional capacity of particular solutions of problem 2 can be lower, equal to and higher than the corresponding conditional capacity of general solutions of problem 1 (for some certain parameter values). This means the complete equality of the solutions of problems 1 and 2 with respect to the analysis of nonlinear PDEs.
Problem 1 is classical, rather well-developed methods for integrating ordinary differential equations [1–11] are used to solve it.
Problem 2 is non-classical, at the present time, the methods for solving it have been developed insufficiently (this circumstance, first of all, is connected with the fact that almost no specialists in the field of ordinary differential equations dealt with this problem). In the literature, relatively few methods of solving such problems are described, as a rule, having a very narrow range of applicability [19–36]. These methods are most often based on the explicit specification of the solution in the form of a sum or ratio of several exponential, hyperbolic, or trigonometric functions (sometimes after some simple point transformation of the equation under consideration) containing free parameters whose values are determined further by the method of undetermined coefficients. The most common methods of this kind are the tanh method [20–23, 26, 34] and the exp-function method [27–33, 35, 36]. An essential restriction of such direct methods is that the solution is sought in explicit form, while the vast majority of known general solutions of nonlinear equations is represented in implicit or parametric form (such conclusion follows from the statistical processing of the text of the most complete handbooks on exact solutions of ODEs [9, 11]).
We can also arrive at the formulation of problems 1 and 2 entirely from other considerations that do not involve the connection of ordinary differential equations with the corresponding partial differential equations. One can, for example, simply consider the ordinary differential equation (7), assuming that it depends on the physico-chemical constants a_i, which play an important role in applications and can vary widely.
In this case, the role of the integration constants, involving into the general or a particular solution of this equation, and the determining physico-chemical constants of the equation a_i, can be considered the same (it can often be more useful to find a particular solution of a rather broad class of equations than to find the general solution of a narrower class of equations).
In what follows, we will investigate ordinary differential equations with varied parameters, without considering a physical or other meaning of these parameters.

Remark 2.1. Using the splitting method to find exact solutions of nonlinear equations of mathematical physics with a generalized separation of variables leads to systems of ordinary differential equations that can contain many free parameters that are not included in the original equation [10, 12, 13]. Examples of this kind will be considered later in Section 4.

3. Method of Differential Constraints and Their Applications

3.1. First-order Differential Constraints. General Description

Recently, the method of differential constraints, which has a large generality and makes it possible to construct exact solutions of nonlinear partial differential equations (the basic ideas of which were first described in [37]), has developed significantly. A detailed description of the method of differential constraints for PDEs and some of its applications can be found in [10, 12, 13, 38–40]. It is important to note that, up to now, this method has been very rarely used to construct exact solutions of nonlinear ordinary differential equations (only differential constraints of a special type were encountered [18, 41–43], see further Section 3.7). Below, a rather general formulation of the method of differential constraints with respect to ordinary differential equations will be given and many specific examples of its application for analyzing and solving problems of the second type will be considered. In doing so, it was possible to describe a simple algorithm that allows us to eventually arrive at algebraic equations with parameters when applying the method of differential constraints to ordinary differential equations of any order (this cannot be done in the case of partial differential equations).

The main idea of the method is to search for exact solutions of a complex (non-integrable) equation by means of a consistency analysis of this equation and an additional simpler (integrable) equation called the differential constraint.

The order of the differential constraint is determined by the order of the the highest derivative (involved in the differential constraint). Usually the order of differential constraint is less than the order of the equation under consideration (the simplest and most common are the first-order differential constraints). The equation under consideration and the differential constraint must contain a set of free parameters (and sometimes arbitrary functions) whose specific values are chosen by analyzing the consistency of the given equation and the differential constraint. After the consistency analysis, all the solutions obtained by integrating the differential constraint will also simultaneously be solutions of the original equation. This method allows us to find particular solutions of the original equation for certain values of the determining parameters.

First, for simplicity, we consider the autonomous ordinary differential equations of the form

$$F(y, y'_x, \dots, y_x^{(n)}; \mathbf{a}) = 0, \tag{10}$$

which do not depend explicitly on the independent variable x and include the vector of free parameters $\mathbf{a} = \{a_1, \dots, a_k\}$ (we recall that similar equations often arise in mathematical physics, see Section 1). For the equation (10), it is necessary to choose the first-order differential constraints of an autonomous type,

$$G(y, y'_x; \mathbf{b}) = 0, \tag{11}$$

depending on the vector of free parameters $\mathbf{b} = \{b_1, \dots, b_s\}$, $s \in \mathbb{N}$.

The successive differentiation of the equation (11) allows us to express the highest derivatives in terms of y and y'_x: $y_x^{(k)} = \varphi_k(y, y'_x; \mathbf{b})$. Substituting these expressions into the original equation (10), we arrive at the first-order equation

$$H(y, y'_x; \mathbf{a}, \mathbf{b}) = 0. \tag{12}$$

Eliminating the derivative y'_x from (11) and (12), we obtain the algebraic/transcendental equation

$$P(y; \mathbf{a}, \mathbf{b}) = 0. \tag{13}$$

Next, we seek the values of the parameters $\mathbf{a}$ and $\mathbf{b}$ for which the equation (13) is satisfied identically for any functions y (some restrictions on the components of the vector $\mathbf{a}$ may arise). After this, the parameter vector $\mathbf{b}$ is expressed in terms of the parameter vector $\mathbf{a}$, that is, $\mathbf{b} = \mathbf{b}(\mathbf{a})$, and is substituted into the differential constraint (11). The result is an ordinary differential equation of the first order

$$g(y, y'_x; \mathbf{a}) = 0 \qquad (g = G|_{\mathbf{b}=\mathbf{b}(\mathbf{a})}), \tag{14}$$

which is consistent with the original equation (10) (in other words, the original equation is a consequence of the equation (14) and therefore inherits all its solutions). Then, we find the general solution of the equation (14) (by solving with respect to the derivative, the equation (14) reduces to an equation with separable variables), which will also be an exact solution of the equation (10) under consideration.

Remark 3.1. If the first-order differential constraint is given in the explicit form $y'_x = h(y;\mathbf{b})$, then its successive differentiation

$$y''_{xx} = (y'_x)'_y y'_x = hh'_y, \quad y'''_{xxx} = (y''_{xx})'_y y'_x = h(hh'_y)'_y, \quad \ldots$$

allows one to express the highest derivatives in terms of y: $y_x^{(k)} = \varphi_k(y;\mathbf{b})$. Eliminating the derivatives in (10) with the help of these expressions and the differential constraint, we immediately arrive at an algebraic/transcendental equation of the form (13).

Remark 3.2. Instead of y'_x one can exclude the dependent variable y from (11) and (12). As a result, we obtain the algebraic/transcendental equation with respect to the derivative

$$Q(y'_x;\mathbf{a},\mathbf{b}) = 0. \tag{15}$$

Next, we seek the values of the parameters $\mathbf{a}$ and $\mathbf{b}$ for which the equation (15) is satisfied identically for any functions y'_x.

Remark 3.3. Any first-order differential constraint, that is consistent with the original ODE, is a particular integral of this ODE.

In a similar way, we can investigate non-autonomous ordinary differential equations (depending explicitly on x), for the construction of exact solutions of which one must use non-autonomous differential constraints of the first order (see also Section 3.3)

3.2. Examples of Differential Constraints of the First Order

The structural form of the nonlinearity of the differential constraint (11) can be chosen in many cases similar to the nonlinearity of the original equation (10) (but with other determining parameters).
Let us illustrate what was said on the specific examples of ODEs of the second, third, fourth and higher orders.

