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η-Einstein Solitons in a δ- Lorentzian Trans Sasakian Manifolds

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Abstract: The object of the present research is to study the δ-Lorentzian Trans-Sasakian manifolds addmitting the η-Einstein Solitons and gradient Einstein soliton. It is shown that a symmetric second order covariant tensor in a δ-Lorentzian Trans Sasakian manifold is a constant multiple of metric tensor. Also an example of η-Einstein soliton in 3-dimensional δ-Lorentzian Trans-Sasakian manifold is provided in the region where δ-Lorentzian Trans-Sasakian manifold expanding.

1. Introduction

Study of differentiable manifolds with Lorentzian metric is a natural and interesting topic in differential geometry. In [1], T. Ikawa and M. Erdogan studied Lorentzian Sasakian manifold. Lorentzian Kenmotsu manifold was introduced by Mihai et al. [2]. Also Lorentzian para contact manifolds were introduced by K. Matsumoto [3]. Trans Lorentzian para Sasakian manifolds have been used by H. Gill and K. K. Dube [4]. In [5] A. Yıldız et al. studied Lorentzian α-Sasakian manifold and Lorentzian β-Kenmotsu manifold studied by Funda et al. in [6]. After that in 2011 S. S Pujar and V. J. Khairnar [7] have initiated the study of Lorentzian Trans-Sasakian manifolds and studied the some basic results with some of its properties. Earlier to this, S. S. Pujar [8] has initiated the study of δ-Lorentzian α Sasakian manifolds and δ-Lorentzian β Kenmotsu manifolds . In [9] U. C. De also studied properties of curvatures in Lorentzian Trans-Sasakian manifolds.
The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relativity. In 1969, Takahashi [10] has introduced the notion of almost contact metric manifolds equipped with pseudo Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are known as (ε)-almost contact metric manifolds. The concept of (ε)-Sasakian manifolds was initiated by Bejancu and Duggal [11]. U. C. De and A. Sarkar [12] studied the notion of (ε)-Kenmotsu manifolds. S.S. Shukla and D. D. Singh [13] extended the study to (ε)-Trans-Sasakian manifolds with indefinite metric. Siddiqi et al. [14] also studied some properties of Indefinite trans-Sasakian manifolds which is closely related to this topic. The semi Riemannian manifolds has the index 1 and the structure vector field ξ is always a time like. This motivated the Thripathi et al. [15] to introduce (ε)-almost para contact structure where the vector filed ξ is space like or time like according as $(\varepsilon)=1$ or $(\varepsilon)=-1$. When M has a Lorentzian metric g, that is, a symmetric non degenerate $(0,2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case gives interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1-dimensional distribution such that g is negative defined on it. Hence odd dimensional manifold is able to have a Lorentzian metric. Inspired by the above results in 2014, S. M Bhati [16] studied the notion of δ-Lorentzian trans-Sasakian manifolds.

In 1982, R. Hamilton [17] introduced that the Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric that is they are stationary points of the Ricci flow is given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

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Definition 1.1. A Ricci soliton (g,V,λ) on a Riemannian manifold is defined by

$$\mathscr{L}_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, $\mathscr{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.
If the vector field V is the gradient of a potential function -ψ , where ψ is some smooth function $\psi : M \to \mathbb{R}$, then g is called a gradient Ricci soliton and equation (1.2) assumes the form