Example 1. Consider the second-order ordinary differential equation with exponential nonlinearity

$$y''_{xx} - ky'_x = a + be^{\lambda y}. \tag{16}$$

We note that equations of this type are encountered in the theory of chemical reactors, combustion theory, and mathematical biology[3].
We supplement the equation (16) with the first-order differential constraint

$$y'_x = \alpha + \beta e^{\mu y}, \tag{17}$$

which is a readily integrable equation with separable variables. The form of the right-hand side of the differential constraint (17) is chosen similar to the form of the right-hand side of the original equation (16).
The equation under consideration and the differential constraint include seven parameters a, b, k, α, β, λ, and μ. The purpose of further research is to determine the parameters of the differential constraint α, β, μ, which must be expressed in terms of a, b, k, λ. At the same time, the conditions that the parameters of the equation (16) must satisfy, are determined simultaneously.
Differentiating (17), and then replacing the first derivative by the right-hand side of (17), we find the second derivative

$$y''_{xx} = \beta\mu e^{\mu y} y'_x = \beta\mu e^{\mu y}(\alpha + \beta e^{\mu y}) = \alpha\beta\mu e^{\mu y} + \beta^2 \mu e^{2\mu y}. \tag{18}$$

Eliminating the first and second derivatives from the considered equation (16) by means of (17) and (18), after elementary transformations we obtain

$$-(a + k\alpha) + \beta(\alpha\mu - k)e^{\mu y} + \beta^2 \mu e^{2\mu y} - be^{\lambda y} = 0.$$

In order for this equality to be satisfied for all functions $y = y(x)$ we must put $\lambda = 2\mu$ and equate to zero the coefficients of the exponents with different powers. As a result, we obtain the simple system of algebraic equations:

$$\lambda = 2\mu, \quad a + k\alpha = 0, \quad \alpha\mu - k = 0, \quad \beta^2\mu - b = 0. \tag{19}$$

If the conditions (19) are satisfied, then the solutions of the equation (17) are also solutions of the more complicated equation (16). The determining system (19), consisting of four equations, includes seven parameters a, b, k, α, β, λ,

[3] Equations (16) and (21) describe exact traveling wave solutions of the Kolmogorov–Petrovskii–Piskunov equation $w_t = w_{zz} + f(w)$ for special types of the kinetic function $f(w)$. In this case, $w = y(x)$, where $x = z + kt$.

and μ. Three parameters b, k, λ of the original equation can be considered quite arbitrary and the other parameters are expressed in terms of them as follows:

$$a=-\frac{2k^2}{\lambda},\quad \alpha=\frac{2k}{\lambda},\quad \beta=\pm\sqrt{\frac{2b}{\lambda}},\quad \mu=\frac{\lambda}{2}. \tag{20}$$

where $b\lambda>0$. We see that for the consistency of the equations (16) and (17), the parameter of the original equation a must be connected in a certain way with two other parameters of the same equation, k and λ. In this case, it is possible to choose two families of the parameters (20) for the differential constraint (17), which give two different one-parameter solutions of the equations (16) and (17).

Integrating the differential constraint (17), which is the first-order ODE with separable variables, finally we obtain two exact solutions of the equation (16) with $a=-2k^2/\lambda$:

$$y=-\frac{2}{\lambda}\ln\left[C\exp(-kx)\mp\sqrt{\frac{b\lambda}{2k^2}}\right],$$

where C is an arbitrary constant.

Example 2. In a similar way, we can study the second-order equation with a power nonlinearity

$$y''_{xx}-ky'_x=ay+by^n. \tag{21}$$

We add the first-order differential constraint

$$y'_x=\alpha y+\beta y^m, \tag{22}$$

the right-hand side of which is similar to the right-hand side of the considered equation (21).

An analysis (similar to that in Example 1) shows that three parameters of the original equation, b, k, n, can be considered quite arbitrary, and the remaining parameters are expressed in terms of them as follows:

$$a=-\frac{2k^2(n+1)}{(n+3)^2},\quad m=\frac{n+1}{2},\quad \alpha=\frac{2k}{n+3},\quad \beta=\pm\sqrt{\frac{2b}{n+1}}, \tag{23}$$

where $k\neq-1$, $k\neq-3$, and $b(n+1)>0$. We see that for the consistency of the equations (21) and (22), the parameter of the original equation a must be connected in a certain way with two other parameters of the same equation, k and n. In this case, we can choose two families of the parameters (23) for the differential constraint (22), which give two different one-parameter solutions of the equations (21) and (22).

Integrating the differential constraint (22) and taking into account the formulas (23), finally we obtain two exact solutions of the equation (21) with $a=-\frac{2k^2(n+1)}{(n+3)^2}$:

$$y=\left[Ce^{\alpha(1-m)x}-(\beta/\alpha)\right]^{\frac{1}{1-m}},$$

where C is an arbitrary constant and the constants m, α, and β are expressed in terms of the parameters of the original equation by the formulas (23).

Example 3. Consider the third-order nonlinear equation

$$y'''_{xxx}=ay^4+by^2+c. \tag{24}$$

We add the first-order differential constraint

$$y'_x=\alpha y^2+\beta. \tag{25}$$

By using (25), we successively find the derivatives

$$\begin{aligned} y''_{xx}&=\alpha yy'_x=2\alpha y(\alpha y^2+\beta)=2\alpha^3y^3+2\alpha\beta y,\\ y'''_{xxx}&=(6\alpha^2y^2+2\alpha\beta)y'_x=(6\alpha^2y^2+2\alpha\beta)(\alpha y^2+\beta)=6\alpha^3y^4+8\alpha^2\beta y^2+2\alpha\beta^2. \end{aligned}$$

For the last equality to coincide with (24), the following relations must hold:

$$a=6\alpha^3,\quad b=8\alpha^2\beta,\quad c=2\alpha\beta^2.$$

Let us solve the first two equations with respect to α and β and substitute the obtained values in the last expression. We have

$$\alpha=\left(\frac{a}{6}\right)^{1/3},\quad \beta=\left(\frac{a}{6}\right)^{-2/3}\frac{b}{8},\quad c=\frac{3b^2}{16a}. \tag{26}$$

It follows that for such c the third-order equation (24) has a particular solution, determined by solving a first-order equation with separable coefficients (25), whose parameters are related to the parameters of the original equation by the first two relations (26).

Example 4. Consider the fourth-order nonlinear equation

$$y''''_{xxxx} = ay^n + by^{2n+3}. \tag{27}$$

We supplement it with a first-order differential constraint of the following form:

$$(y'_x)^2 = \alpha y^m + \beta. \tag{28}$$

Successively differentiating (28), we find the derivatives

$$\begin{aligned} y''_{xx} &= \tfrac{1}{2}\alpha m y^{m-1} \qquad \text{(after a reduction of } y'_x\text{)},\\ y'''_{xxx} &= \tfrac{1}{2}\alpha m(m-1)y^{m-2}y'_x,\\ y''''_{xxxx} &= \tfrac{1}{2}\alpha m(m-1)y^{m-2}y''_{xx} + \tfrac{1}{2}\alpha m(m-1)(m-2)y^{m-3}(y'_x)^2\\ &= \tfrac{1}{2}\alpha\beta m(m-1)(m-2)y^{m-3} + \tfrac{1}{4}\alpha^2 m(m-1)(3m-4)y^{2m-3}. \end{aligned} \tag{29}$$

A comparison of the right-hand side of the equation (27) and the right-hand side of the last equality (29) allows us to make the following conclusions about the consistency of the system (27)–(28).

1°. For any values of the parameters of the original equation (27), satisfying the inequality $b(n+2)(n+3)(3n+5) > 0$, we can find the parameters of the differential constraint (28) by the formulas

$$m = n+3, \quad \alpha = \pm 2\sqrt{\frac{b}{(n+2)(n+3)(3n+5)}}, \quad \beta = \frac{2a}{\alpha(n+1)(n+2)(n+3)}.$$

2°. If $b = 0$, $n = -\frac{5}{3}$, then we have

$$m = \frac{4}{3}, \quad \beta = -\frac{27a}{4\alpha}, \quad \alpha \neq 0 \text{ is an arbitrary constant.}$$

In this case, the solution of the equation (28) will depend on two arbitrary constants (the arbitrary constant α plays the role of an additional constant of integration).

Remark 3.4. We can obtain the general solution of the equation (27) for $b = 0$, $n = -\frac{5}{3}$ (see [9], p. 659).