$$\nabla\nabla\psi = S + \lambda g. \tag{1.3}$$

It is well known that the quantity $a(g,\psi) := R + \left|\nabla\psi^2\right| - \psi$ must be constant on M and it is often called the *auxiliary constant*. When ψ is constant the gradient Ricci soliton is simply an *Einstein manifold*. Thus Ricci solitons are natural extensions of Einstein metrics, an Einstein manifold with constant potential function is called a trivial gradient Ricci soliton. Gradient Ricci solitons play an important role in Hamiltonian Ricci flow as they correspond to self-similar solutions, and often arise as singularity models. They are also related to smooth metric measure spaces, since equation (1.3) is equivalent to ∞-Bakry-Emery Ricci tensor $Ric\psi = 0$. In physics, a smooth metric space $(M,g,e^{\psi},dvol)$ with $Ric\psi = \lambda$g is called quasi-Einstein manifold. Therefore it is important to study geometry and topology of gradient Ricci solitons and their classifications.
In general one cannot expect potential function ψ to grow or decay linearly along all directions at infinity, because of the product property: the product of any two gradient steady Ricci solitons is also a gradient steady Ricci soliton. Consider for example (R,g,ψ), where g is the standard Euclidean metric, $\psi(x_1,x_2) = x_1$. ψ is constant along x_2 direction, so without additional conditions, ψ may not have linear growth at infinity.
On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics and control theory.
In 2016, G. Catino and L. Mazzieri introduced the notion of Einstein solitons [18], which generate self-similar solutions to Einstein flow

$$\frac{\partial g}{\partial t} = -2\left(S - \frac{scal}{2}g\right). \tag{1.4}$$

The interest in studying this equation from different points of view arises from concrete physical problems. On the other hand, gradient vector fields play a central role in Morse-Smale theory. In what follows, after characterizing the manifold of constant scalar curvature via the existence of η-Einstein solitons, we focus on the case when the potential vector field ξ is of gradient type i.e., $\xi = grad(f)$, for f a nonconstant smooth function on M, and give the Laplacian equation satisfied by f. Under certain assumptions, the existence of an η-Einstein soliton implies that the manifold is quasi-Einstein. Remark that quasi-Einstein manifolds arose during the study of exact solutions of Einstein field equations.

In 1925, H. Levy [19] obtained the necessary and sufficient conditions for the existence of such tensors. Later, R. Sharma [20] initiated the study of Ricci solitons in contact Riemannian geometry. After that, Tripathi [21], Nagaraja et al. [22] and others like C. S. Bagewadi et al. ([23], [24]) extensively studied Ricci soliton in almost (ε)-contact metric manifolds. In 2009, J. T. Cho and M. Kimura [25] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. Later η-Ricci solitons in (ε)-almost paracontact metric manifolds have been studied by A. M. Blaga et al. [26]. Recently, A. M. Blaga and various others authors also have been studied η-Ricci solitons in manifolds with different structures (*see* [27], [28], [29]), recently K. Venu et al. [30] study the η-Ricci solitns in Trans-Sasakian manifold. It is natural and interesting to study η-Einstein soliton in δ-Lorentzian tTrans-Sasakian manifolds not as real hypersurfaces of complex space forms but a special contact structures. In this paper we derive the condition for a 3 dimensional δ-Lorentzian Trans-Sasakian manifold as an η-Einstein soliton and derive expression for the scalar curvature. Moreover, in the last section studied the gradient Ricci soliton for a 3 dimensional δ-Lorentzian Trans-Sasakian manifolds.

2. Preliminaries

Let M be an δ-almost contact metric manifold equipped with δ-almost contact metric structure (ϕ,ξ,η,g,δ) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \eta\circ\phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X,\phi Y) = g(X,Y) + \delta\eta(X)\eta(Y), \quad \eta(X) = \delta g(X,\xi), \quad g(\xi,\xi) = -\delta, \tag{2.2}$$

for all $X,Y \in M$, where δ is $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\phi, \xi, \eta, g, \delta)$ on M is called the δ Lorentzian structure on M. If $\delta = 1$ and this is usual Lorentzian structure on M, the vector field ξ is the time like, that is M contains a time like vector field. In [31], Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He proved that they could be divided into three classes: (i)-homogeneous normal contact Riemannian manifolds with $c > 0$, (ii)-global Riemannian products of a line or a circle with a Kahler manifold of constant holomorphic sectional curvature if $c = 0$ and (iii)-a warped product space $R \times_f C^n$ if $c < 0$. It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure. The manifolds of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu [31] characterized the differential geometric properties of the manifolds of class (iii) which are nowadays called Kenmotsu manifolds[31].