Example 5. For the fourth-order equation with exponential nonlinearity

$$y''''_{xxxx} - cy''_{xx} = ae^{\lambda y} + be^{2\lambda y} \tag{30}$$

we can use the differential constraint

$$(y'_x)^2 = \alpha + \beta e^{\lambda y}. \tag{31}$$

The analysis shows that for any values of the parameters of the original equation (30), which are satisfied the condition $b\lambda > 0$, two families of the parameters of the differential constraint (31) can be found by the formulas

$$\alpha = \pm\frac{a}{\lambda^2}\left(\frac{3\lambda}{b}\right)^{1/2} + \frac{c}{\lambda^2}, \quad \beta = \pm\frac{2}{\lambda}\left(\frac{b}{3\lambda}\right)^{1/2}.$$

Here, either upper signs or lower signs are taken.

Example 6. The nonlinear $2n$ th-order equation of the form

$$y_x^{(2n)} + a[yy''_{xx} - (y'_x)^2] = b \tag{32}$$

admits the differential constraint

$$(y'_x)^2 = \alpha y^2 + \beta y + \gamma. \tag{33}$$

Successively differentiating (33), we have

$$y''_{xx} = \alpha y + \tfrac{1}{2}\beta, \ \ldots, \ y_x^{(2n)} = \alpha^n y + \tfrac{1}{2}\alpha^{n-1}\beta.$$

Substituting these relations into (32) and taking into account (33), we find the coefficients of the differential constraint:

$$\alpha \text{ is any}, \quad \beta = 2\alpha^n/a, \quad \gamma = (\alpha^{2n-1} - ab)/a^2.$$

Example 7. The differential equation

$$y_x^{(2n)} = yf\big(yy''_{xx} - (y'_x)^2\big),$$

where $f(\zeta)$ is an arbitrary function, admits the differential constraint (33) with $\beta = 0$.

Example 8. The nth-order nonlinear equation of the form

$$y_x^{(n)} = ae^{\lambda y} y_x^{(m)}, \quad 0 \le m < n, \tag{34}$$

where $y_x^{(0)} = y$, admits the first-order differential constraint

$$y'_x = be^{\mu y}. \tag{35}$$

The successive differentiation (35) yields $y_x^{(m)} = b^m \mu^{m-1}(m-1)!\, e^{m\mu y}$ with $m = 1, 2, \ldots$ Substituting the relations obtained in (34) and taking into account (35), finally we find the coefficients of the differential constraint

$$\mu = \frac{\lambda}{n}, \qquad b = \left[\frac{an^{n-1}}{\lambda^{n-1}(n-1)!}\right]^{\frac{1}{n}} \qquad \text{if} \quad m = 0;$$

$$\mu = \frac{\lambda}{n-m}, \quad b = \frac{\lambda}{n-m}\left[\frac{a(m-1)!}{(n-1)!}\right]^{\frac{1}{n-m}} \quad \text{if} \quad 1 \le m < n.$$

Remark 3.5. It follows from (35) that $y''_{xx}/(y'_x)^2 = \text{const}$. Therefore more general, than (34), a differential equation of the form

$$y_x^{(n)} = e^{\lambda y} f\big(y''_{xx}/(y'_x)^2\big) y_x^{(m)},$$

where $f(\zeta)$ is an arbitrary function, also admits the differential constraint (35).

The nonlinear ODEs and some other equations of the second and higher orders, considered in Examples 1–8, whose exact solutions can be found with the help of the first-order differential constraints, are presented in Table 1.

3.3. Differential Constraints of Arbitrary Order. General Consistency Method for an Overdetermined System of Two ODEs

In the general case, a differential constraint can be an ordinary differential equation of arbitrary order. Therefore, it becomes necessary to investigate the consistency of an overdetermined system of two ordinary differential equations. The general algorithm for analyzing such systems is described below.

1°. *The case of ODEs of the same order.* Consider first two ordinary differential equations of the same order

$$F_1(x, y, y'_x, \ldots, y_x^{(n)}) = 0, \tag{36}$$

$$F_2(x, y, y'_x, \ldots, y_x^{(n)}) = 0 \tag{37}$$

(here and in what follows, it is assumed that the equations depend on the free parameters, which are omitted for brevity). We exclude the highest derivative from them (it is considered that one of the equations under consideration can be solved with respect to the highest derivative $y_x^{(n)}$, and then we can substitute the obtained expression into another equation). We obtain the $(n-1)$ st-order equation

$$G_1(x, y, y'_x, \ldots, y_x^{(n-1)}) = 0. \tag{38}$$

We differentiate (38) with respect to x, and then, by means of the resulting equation, we exclude the derivative $y_x^{(n)}$ in any of the original equations (36) and (37). We arrive at the another $(n-1)$ st-order equation

$$G_2(x, y, y'_x, \ldots, y_x^{(n-1)}) = 0. \tag{39}$$

Thus, the analysis of two nth-order equations (36)–(37) reduces to the analysis of two $(n-1)$ st-order equations (38)–(39). Continuing in a similar manner to lower the order of the equations, finally we arrive at one algebraic/transcendental equation (since two first-order differential equations reduce to one algebraic equation).

No.	Differential equation	Differential constraint
1	$y''_{xx} = ay^n + by^{2n+1}$	$y'_x = \alpha + \beta y^{n+1}$
2	$y''_{xx} = ay^n + by^m$	$(y'_x)^2 = \alpha y^{n+1} + \beta y^{m+1} + \gamma$ if $n, m \neq -1$
3	$y''_{xx} = ae^{\lambda y} + be^{2\lambda y}$	$y'_x = \alpha + \beta e^{\lambda y}$
4	$y''_{xx} = ae^{\lambda y} + be^{\mu y} + c$	$(y'_x)^2 = \alpha e^{\lambda y} + \beta e^{\mu y} + cy + \gamma$
5	$y''_{xx} = a\cos(ky) + b\sin(ky)$	$(y'_x)^2 = \alpha\cos(ky) + \beta\sin(ky) + \gamma$
6	$y''_{xx} - ky'_x = ay + by^n$	$y'_x = \alpha y + \beta y^m$ with $m = \frac{1}{2}(n+1)$
7	$y''_{xx} - ky'_x = ay + by^n + cy^{2n-1}$	$y'_x = \alpha y + \beta y^n$
8	$y''_{xx} - ky'_x = ay^{n-1} + by^n + cy^{2n-1}$	$y'_x = \alpha + ky + \beta y^n$
9	$y''_{xx} - ky'_x = a + be^{\lambda y}$	$y'_x = \alpha + \beta e^{\mu y}$ with $\mu = \frac{1}{2}\lambda$
10	$y''_{xx} - ky'_x = a + be^{\lambda y} + ce^{2\lambda y}$	$y'_x = \alpha + \beta e^{\lambda y}$
11	$y''_{xx} - ky^{n-1}y'_x = ay + by^n + cy^{2n-1}$	$y'_x = \alpha y + \beta y^n$
12	$y''_{xx} - ke^{\lambda y}y'_x = ae^{\lambda y} + be^{2\lambda y}$	$y'_x = \alpha + \beta e^{\lambda y}$
13	$yy''_{xx} - k(y'_x)^2 = ay^n + by^m + c$	$(y'_x)^2 = \alpha y^n + \beta y^m + \gamma$ if $n, m \neq -1$
14	$y''_{xx} - k(y'_x)^2 = ae^{\lambda y} + be^{\mu y} + c$	$(y'_x)^2 = \alpha e^{\lambda y} + \beta e^{\mu y} + \gamma$
15	$y''_{xx} - k(y'_x)^2 = a\cos(ky) + b\sin(ky) + c$	$(y'_x)^2 = \alpha\cos(ky) + \beta\sin(ky) + \gamma$
16	$y'''_{xxx} = a + by^2 + cy^4$	$y'_x = \alpha + \beta y^2$
17	$y'''_{xxx} = ay^n + by^{2n+2} + cy^{3n+4}$	$y'_x = \alpha + \beta y^{n+2}$
18	$y'''_{xxx} = ae^{3\lambda y} + be^{2\lambda y} + ce^{\lambda y}$	$y'_x = \alpha e^{\lambda y} + \beta$
19	$y'''_{xxx} = (ay^n + b)y'_x$	$(y'_x)^2 = \alpha y^{n+2} + \beta y^2 + \gamma y + \delta$
20	$y'''_{xxx} = (ae^{\lambda y} + be^{\mu y} + c)y'_x$	$(y'_x)^2 = \alpha e^{\lambda y} + \beta e^{\mu y} + \gamma y^2 + \delta y + \sigma$
21	$y'''_{xxx} = [a\cos(\lambda y + \mu) + b]y'_x$	$(y'_x)^2 = \alpha\cos(\lambda y + \mu) + \beta y^2 + \gamma y + \delta$
22	$y''''_{xxxx} = ay^n + by^{2n+3}$	$(y'_x)^2 = \alpha y^{n+3} + \beta$
23	$y''''_{xxxx} = ae^{\lambda y} + be^{2\lambda y}$	$(y'_x)^2 = \alpha + \beta e^{\lambda y}$
24	$y''''_{xxxx} = a(y'_x)^4 + b(y'_x)^2 + c$	$(y'_x)^2 = \alpha + \beta e^{\mu y}$
25	$y''''_{xxxx} - cy''_{xx} = ae^{\lambda y} + be^{2\lambda y}$	$(y'_x)^2 = \alpha + \beta e^{\lambda y}$
26	$y''''_{xxxx} = a[yy''_{xx} - (y'_x)^2] + by + c$	$y'_x = \alpha + \beta y$
27	$y''''_{xxxx} = ayy''_{xx} + b(y'_x)^2 + cy^2 + dy + p$	$(y'_x)^2 = \alpha y^2 + \beta y + \gamma$
28	$y''''_{xxxx} = a(y''_{xx})^2 + by^2 + c$	$(y'_x)^2 = \alpha y^2 + \beta y + \gamma$
29	$y^{(n)}_x = ae^{\lambda y}$	$y'_x = be^{\mu y}$ with $\mu = \lambda/n$
30	$y^{(n)}_x = ae^{\lambda y}y'_x$	$y'_x = be^{\mu y}$ with $\mu = \lambda/(n-1)$
31	$y^{(n)}_x = a[yy''_{xx} - (y'_x)^2] + by + c$	$y'_x = \alpha + \beta y$
32	$y^{(2n)}_x = a(y''_{xx})^2 + byy''_{xx} + c(y'_x)^2 + dy^2 + py + q$	$(y'_x)^2 = \alpha y^2 + \beta y + \gamma$
33	$y^{(n)}_x = ae^{\lambda y}[y^{(m)}_x]^k$	$y'_x = be^{\mu y}$ with $\mu = \lambda/(n - km)$