In [32], Gray and Harvella introduced the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds. The class $C_6 \oplus C_5$ [33] coincides with the class of Trans-Sasakian structure of type (α, β). In fact, the local nature of the two sub classes, namely C_6 and C_5 of Trans-Sasakian structures are characterized completely.
An almost contact metric structure on M is called a Trans-Sasakian (see [34], [35]) if $(M \times R, J, G)$ belongs to the class W_4. If we set

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi(X) - f\xi, \eta(X)\frac{d}{dt}\right),$$

then J is the almost product structure on $M \times R$, that is $J^2 = I$, where X is any vector fields on M, f smooth functions on $M \times R$ and $\frac{d}{dt}$ is the basis vector field of R and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.3}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensure by the above discussion.
With the above literature we now define the δ-Lorentzian Trans-Sasakian manifolds M [16] as follows:

Definition 2.1. A δ-Lorentzian manifold M with structure $(\phi, \xi, \eta, g, \delta)$ is said to be δ-Lorentzian Trans-Sasakian manifold of type (α, β) if it satisfies the condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \delta\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X) \tag{2.4}$$

for any vector fields X and Y on M.

If $\delta = 1$, then the δ-Lorentzian Trans Sasakian manifold is the usual Lorentzian trans Sasakian manifold of type (α, β) [16]. δ-Lorentzian Trans Sasakian manifold of type $(0,0)$, $(0,\beta)$ $(\alpha,0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, the δ-Lorentzian Trans Sasakian manifolds reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds, respectively.
Form (2.4), we have

$$\nabla_X \xi = \delta\{-\alpha\phi(X) - \beta(X + \eta(X)\xi\}, \tag{2.5}$$

and

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta[g(X,Y) + \delta\eta(X)\eta(Y)]. \tag{2.6}$$

In a δ-Lorentzian Trans Sasakian manifold M, we have the following relations:

$$R(X,Y)\xi = (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{2.7}$$

$$+ \delta[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y]$$

$$S(X,\xi) = [((n-1)(\alpha^2 + \beta^2) - (\xi\beta)]\eta(X) + \delta((\phi X)\alpha) + (n-2)\delta(X\beta), \tag{2.8}$$

$$Q\xi = \delta(n-1)(\alpha^2 + \beta^2) - (\xi\beta))\xi + \delta\phi(grad\alpha) - \delta(n-2)(grad\beta), \tag{2.9}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.
Further in an δ-Lorentzian Trans-Sasakian manifold , we have

$$\delta\phi(grad\alpha) = \delta(n-2)(grad\beta), \tag{2.10}$$

and

$$2\alpha\beta - \delta(\xi\alpha) = 0. \tag{2.11}$$

Using (2.7) and (2.10), for constants α and β , we have

$$R(\xi,X)Y = (\alpha^2+\beta^2)[\delta g(X,Y)\xi - \eta(Y)X], \tag{2.12}$$

$$R(X,Y)\xi = (\alpha^2+\beta^2)[\eta(Y)X - \eta(X)Y], \tag{2.13}$$

$$\eta(R(X,Y)Z) = \delta(\alpha^2+\beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.14}$$

$$S(X,\xi) = [((n-1)(\alpha^2+\beta^2) - \delta(\xi\beta)]\eta(X), \tag{2.15}$$

$$Q\xi = [(n-1)(\alpha^2+\beta^2) - (\xi\beta)]\xi. \tag{2.16}$$

An important consequence of (2.5) is that ξ is a geodesic vector field.

$$\nabla_\xi \xi = 0. \tag{2.17}$$

For arbitrary vector field X, we have that

$$d\eta(\xi,X) = 0. \tag{2.18}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (2.13), we have

$$K_\xi = g(R(\xi,X),\xi,X) = (\alpha^2+\beta^2) - \delta(\xi\beta). \tag{2.19}$$

It follows from (2.19) that ξ-sectional curvature does not depend on X.