Table 1. Some nonlinear differential equations with parameters and the corresponding first-order differential constraints, which make it possible to find their exact solutions.

Analysis of the algebraic equation obtained does not represent fundamental difficulties and is carried out also as it was done earlier in Sections 3.1 and 3.2 for the case of a first-order differential constraint.

2°. *The case of ODEs of different order*. Let two ordinary differential equations have different orders

$$F_1(x,y,y'_x,\dots,y_x^{(n)})=0, \tag{40}$$

$$F_2(x,y,y'_x,\dots,y_x^{(m)})=0, \tag{41}$$

where $m<n$. Then, differentiating the equation (41) $n-m$ times, we reduce the system (40)–(41) to a system of the form (36)–(37), in which both equations have the same order n.

It is important to note that an integrable differential constraint of any order, which is consistent with the original ODE, makes it possible to constructively build its particular solutions.

Remark 3.6. The cases of specifying the explicit form of the solution $y=f(x,\mathbf{a})$ with the varied parameter vector $\mathbf{a}=\{a_1,\dots,a_k\}$ (or the implicit form of the solution $f(x,y,\mathbf{a})=0$ with the parameter vector $\mathbf{a}$) can be treated as degenerate cases of zero-order differential constraint (41) for $m=0$. Therefore, the tanh-function method [20–23, 26, 34] and the exp-function method [27–33, 35, 36] and their various modifications and generalizations, as well as other direct methods of this type can be considered as degenerate particular cases of the method of differential constraints.

3.4. Examples of Differential Constraints of the Second and Higher Orders Containing Free Parameters

Example 9. Consider the fourth-order equation with quadratic nonlinearity

$$y''''_{xxxx}=a(y''_{xx})^2-by^2+c. \tag{42}$$

We add to (42) a linear differential constraint of the second order

$$y''_{xx}=\alpha y+\beta. \tag{43}$$

Differentiating (43) twice, we have $y''''_{xxxx}=\alpha^2y+\alpha\beta$. Eliminating the derivatives in (42) with the help of this expression and the equation (43), we obtain the quadratic equation with respect to y,

$$(a\alpha^2-b)y^2+\alpha(2a\beta-\alpha)y+c-\alpha\beta+a\beta^2=0,$$

which will be identically satisfied if its coefficients are equal to zero:

$$a\alpha^2-b=0,\quad \alpha-2a\beta=0,\quad c=\alpha\beta-a\beta^2.$$

Two parameters a, b of the original equation can be considered arbitrary, and the remaining constants are expressed in terms of them as follows:

$$c=\frac{b}{4a^2},\quad \alpha=\pm\sqrt{\frac{b}{a}},\quad \beta=\pm\frac{1}{2a}\sqrt{\frac{b}{a}}.$$

Example 10. Consider another fourth-order equation with quadratic nonlinearity

$$y''''_{xxxx}=a(y''_{xx})^2+b(y'_x)^2+c. \tag{44}$$

We add to (44) a nonlinear differential constraint of the second order

$$y''_{xx}=\alpha(y'_x)^2+\beta, \tag{45}$$

which is an autonomous equation that is integrable in quadratures. Successively differentiating (45), we find the derivatives

$$\begin{aligned}
y'''_{xxx}&=2\alpha y'_xy''_{xx}=2\alpha y'_x[\alpha(y'_x)^2+\beta]=2\alpha^2(y'_x)^3+2\alpha\beta y'_x,\\
y''''_{xxxx}&=[6\alpha^2(y'_x)^2+2\alpha\beta]y''_{xx}=[6\alpha^2(y'_x)^2+2\alpha\beta][\alpha(y'_x)^2+\beta]\\
&=6\alpha^3(y'_x)^4+8\alpha^2\beta(y'_x)^2+2\alpha\beta^2.
\end{aligned} \tag{46}$$

Substituting (45) and (46) into (44), we obtain the biquadratic equation with respect to the derivative

$$(6\alpha^3-a\alpha^2)(y'_x)^4+(8\alpha^2\beta-2a\alpha\beta-b)(y'_x)^2+2\alpha\beta^2-c=0,$$

which will be identically satisfied if we put

$$6\alpha^3-a\alpha^2=0,\quad 8\alpha^2\beta-2a\alpha\beta-b=0,\quad 2\alpha\beta^2-c=0.$$

Two parameters a, b of the original equation can be considered arbitrary, and the remaining constants are expressed in terms of them as follows:

$$c = 27a^{-3}b^2, \quad \alpha = \tfrac{1}{6}a, \quad \beta = -9a^{-2}b.$$

Example 11. The mn th-order equation with a quadratic nonlinearity

$$y_x^{(mn)} = a[y_x^{(n)}]^2 + byy_x^{(n)} + cy_x^{(n)} + dy^2 + ky + p \qquad (m \text{ is a positive integer}),$$

which generalizes the equation (42), can be investigated with the aid of the nth-order linear differential constraint

$$y_x^{(n)} = \alpha y + \beta.$$

3.5. Some Second-order Differential Constraints Containing Arbitrary Functions

We now present several differential constraints of the second order of a sufficiently general form, which are integrated in quadratures.

Differential constraint 1. An autonomous differential constraint of the second order of the form

$$y''_{xx} = f(y), \tag{47}$$

containing an arbitrary function $f(y)$, is equivalent to an autonomous differential constraint of the first order

$$(y'_x)^2 = \Psi(y), \tag{48}$$

where $\Psi(y) = 2\int f(y)\,dy + C$ and C is an arbitrary constant. This is proved by differentiating the expression (48) and comparing the resulting equation with the original differential constraint.
Taking this into account, the differential constraint of the second order (43) in Example 7 could be replaced by an equivalent differential constraint of the first order $(y'_x)^2 = \alpha y^2 + 2\beta y + \gamma$, where γ is an additional free parameter. However, in this case the differential constraint (43), in view of its linearity, is easier to integrate.

Differential constraint 2. A more general than (47), autonomous differential constraint of the second order

$$y''_{xx} = f(y)(y'_x)^2 + g(y), \tag{49}$$

containing two arbitrary functions $f(y)$ and $g(y)$, is equivalent to an autonomous differential constraint of the first order of the form (48), where

$$\Psi(y) = 2F(y)\int \frac{g(y)}{F(y)}\,dy + CF(y), \qquad F(y) = \exp\Big[2\int f(y)\,dy\Big],$$

and C is an arbitrary constant.
Taking this into account, the differential constraint of the second order (45) in Example 9 could be replaced by an equivalent differential constraint of the first order $(y'_x)^2 = \gamma e^{2\alpha y} - (\beta/\alpha)$, where γ is an additional free parameter.

Differential constraint 3. An autonomous differential constraint of the second order

$$y''_{xx} = f(y)(y'_x)^2 + g(y)y'_x, \tag{50}$$

containing two arbitrary functions $f(y)$ and $g(y)$, is equivalent to an autonomous differential constraint of the first order

$$y'_x = \Theta(y),$$

where

$$\Theta(y) = E(y)\int \frac{g(y)}{E(y)}\,dy + CE(y), \qquad E(y) = \exp\Big[\int f(y)\,dy\Big],$$

and C is an arbitrary constant.

3.6. Using Point Transformations in Combination with Differential Constraints

In some cases, it is first useful to reduce the ODE of interest, with a point transformation, to another equation (simpler or more convenient for investigation), which can then be analyzed using a suitable differential constraint. With this approach, solutions to the autonomous equation (10) are sought in the form

$$y = G(u; \mathbf{b}), \tag{51}$$

where G is a given function and $u = u(x)$ is a function satisfying the first-order differential equation (the differential constraint)

$$H(u, u'_x; \mathbf{c}) = 0. \tag{52}$$

The functions G and H in (51) and (52) depend on the vectors of free parameters $\mathbf{b}$ and $\mathbf{c}$.
The introduction of the new variable u, defined by the relation (51), reduces the equation (10) to a new ODE with one differential constraint (52); this creates the standard situation discussed in Section 3.1 (specific examples see also in Section 3.2).

Example 12. We consider an equation with nonlinearities of power type

$$y''_{xx} + (a_1 + a_2 y^{n-1}) y'_x = b_1 y + b_2 y^n + b_3 y^{2n-1}, \quad n \neq 1. \tag{53}$$

First, we make the change of variable $y = u^p$, with the exponent p to be determined. Then, on multiplying the result by u^{2-p}, we get

$$p u u''_{xx} + p(p-1)(u'_x)^2 + p(a_1 u + a_2 u^k) u'_x = b_1 u^2 + b_2 u^{k+1} + b_3 u^{2k}, \quad k = p(n-1) + 1. \tag{54}$$

Case 1. In order to obtain an equation with a quadratic nonlinearity, one must set $k = 0$, from where follows $p = \frac{1}{1-n}$. Thus, the change of variable $y = u^{\frac{1}{1-n}}$ reduces the equation (53) to the form

$$u u''_{xx} + s(u'_x)^2 + a_1 u u'_x + a_2 u'_x + b_1(n-1)u^2 + b_2(n-1)u + b_3(n-1) = 0, \quad s = \frac{n}{1-n}. \tag{55}$$

1.1. We add the linear differential constraint

$$u'_x = \alpha u + \beta \tag{56}$$

to the equation (55). Eliminating the derivatives in (55) with this expression, we obtain that the second-degree polynomial in u vanishes. Equating the coefficients of the polynomial to zero, we arrive at the system of three algebraic equations:

$$\begin{aligned} (s+1)\alpha^2 + a_1\alpha + b_1(n-1) &= 0, \\ (2s+1)\alpha\beta + a_2\alpha + a_1\beta + b_2(n-1) &= 0, \\ s\beta^2 + a_2\beta + b_3(n-1) &= 0. \end{aligned} \tag{57}$$

The first quadratic equation in the system (57) serves to determine α (in a wide range of the parameters a_1, b_1, and n, it has two distinct roots). Similarly, the last quadratic equation in (57) serves to determine β (in a wide range of the parameters a_2, b_3, and n, it has two distinct roots). Therefore, in the general case, the second equation in (57) determines four admissible values of the coefficient b_2, for which there exist solutions of the form $u = Ce^{\alpha x} - (\beta/\alpha)$ satisfying the differential constraint (56).

1.2. Other particular solutions of the equation (55) can be obtained with the help of the differential constraint

$$u'_x = \alpha u + \beta u^{1/2} + \gamma. \tag{58}$$

We use this relation to eliminate the derivatives from (55) to obtain an algebraic equation of the fourth degree for $\xi = u^{1/2}$. Equating its coefficients to zero results in the system consisting of five algebraic equations

$$\begin{aligned} (s+1)\alpha^2 + a_1\alpha + b_1(n-1) &= 0, \\ \beta\big[(2s + \tfrac{3}{2})\alpha + a_1\big] &= 0, \\ (s + \tfrac{1}{2})(\beta^2 + 2\alpha\gamma) + a_1\gamma + a_2\alpha + b_2(n-1) &= 0, \\ \beta\big[(2s + \tfrac{1}{2})\gamma + a_2\big] &= 0, \\ s\gamma^2 + a_2\gamma + b_3(n-1) &= 0. \end{aligned} \tag{59}$$

For $\beta = 0$, the differential constraint (58), up to notation, coincides with (56), therefore we further assume that $\beta \neq 0$.

From the second, third, and fourth equations (59), we find the coefficients of the differential constraint (58):

$$\alpha=-\frac{a_1}{2s+\frac{3}{2}},\quad \gamma=-\frac{a_2}{2s+\frac{1}{2}},\quad \beta=\pm\left[\frac{b_2(1-n)-a_1\gamma-a_2\alpha-2\alpha\gamma(s+\frac{1}{2})}{s+\frac{1}{2}}\right]^{1/2}.$$

The first and last equations (59) impose the following two restrictions on the coefficients of the equation (55):

$$b_1=\frac{a_1^2(s+\frac{1}{2})}{(n-1)(2s+\frac{3}{2})^2}=-\frac{2a_1^2(n+1)}{(n+3)^2},\quad b_3=\frac{a_2^2(s+\frac{1}{2})}{(n-1)(2s+\frac{1}{2})^2}=-\frac{2a_2^2(n+1)}{(3n+1)^2},$$

for which the relation $s=n/(1-n)$ is taken into account.

1.3. For $a_1=a_2=0$, we can add the nonlinear differential constraint

$$(u'_x)^2=\alpha u^2+\beta u+\gamma \tag{60}$$

to the equation (55). A simple analysis shows that the coefficients of the differential constraint (60) can be expressed in terms of the coefficients of the equation (55) as follows:

$$\alpha=b_1(1-n)^2,\quad \beta=\frac{2b_2(1-n)^2}{1+n},\quad \gamma=\frac{b_3(1-n)^2}{n}.$$

Remark 3.7. A differential constraint of the form (60) with $\alpha=0$ determines the quadratic solution

$$u=\frac{1}{4}\beta x^2+Cx+\frac{C^2-\gamma}{\beta},$$

where C is an arbitrary constant.