3. η-Einstein solitons on $(M,\phi,\xi,\eta,g,\delta)$

In the study of the η-Einstein soliton equation we will consider certain assumptions, one essential condition being $\nabla\xi = I_\xi(M) + \eta\otimes\xi$ which naturally arises in different geometry of δ-Lorentzian Trans-Sasakian manifolds.
An important geometrical object in studying Ricci solitons is a symmetric $(0,2)$- tensor field which is parallel with respect to the Levi-Civita connection.
Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇ that is $\nabla h = 0$. Applying the Ricci commutation identity [36]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{3.1}$$

we obtain the relation

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0. \tag{3.2}$$

Replacing $Z = W = \xi$ in (3.2) and using (2.7) and also use the symmetricness of h, we have

$$\begin{aligned} &2(\alpha^2+\beta^2)[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] + 2\delta[(Y\alpha)h(\phi X,\xi) - (X\alpha)h(\phi Y,\xi)] \\ &+ 2\delta[(Y\beta)h(\phi^2 X,\xi) - (X\beta)h(\phi^2 Y,\xi)] + 4\alpha\beta[\eta(Y)h(\phi X,\xi) - \eta(X)h(\phi Y,\xi)] = 0. \end{aligned} \tag{3.3}$$

Putting $X = \xi$ in (3.3) and by virtue of (2.1), we obtain

$$-2[(\delta\xi\alpha - 2\alpha\beta]h(\phi Y,\xi) + 2[(\alpha^2+\beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{3.4}$$

By using (2.11) in (3.4), we have

$$[(\alpha^2+\beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{3.5}$$

Suppose $(\alpha^2+\beta^2) - \delta(\xi\beta) \neq 0$, it results

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{3.6}$$

Now, we can call a regular δ-Lorentzian Trans-Sasakian manifold with $(\alpha^2+\beta^2) - \delta(\xi\beta) \neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of δ-Lorentzian Trans-Sasakian manifolds.
Differentiating (3.6) covariantly with respect to X, we have

$$\begin{aligned} (\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) + h(Y,\nabla_X\xi) = [\delta g(\nabla_X Y,\xi) + \delta g(Y,\nabla_X\xi)]h(\xi,\xi) \\ + \eta(Y)[(\nabla_X h)(Y,\xi) + 2h((\nabla_X\xi,\xi)]. \end{aligned} \tag{3.7}$$

By using the parallelism $\nabla h = 0$, $\eta(\nabla_X\xi) = 0$ and by the virtue of (3.6) in (3.7), we get

$$h(Y,\nabla_X\xi) = \delta g(Y,\nabla_X\xi)h(\xi,\xi).$$

Now using (2.5) in the above equation, we get

$$-\alpha h(Y,\phi X)+\beta\delta h(Y,X)=-\alpha g(Y,\phi X)h(\xi,\xi)+\beta\delta g(Y,X)h(\xi,\xi). \tag{3.8}$$

Replacing X by ϕX in (3.8) and after simplification, we get

$$h(X,Y)=\delta g(X,Y)h(\xi,\xi), \tag{3.9}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (3.6). Now by considering the above equations, we can give the conclusion:

Theorem 3.1. *Let $(M,\phi,\xi,\eta,g,\delta)$ be an δ-Lorentzian Trans Sasakian manifold with non-vanishing ξ-sectional curvature and endowed with a tensor field $h\in\Gamma(T_2^0(M))$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.*

Definition 3.2. Let $(M,\phi,\xi,\eta,g,\delta)$ be an δ-almost contact metric manifold. Consider the equation

$$\mathscr{L}_\xi g+2S+(2\lambda-scal)g+2\mu\eta\otimes\eta=0, \tag{3.10}$$

where $\mathscr{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g and λ and μ are real constants. For $\mu\neq 0$, the data (g,ξ,λ,μ) will be called η-Einstein soliton.