Case 2. We now set $k=2$ in the equation (54), which implies $p=\frac{1}{n-1}$, which corresponds to the substitution $y=u^{\frac{1}{n-1}}$. As a result, we arrive at the equation with a fourth-order nonlinearity

$$uu''_{xx}+c(u'_x)^2+a_1uu'_x+a_2u^2u'_x=b_1(n-1)u^2+b_2(n-1)u^3+b_3(n-1)u^4, \tag{61}$$

where $c=(2-n)/(n-1)$. The exact solutions of this equation can be found using the quadratic differential constraint

$$u'_x=\alpha u^2+\beta u+\gamma. \tag{62}$$

We use this relation to eliminate the derivatives from (61) and to obtain an algebraic equation of the fourth degree for u. Equating its coefficients to zero results in the system consisting of five algebraic equations

$$\begin{aligned}(c+2)\alpha^2+a_2\alpha&=b_3(n-1),\\(2c+3)\alpha\beta+a_1\alpha+a_2\beta&=b_2(n-1),\\(c+1)(\beta^2+2\alpha\gamma)+a_1\beta+a_2\gamma&=b_1(n-1),\\\gamma[(2c+1)\beta+a_1]&=0,\\c\gamma^2&=0.\end{aligned} \tag{63}$$

The two cases $c=0$ and $\gamma=0$ need to be considered; these correspond to solutions of the last equation.
Let us consider only the first case $c=0$, which corresponds to the value $n=2$. To simplify the calculations, we also set $a_2=0$. The coefficients of the differential constraint (62) are determined from the first, third, and fourth equations of (63):

$$\alpha=\pm\sqrt{b_3/2},\quad \beta=-a_1,\quad \gamma=\pm\frac{b_1}{\sqrt{2b_3}}. \tag{64}$$

The second equation of (63) defines the relation between the coefficients: $a_1\sqrt{2b_3}\pm b_2=0$ (either the upper or lower signs must be taken in all formulas). The desired solution is determined by integrating the separable equation (62) taking into account (64).

Case 3. On setting $k=3$ in (55), which implies $p=\frac{2}{n-1}$ and $y=u^{\frac{2}{n-1}}$, we can also look for a solution in the more complex form

$$u=\alpha_0+\alpha_1v+\alpha_2v^2,\quad v'_x=\beta_0+\beta_1v+\beta_2v^2.$$

The subsequent calculations are omitted here.

Example 13. Now, we consider an equation with nonlinearities of exponential type

$$y''_{xx} + (a_1 + a_2 e^{\lambda y}) y'_x = b_1 + b_2 e^{\lambda y} + b_3 e^{2\lambda y}. \tag{65}$$

Here, we make the change of variable $y = (\mu/\lambda)\ln u$, where the parameter μ to be determined. Then, on multiplying the result by λu^2, we get

$$\mu u u''_{xx} - \mu (u'_x)^2 + \mu (a_1 u + a_2 u^{\mu+1}) u'_x = b_1 \lambda u^2 + b_2 \lambda u^{\mu+2} + b_3 u^{2\mu+2}. \tag{66}$$

Denoting $\mu = k-1$ in (66), we obtain an equation of the form (54) with the same nonlinearities, but with different coefficients. Therefore, in order to construct exact solutions of the equation (66) the same differential constraints, as in Example 12, can be considered. In particular, setting $\mu = -1$ in (66) we have the equation with a quadratic nonlinearity

$$u u''_{xx} - (u'_x)^2 + (a_1 u + a_2) u'_x + b_1 \lambda u^2 + b_2 \lambda u + b_3 = 0, \tag{67}$$

which only differs from equation (55) in coefficients. Differential constraints (56), (58), and (60) allow one to find particular solutions to the equation (67) (details are omitted).

3.7. Using Several Differential Constraints

In some cases, several differential constraints, containing an additional required function, can be added to the equation under consideration. For definiteness, we return to the n th-order autonomous equation (10). We supplement it with two first-order differential constraints

$$y = G(u, u'_x; \mathbf{b}), \tag{68}$$
$$H(u, u'_x; \mathbf{c}) = 0, \tag{69}$$

where $\mathbf{b}$ and $\mathbf{c}$ are vectors of free parameters. Substituting (68) into (10), we obtain the $(n+1)$ st-order equation for the function $u = u(x)$:

$$F_1(u, u'_x, \ldots, u_x^{(n+1)}; \mathbf{a}, \mathbf{b}) = 0. \tag{70}$$

This equation, together with the differential constraint (69), is analyzed by the method described in Section 3.1. A small (but not fundamental) difference is that the order of the equation (70) is higher than the order of the original equation (10).

Example 14. In [41–43] (see also [18]) the differential constraint (69) was chosen in one of the following three types:

$$u'_x + u^2 - c_1 u - c_2 = 0, \tag{71}$$
$$(u'_x)^2 - 4u^3 - c_1 u^2 - c_2 u - c_3 = 0, \tag{72}$$
$$(u'_x)^2 - u^4 - c_1 u^3 - c_2 u^2 - c_3 u - c_4 = 0, \tag{73}$$

and the differential constraint (68) was chosen in the class of functions

$$y = \sum_{k=0}^{K} c_{1k} u^k + u'_x \sum_{l=0}^{L} c_{2l} u^l + \sum_{m=1}^{M} c_{3m} \left(\frac{u'_x}{u}\right)^m. \tag{74}$$

In (74), for the differential constraint (71) we assumed that $K = M$, $c_{2l} = 0$ $(l = 1, \ldots, L)$. As a result, a number of new exact solutions of nonlinear equations of the second, third, and fourth orders were obtained.

Remark 3.8. All the equations (71)–(73) are reduced to the equations with separable variables (their solutions are expressed in elementary functions or in quadratures). The solution of equation (72) can be expressed in terms of the Weierstrass function $\wp = \wp(z, g_2, g_3)$, and the solution of (73), in terms of the Jacobi elliptic function.

The differential constraints (68) and (69) can involve higher derivatives of u with respect to x.

Example 15. With the G'/G-expansion method [44–47], one looks for particular solutions to autonomous equations using two first- and second-order differential constraints of the special form[4]

$$y = \sum_{k=0}^{n} b_k \left(\frac{u'_x}{u}\right)^k, \tag{75}$$

$$u''_{xx} - c_1 u'_x - c_0 u = 0. \tag{76}$$

[4] In the original paper [44] and subsequent publications, the notation $u = G$ was used.

Differential constraints (75)–(76) can be simplified using the substitution $\xi = u'_x/u$. As a result, they are reduced to a point transformation of the polynomial type in combination with the first-order differential constraint of the Riccati type:

$$y = \sum_{k=0}^{n} b_k \xi^k,$$
$$\xi'_x + \xi^2 - c_1\xi - c_0 = 0.$$

In [48] it was shown that searching for particular solutions of an ODE by applying the G'/G-expansion method, based on the differential constraints (75)–(76) with $c_1^2 + 4c_0 > 0$, leads to the same results as the tanh method. For $c_1^2 + 4c_0 < 0$, the G'/G-expansion method is equivalent to the tan method.

4. Overdetermined Systems of ODEs Arising from a Generalized Separation of Variables in Nonlinear PDEs

4.1. Generalized Separation of Variables by the Splitting Method

The construction of exact solutions of nonlinear equations of mathematical physics with generalized or functional separation of variables using the splitting method [10, 12, 13] often leads to overdetermined systems of ordinary differential equations which can contain many free parameters that are not included in the original equation.

Let us briefly describe the main idea of the splitting method for equations that depend on two independent variables x and y. Let there be a functional-differential equation (reduced from a PDE) of the special form

$$\Phi_1\Psi_1 + \cdots + \Phi_n\Psi_n = 0, \tag{77}$$

where

$$\begin{aligned} &\Phi_i = \Phi_i(x, \varphi_1, \ldots, \varphi_m; \varphi'_1, \ldots, \varphi'_m; \varphi''_1, \ldots, \varphi''_m; \ldots), \quad \varphi_j = \varphi_j(x);\\ &\Psi_i = \Psi_i(y, \psi_1, \ldots, \psi_m; \psi'_1, \ldots, \psi'_m; \psi''_1, \ldots, \psi''_m; \ldots), \quad \psi_j = \psi_j(y); \end{aligned} \tag{78}$$

$i, j = 1, \ldots, m$; φ_j and ψ_j are the required functions, and the corresponding derivatives are denoted by the usual prime notation.

Functional-differential equation (77)–(78) can have solutions only when the quantities Φ_i and Ψ_i are related by linear relations of the form

$$\sum_i A_{pi}\Phi_i = 0, \quad \sum_i B_{qi}\Psi_i = 0, \quad 2 \le p, q \le n, \tag{79}$$

where A_{pi} and B_{qi} are some constants (for details see in [10, 12, 13]).

The relations (79) together with (78) are systems of ODEs for determining the functions φ_j and ψ_j (these systems are most often overdetermined, but can also be determined or underdetermined). In degenerate cases, the simplest relations of the form $\Phi_r = 0$ and $\Psi_s = 0$ ($1 \le r, s \le n$), corresponding to degenerate solutions, can also be added to (79).

To illustrate the appearance of the functional-differential equations of the form (77)–(78) and their solutions by the splitting method, we consider a nontrivial example of constructing exact solutions of a nonlinear PDE of hydrodynamic type by applying the method of generalized separation of variables.

Example 16. Consider the $2k$ th-order nonlinear PDE

$$w_y(\Delta w)_x - w_x(\Delta w)_y = c\Delta^k w, \quad \Delta w = w_{xx} + w_{yy}, \tag{80}$$

where $k \ge 2$ is a natural number. For $k = 2$, the equation (80) is derived from the two-dimensional Navier–Stokes equations by introducing the stream function w, with the subsequent elimination of pressure [49, 50] (in this case, c is the kinematic viscosity of the liquid).

We seek the exact separable solutions of the equation (80) in the form

$$w = \varphi(x) + \psi(y). \tag{81}$$

Substituting (81) into (80), we obtain the functional-differential equation

$$\psi'_y\varphi'''_{xxx} - \varphi'_x\psi'''_{yyy} = c\varphi_x^{(2k)} + c\psi_y^{(2k)}, \tag{82}$$

which can be conveniently represented in the form

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 + \Phi_4\Psi_4 = 0, \tag{83}$$

No.	Function $\varphi=\varphi(x)$	Function $\psi=\psi(y)$	Coefficients in (86)
1	$\varphi=a_2x^2+a_1x$	$\psi=b_2y^2+b_1y$	$A_1=A_2=A_3=A_4=0$
2	$\varphi=a_3x^3+a_2x^2+a_1x$	$\psi=0$	$A_1=A_2=A_3=0,\ A_4=6a_3/c$
3	$\varphi=a_2e^{\lambda x}+a_1cx$	$\psi=c\lambda^{2k-3}y$	$A_1=\lambda^{2k-1},\ A_2=-a_1\lambda^{2k-1},\ A_3=\lambda^2,\ A_4=-a_1\lambda^2$
4	$\varphi=a_2e^{\lambda x}-c\lambda^{2k-3}x$	$\psi=b_2e^{\lambda y}+c\lambda^{2k-3}y$	$A_1=A_4=\lambda^{2k-1},\ A_2=\lambda^{4k-4},\ A_3=\lambda^2$
5	$\varphi=a_2e^{\lambda x}+c\lambda^{2k-3}x$	$\psi=b_2e^{-\lambda y}+c\lambda^{2k-3}y$	$A_1=\lambda^{2k-1},\ A_2=-\lambda^{4k-4},\ A_3=\lambda^2,\ A_4=-\lambda^{2k-1}$

Table 2. Exact solutions of the overdetermined system of ODEs (86), where a_i, b_i, and λ are arbitrary constants; the additive integration constants of the functions φ and ψ are omitted.

where Φ_i are functions depending on x, and Ψ_i are functions depending on y:

$$\begin{aligned} &\Phi_1=\varphi_x^{(2k)}, \quad \Phi_2=\varphi'''_{xxx}, \quad \Phi_3=\varphi'_x, \quad \Phi_4=c;\\ &\Psi_1=c, \quad \Psi_2=-\psi'_y, \quad \Psi_3=\psi'''_{yyy}, \quad \Psi_4=\psi_y^{(2k)} \end{aligned} \tag{84}$$

(for clarity, the functions Φ_i are written out as the order of the derivative decreases).
It is easy to verify that the functional equation (83) can be satisfied if we put [10, 13]:

$$\begin{aligned} &\Phi_1=A_1\Phi_3+A_2\Phi_4, \quad \Phi_2=A_3\Phi_3+A_4\Phi_4,\\ &\Psi_3=-A_1\Psi_1-A_3\Psi_2, \quad \Psi_4=-A_2\Psi_1-A_4\Psi_2, \end{aligned} \tag{85}$$

where A_1, A_2, A_3, A_4 are arbitrary constants, and Φ_3, Φ_4, Ψ_1, Ψ_2 are arbitrary functions.
Substituting (84) into (85), we obtain an overdetermined system of ODEs for determining the functions φ and ψ:

$$\begin{aligned} &\varphi_x^{(2k)}=A_1\varphi'_x+A_2c, \quad \varphi'''_{xxx}=A_3\varphi'_x+A_4c,\\ &\psi'''_{yyy}=-A_1c+A_3\psi'_y, \quad \psi_y^{(2k)}=-A_2c+A_4\psi'_y. \end{aligned} \tag{86}$$

Since the system (86) consists of linear equations with constant coefficients, its analysis is elementary and is determined by solutions of simpler equations of the third order.
In Table 2, we give the exact solutions of the system (86) (we omit solutions that can be obtained from the solutions indicated by means of changing notation $x \leftrightarrows y$ and $w \to -w$). In the particular case $k=2$, the solutions described turn into the solutions obtained in [49].
There are also degenerate solutions of the functional equation (83), corresponding to $\Phi_1=\Phi_2=0$ and $\Psi_3=\Psi_4=0$, which give new solutions of the original PDE (80). In the second case, from (82) we have $\psi(y)=ay$, where a is an arbitrary constant. For the function $\varphi(x)$ we obtain the linear equation with constant coefficients $a\varphi'''_{xxx}=c\varphi_x^{(2k)}$, which admits a simple particular solution

$$\varphi(x)=b_3e^{\lambda x}+b_2x^2+b_1x, \quad \lambda=(a/c)^{1/(2k-3)},$$

where b_1, b_2, and b_3 are arbitrary constants.

4.2. Generalized Separation of Variables Using Invariant Subspaces

Systems of overdetermined ODEs with arbitrary parameters arise also in the construction of exact solutions of nonlinear equations of mathematical physics on the basis of the use of invariant linear subspaces of nonlinear differential operators.
Let us briefly describe the essence of the method [17], which allows us to construct exact solutions of nonlinear PDEs by finding invariant subspaces (see also [10, 12, 13], where this method is called the Titov–Galaktionov method).
We consider the nonlinear evolution equation

$$w_t=F[w], \tag{87}$$

where $F[w]$ is a nonlinear differential operator with respect to x:

$$F[w]\equiv F\big(w,w_x,\dots,w_x^{(m)}\big). \tag{88}$$

The finite-dimensional linear subspace

$$\mathscr{L}_n=\big\{\varphi_1(x),\dots,\varphi_n(x)\big\}, \tag{89}$$

formed by linear combinations of linearly independent functions $\varphi_1(x)$, ..., $\varphi_n(x)$, is said to be invariant with respect to the operator F if for arbitrary constants $C_1,\dots,C_n$ we have the equality

$$F\left[\sum_{i=1}^{n} C_i\varphi_i(x)\right] = \sum_{i=1}^{n} f_i(C_1,\dots,C_n)\varphi_i(x). \tag{90}$$

Let the linear subspace (89) be invariant with respect to the operator F. Then the equation (87) admits a solution with generalized separation of variables of the form

$$w(x,t) = \sum_{i=1}^{n} \psi_i(t)\varphi_i(x), \tag{91}$$

where the functions $\psi_1(t),\dots,\psi_n(t)$ are described by the autonomous system of ordinary differential equations

$$\psi_i' = f_i(\psi_1,\dots,\psi_n), \qquad i = 1,\dots,n, \tag{92}$$

and the prime denotes the derivative with respect to t.
The most complicated part of the described method consists in determining linear subspaces that are invariant with respect to a given nonlinear operator F. We show how this can be done with a specific example of independent interest.