Remark 3.3. If the scalar curvature $scal$ of the manifold is constant, then the η-Einstein soliton $(g,\xi,\lambda,\frac{scal}{2},\mu)$ reduces to an η-Ricci soliton and, moreover, if $\mu=0$, to a Ricci soliton $(g,\xi,\lambda,\frac{scal}{2})$. Therefore, the two concepts of η-Einstein soliton [18] and η-Ricci soliton are distinct on manifolds of non-constant scalar curvature.

Writing $\mathscr{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X,Y)=-g(\nabla_X\xi,Y)-g(X,\nabla_X\xi)-(2\lambda-scal)g(X,Y)-2\mu\eta(X)\eta(Y), \tag{3.11}$$

for any $X,Y\in\chi(M)$.

Definition 3.4. The data (g,ξ,λ,μ) which satisfies the equation (3.10) is said to be η- *Einstein soliton* on M and it is called shrinking, steady or expanding according as $\lambda<0$, $\lambda=0$ or $\lambda>0$ respectively [25].

Now, from (2.5) , the equation (3.10) becomes:

$$S(X,Y)=-\left(\lambda-\frac{scal}{2}+\beta\delta\right)g(X,Y)+(\beta\delta-\mu)\eta(X)\eta(Y). \tag{3.12}$$

The above equations yields

$$S(X,\xi)=-\left(\lambda-\frac{scal}{2}+\mu\right)\eta(X) \tag{3.13}$$

$$QX=-\left(\lambda-\frac{scal}{2}+\beta\delta\right)X+(\beta\delta-\mu)\xi \tag{3.14}$$

$$Q\xi=-\left(\lambda-\frac{scal}{2}+\mu\right)\xi \tag{3.15}$$

$$r=\frac{2}{2+(2\lambda+1)n}(-(n-1)\beta\delta-\mu), \tag{3.16}$$

where r is the scalar curvature. Of the two natural situations regrading the vector field $V: V\in Span\{\xi\}$ and $V\perp\xi$, we investigate only the case $V=\xi$.

Our interest is in the expression for $\mathscr{L}_\xi g+2S+2\mu\eta\otimes\eta$. A direct computation gives

$$\mathscr{L}_\xi g(X,Y)=2\beta\delta[g(X,Y)+\eta(X)\eta(Y)]. \tag{3.17}$$

In a 3-dimensional δ-Lorentzian Trans-Sasakian manifold the Riemannian curvature tensor is given by

$$R(X,Y)Z=g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y \tag{3.18}$$

$$-\frac{r}{2}[g(Y,Z)X-g(X,Z)Y].$$

Putting $Z=\xi$ in (3.18) and using (2.7) and (2.8) for 3-dimensional δ-Lorentzian Trans-Sasakian manifold, we get

$$\begin{aligned}(\alpha^2+\beta^2)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y]\\ +\delta[(Y\alpha)\phi X-(X\alpha)\phi Y]+\delta[(Y\beta)\phi^2X-(X\beta)\phi^2Y]\\ =[(\alpha^2+\beta^2)-(\xi\beta)][\eta(Y)X-\eta(X)Y]\\ +\delta\eta(Y)QX-\delta\eta(X)QY-\delta[((\phi Y)\alpha)X+(Y\beta)X]\\ +\delta[((\phi X)\alpha)Y+(X\beta)Y].\end{aligned} \tag{3.19}$$

Again, putting $Y=\xi$ in the (3.19) and using (2.1) and (2.11), we obtain

$$QX=\left[\frac{r}{2}+(\xi\beta)-(\alpha^2+\beta^2)\right]X+\left[\frac{r}{2}+(\xi\beta)-3(\alpha^2+\beta^2)\right]\eta(X)\xi. \tag{3.20}$$

From (3.20), we have

$$\begin{aligned}S(X,Y)=\left[\frac{r}{2}+(\xi\beta)-(\alpha^2+\beta^2)\right]g(X,Y)\\ +\left[\frac{r}{2}+(\xi\beta)-3(\alpha^2+\beta^2)\right]\delta\eta(X)\eta(Y).\end{aligned} \tag{3.21}$$