Example 17. Consider the $2k$ th-order nonlinear evolution PDE

$$w_t = aw_x^{(2k)} + w_x^2 + bw^2 + cw, \quad k \geq 1. \tag{93}$$

In this case $F[w] = aw_x^{(2k)} + w_x^2 + bw^2 + cw$. We look for invariant subspaces in the form $\mathscr{L}_2 = \{1, \varphi(x)\}$. We have

$$F[C_1 + C_2\varphi(x)] = C_2 a\varphi_x^{(2k)} + C_2^2[(\varphi_x')^2 + b\varphi^2] + C_1^2 b + C_1 c + C_2(2C_1 b + c)\varphi. \tag{94}$$

In order that the right-hand side of the relation (94) has the form of the right-hand side in (90), we have to put

$$\begin{aligned} (\varphi_x')^2 + b\varphi^2 &= p_1 + p_2\varphi, \\ \varphi_x^{(2k)} &= p_3 + p_4\varphi, \end{aligned} \tag{95}$$

where p_1, p_2, p_3, p_4 are arbitrary constants.
Thus, the construction of a linear subspace that is invariant with respect to the given nonlinear operator $F[w] = aw_x^{(2k)} + w_x^2 + bw^2 + cw$, leads to an overdetermined system of two ODEs (95) with four arbitrary parameters p_m, which do not involve into the original equation (93). We use the method of investigation of such systems, described in Section 3.3.
Differentiating the first equation of the system (95), after simplifying by φ_x', we obtain the second-order linear equation

$$\varphi_{xx}'' = -b\varphi + \tfrac{1}{2}p_2. \tag{96}$$

Successively differentiating this equation and eliminating the second derivatives, we find

$$\varphi_x^{(2k)} = (-b)^{k-1}(-b\varphi + \frac{1}{2}p_2).$$

Eliminating, by means of this relation, the highest derivative in the second equation (95), we have $[(-b)^k - p_4]\varphi + \frac{1}{2}(-b)^{k-1}p_2 - p_3 = 0$. Equating the coefficient of the function φ to zero, we obtain the conditions on the coefficients

$$p_3 = \tfrac{1}{2}(-b)^{k-1}p_2, \quad p_4 = (-b)^k, \tag{97}$$

at which the overdetermined system (95) will be consistent.
To construct exact solutions of the first equation (95), it is first convenient to use its corollary, equation (96), and then determine the constant p_1 from the first equation (95). A simple analysis shows that the three cases are possible, which are summarized in Table 3.
In the first case in Table 3 (for $b = 0$) from the formulas (97) for $k \geq 2$ it follows that $p_3 = p_4 = 0$, which corresponds to the degenerate solution of the original equation (93) in the form of a quadratic polynomial with respect to x (whose coefficients depend on t).
For $b > 0$, taking into account equations (95), formulas (97), and the data given in line No. 2 in Table 3, we transform the relation (94) to the form

$$F[C_1 + C_2\varphi] = b(\alpha^2 + \beta^2)C_2^2 + bC_1^2 + cC_1 + [2bC_1 + a(-b)^k + c]C_2\varphi, \tag{98}$$

No.	Coefficient b	Function $\varphi = \varphi(x)$	Coefficients p_1 and p_2
1	$b = 0$	$\varphi = \alpha x^2 + \beta x + \gamma$	$p_1 = \beta^2 - 4\alpha\gamma$, $p_2 = 4\alpha$
2	$b = \lambda^2 > 0$	$\varphi = \alpha\cos(\lambda x) + \beta\sin(\lambda x)$	$p_1 = b(\alpha^2 + \beta^2)$, $p_2 = 0$
3	$b = -\lambda^2 > 0$	$\varphi = \alpha\exp(-\lambda x) + \beta\exp(\lambda x)$	$p_1 = 4b\alpha\beta$, $p_2 = 0$

Table 3. Exact solutions of the overdetermined system of ODEs (95), where α, β, and γ are arbitrary constants.

where $\varphi = \alpha\cos(\lambda x) + \beta\sin(\lambda x)$ and $\lambda = \sqrt{b}$. Therefore, for $b > 0$, the equation (93) admits an exact solution with a generalized separation of variables of the form

$$w = \psi_1(t) + \psi_2(t)[\alpha\cos(\lambda x) + \beta\sin(\lambda x)], \qquad \lambda = \sqrt{b}, \tag{99}$$

where α and β are arbitrary constants, and the functions $\psi_1(t)$ and $\psi_2(t)$ are described by the autonomous system of ordinary differential equations

$$\begin{aligned} \psi_1' &= b\psi_1^2 + b(\alpha^2 + \beta^2)\psi_2^2 + c\psi_1, \\ \psi_2' &= 2b\psi_1\psi_2 + [a(-b)^k + c]\psi_2. \end{aligned} \tag{100}$$

The case $b < 0$, corresponding to line No. 3 in Table 3, is considered in a similar way.

Remark 4.1. The nonlinear differential operator on the right-hand side of the equation (87) can also depend explicitly on the independent variables, $F[w] \equiv F\left(x, t, w, w_x, \dots, w_x^{(m)}\right)$. In this case, the right-hand side in the relation (90) should be replaced by $f_i(t, C_1, \dots, C_n)$, and the right-hand sides of the system (92), respectively, by $f_i(t, \psi_1, \dots, \psi_n)$.

From Remark 4.1, in particular, it follows that the equation (93) with variable coefficients $a = a(t)$ and $c = c(t)$ for $b > 0$ admits the exact solution (99), where the functions $\psi_1(t)$ and $\psi_2(t)$ are described by the non-autonomous system of ordinary differential equations (100), in which it is necessary to set $a = a(t)$ and $c = c(t)$.

The described method can also be used to construct exact solutions for some nonlinear delay PDEs.

Example 18. It can be shown that the $2k$ th-order nonlinear delay PDE

$$w_t = aw_x^{(2k)} + w_x^2 + bw^2 + c\bar{w}, \qquad \bar{w} = w(x, t - \tau)$$

with the delay time $\tau > 0$ (for $\tau = 0$ this equation is transformed into (93)) admits an exact solution of the form (99), where α and β are arbitrary constants, and the functions $\psi_1(t)$ and $\psi_2(t)$ are described by the system of ordinary differential equations with delay

$$\begin{aligned} \psi_1' &= b\psi_1^2 + b(\alpha^2 + \beta^2)\psi_2^2 + c\bar{\psi}_1, \\ \psi_2' &= 2b\psi_1\psi_2 + a(-b)^k\psi_2 + c\bar{\psi}_2, \end{aligned}$$

$\bar{\psi}_1 = \psi_1(t - \tau)$ and $\bar{\psi}_2 = \psi_2(t - \tau)$.

Generalized and functional separable solutions of other nonlinear delay PDEs can be found in [51–55].

Remark 4.2. Instead of w_t, in the left-hand side of the equation (87) there can be any linear differential operator with respect to t of the form $L[w] \equiv \sum\limits_{i=1}^{s} a_i(t) w_t^{(i)}$. In this case, the derivatives ψ_i' in the left-hand sides of the system (92) must be replaced by $L[\psi_i]$.

5. Brief Conclusions

Various situations are considered when exact solutions of nonlinear ODEs and PDEs are constructed by analyzing overdetermined systems of ordinary differential equations with parameters. The method of investigation of overdetermined systems of two ODEs of any order on consistency, which eventually leads to algebraic equations with parameters, is described. A general description of the method of differential constraints is given and many specific examples of applying this method for obtaining exact solutions of ODEs are considered. It is shown that the construction of exact generalized separable solutions of nonlinear PDEs by applying splitting method and the Titov–Galaktionov method can also lead to overdetermined ODEs with parameters. New exact solutions of several nonlinear partial differential equations of higher order are obtained.

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