Equation (3.21) shows that a 3-dimensional δ-Lorentzian Trans-Sasakian manifold is η-*Einstein*.
Next, we consider the equation

$$h(X,Y)=(\mathscr{L}_\xi g)(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y). \tag{3.22}$$

By Using (3.17) and (3.21) in (3.22), we have

$$\begin{aligned}h(X,Y)=\left[r-4(\alpha^2+\beta^2)+2\beta\delta\right]g(X,Y)\\ +\left[8(\alpha^2+\beta^2)-2\beta\delta-r\right]\delta\eta(X)\eta(Y)+2\mu\eta(X)\eta(Y).\end{aligned} \tag{3.23}$$

Putting $X=Y=\xi$ in (2.3), we get

$$h(\xi,\xi)=2[2\delta(\alpha^2+\beta^2)-2\mu]. \tag{3.24}$$

Now, (3.9) becomes

$$h(X,Y)=2[2\delta(\alpha^2+\beta^2)-2\mu]\delta g(X,Y). \tag{3.25}$$

From (3.22) and (3.25), it follows that g is an η-Ricci soliton. Therefore, we can state as:

Theorem 3.5. *Let $(M,\phi,\xi,\eta,g,\delta)$ be a 3-dimensional δ-Lorentzian Trans-Sasakian manifold, then $\left(g,\xi,\lambda,\frac{scal}{2}\mu\right)$ yields an η-Einstein soliton on M.*

Let V be point-wise collinear with ξ. i.e., $V=b\xi$, where b is a function on the 3-dimensional δ-Lorentzian Trans-Sasakian manifold. Then

$$g(\nabla_X b\xi,Y)+g(\nabla_Y b\xi,X)+2S(X,Y)+(2\lambda-scal)g(X,Y)+2\mu\eta(X)\eta(Y)=0$$

or

$$\begin{aligned}bg((\nabla_X\xi,Y)+(Xb)\eta(Y)+bg(\nabla_Y\xi,X)+(Yb)\eta(X)\\ +2S(X,Y)+(2\lambda-scal)g(X,Y)+2\mu\eta(X)\eta(Y)=0.\end{aligned}$$

Using (2.5), we obtain

$$\begin{aligned}bg(-\delta\alpha\phi X-\beta\delta(X+\eta(X)\xi,Y)+(Xb)\eta(Y)+bg(-\delta\alpha\phi Y-\beta\delta(Y+\eta(Y)\xi,X)\\ +(Yb)\eta(X)+2S(X,Y)+(2\lambda-scal)g(X,Y)+2\mu\eta(X)\eta(Y)=0.\end{aligned}$$

which yields

$$\begin{aligned}-2b\beta\delta g(X,Y)-2b\beta\delta\eta(X)\eta(Y)+(Xb)\eta(Y)\\ +(Yb)\eta(X)+2S(X,Y)+(2\lambda-scal)g(X,Y)+2\mu\eta(X)\eta(Y)=0.\end{aligned} \tag{3.26}$$

Replacing Y by ξ in (3.26), we obtain

$$(Xb)+(\xi b)\eta(X)+2[2(\alpha^2+\beta^2)-(\xi\beta)+\lambda-\frac{scal}{2}+\mu-2b\beta\delta]\eta(X). \tag{3.27}$$

Again putting $X=\xi$ in (3.27), we obtain

$$\xi b=-2(\alpha^2+\beta^2)+(\xi\beta)-\lambda-\frac{scal}{2}-\mu+2b\beta\delta.$$

Plugging this in (3.27), we get

$$(Xb)+2[2(\alpha^2+\beta^2)-(\xi\beta)+\lambda-\frac{scal}{2}+\mu-2b\beta\delta]\eta(X)=0,$$

or

$$db=-\left\{\lambda-\frac{scal}{2}+\mu-(\xi\beta)+2(\alpha^2+\beta^2)-2b\beta\delta\right\}\eta. \tag{3.28}$$

Applying d on (3.28), we get $\{\lambda-\frac{scal}{2}+\mu-(\xi\beta)+2(\alpha^2+\beta^2)-2b\beta\delta\}\,d\eta$. Since $d\eta\neq 0$ we have

$$\lambda-\frac{scal}{2}+\mu-(\xi\beta)+2(\alpha^2+\beta^2)-2b\beta\delta=0. \tag{3.29}$$

Equation(3.29) in (3.28) yields b as a constant. Therefore from (3.26), it follows that

$$S(X,Y)=-\left(\lambda-\frac{scal}{2}+2b\beta\delta\right)g(X,Y)+(2b\beta\delta-\mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for constant $2b\beta\delta$. This leads to the following:

Theorem 3.6. *If in a 3-dimensional δ-Lorentzian Trans-Sasakian manifold the metric g is an η-Einstein soliton and V is positive collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided $b\beta\delta$ is a constant.*

Taking $X=Y=\xi$ in (3.9) and (3.21) and comparing, we get

$$\lambda=\frac{scal}{2}-2(\alpha^2+\beta^2)-(\xi\beta)+\mu-2b\beta\delta=-2K_\xi+\frac{scal}{2}-\mu. \tag{3.30}$$

From (3.16) and (3.30), we obtain

$$r=scal+6(\alpha^2+\beta^2)-3(\xi\beta)-2\beta\delta+2\mu. \tag{3.31}$$

Since λ is a constant, it follows from (3.30) that K_ξ is a constant.

Theorem 3.7. *Let (g,ξ,μ) be an η-Einstein soliton in $(M,\phi,\xi,\eta,g,\delta)$ a 3-dimensional δ-Lorentzian Trans-Sasakian manifold. Then the scalar $\lambda-\frac{scal}{2}+\mu=-2K_\xi$, $r=6K_\xi+2\mu-3(\xi\beta)-2b\beta\delta+scal$.*

Remark 3.8. For $\mu=0$, (3.30) reduces to $\lambda=-2K_\xi+\frac{scal}{2}$, so Einstein soliton in a 3-dimensional δ-Lorentzian Trans-Sasakian manifold is shrinking.

Example 3.9. Consider the three dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3|z\neq 0\}$, where (x,y,z) are the cartesian coordinates in $\mathbb{R}^3$ and let the vector fields be given as follows:

$$e_1=\frac{e^x}{z^2}\frac{\partial}{\partial x},\quad e_2=\frac{e^y}{z^2}\frac{\partial}{\partial y},\quad e_3=\frac{-(\delta)}{2}\frac{\partial}{\partial z},$$

where e_1,e_2,e_3 are linearly independent at each point of M. Let g be the Riemannain metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=-\delta,\quad g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0,$$

where $\delta^2=1$ so that $\delta=\pm 1$.
Let η be the 1-form defined by $\eta(X)=\delta g(X,\xi)$ for any vector field X on M and ϕ be the (1,1) tensor field defined by

$$\phi(e_1)=e_2,\quad \phi(e_2)=-e_1,\quad \phi(e_3)=0.$$

Then by using the linearity of ϕ and g, we have $\phi^2X=X+\eta(X)\xi$, with $\xi=e_3$.
Further $g(\phi X,\phi Y)=g(X,Y)+\delta\eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3=\xi$, the structure

defines an (δ)-almost contact structure in $\mathbb{R}^3$.
Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y,Z) = Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])$$
$$-g(Y,[X,Z])+g(Z,[X,Y]),$$

which is know as Koszul's formula.

$$\nabla_{e_1}e_3 = -\frac{(\delta)}{z}e_1, \quad \nabla_{e_2}e_3 = -\frac{(\delta)}{z}e_2, \quad \nabla_{e_1}e_2 = 0,$$

using the above relation, for any vector X on M, we have $\nabla_X\xi = \delta[-\alpha\phi X-\beta(X+\eta(X)\xi)]$, where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence (ϕ,ξ,η,g) structure defines the δ-Lorentzian Trans-Sasakian structure in $\mathbb{R}^3$.
Here ∇ be the Levi-Civita connection with respect to the metric g, then using Koszul's formula we have

$$[e_1,e_2] = 0, \quad [e_1,e_3] = -\frac{(\delta)}{z}e_1, \quad [e_2,e_3] = -\frac{(\delta)}{z}e_2.$$

Since $g(e_1,e_2) = 0$. Thus we have

$$\nabla_{e_1}e_3 = -\frac{(\delta)}{z}e_1+e_2, \quad \nabla_{e_1}e_2 = 0$$
$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -\frac{(\delta)}{z}e_2, \quad \nabla_{e_2}e_3 = -\frac{(\delta)}{z}e_2-e_1$$
$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = -\frac{(\delta)}{z}e_1+e_2.$$

The manifold M satisfies (2.5) with $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence M is an δ-Lorentzian Trans-Sasakian manifold. Then the non-vanishing components of the curvature tensor fields are computed as follows:

$$R(e_1,e_3)e_3 = \frac{(\delta)}{z^2}e_1, \quad R(e_3,e_1)e_3 = -\frac{(\delta)}{z^2}e_1,$$
$$R(e_2,e_3)e_3 = \frac{(\delta)}{z^2}e_1, \quad R(e_3,e_2)e_3 = -\frac{(\delta)}{z^2}e_1.$$

From the above expression of the curvature tensor we can also obtain Ricci tensor

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = \frac{(\delta^2)}{z^2}$$

since $g(e_1,e_3) = g(e_1,e_2) = 0$.
Therefore, we have

$$S(e_i,e_i) = \frac{(\delta)}{z^2}g(e_i,e_i),$$

and the scalar curvature $scal = 3\frac{(\delta^2)}{z^2}$ for $i = 1,2,3$, and $\alpha = \frac{1}{z}$, $\beta = -\frac{1}{z}$. Hence M is also an $Einstein$ manifold. In this case, from (3.11), computed (e_i,e_i) as follows:

$$2[g(e_i,e_i)-\eta(e_i)\eta(e_i)]+2S(e_i,e_i)+(2\lambda-scal)g(e_i,e_i)+2\mu\eta(e_i)\eta(e_i) = 0$$

for all $i \in \{1,2,3\}$, and we have

$$2(1-\delta_{i3})+2\frac{\delta}{z^2}+(2\lambda-\frac{\delta}{z^2})+2\mu\delta_{i3} = 0$$

for all $i \in \{1,2,3\}$. Therefore $\lambda = \frac{(\delta)^2}{2z^2}-1$ and $\mu = -\frac{(\delta)^2}{2z^2}+\frac{1}{2}$, the data $(g,\xi,\lambda,\frac{scal}{2},\mu)$ is an η-Einstein soliton on (M,ϕ,ξ,η,g).

4. Gradient Einstein Ricci Solitons in 3-dimensional δ-Lorentzian Trans-Sasakian manifold

If the vector field V is the gradient of a potential function -ψ then g is called a gradient Einstein Ricci soliton [37] and (1.2) assumes the form

$$\nabla\nabla\psi = S + \left(\lambda - \frac{scal}{2}\right) g. \tag{4.1}$$

This reduces to

$$\nabla_Y D\psi = QY + \left(\lambda - \frac{scal}{2}\right) Y, \tag{4.2}$$

where D denotes the gradient operator of g. From (4.2) it follows

$$R(X,Y)D\psi = (\bar{\nabla}_X Q)Y - (\bar{\nabla}_Y Q)X. \tag{4.3}$$

Differentiating (3.20) we get

$$(\nabla_W Q)X = \frac{dr(W)}{2}(X - \eta(X)\xi)) - (\frac{r}{2} - 3(\alpha^2 + \beta^2))(\alpha(g(\phi W, X) \tag{4.4}$$

$$+ \beta\delta g(W,X) - \delta\beta\eta(X)\eta(W)) + \eta(X)\nabla_W \xi.$$

In (4.4) replacing $W = \xi$, we obtain

$$(\nabla_\xi Q)X = \frac{dr(\xi)}{2}(X - \eta(X)\xi)). \tag{4.5}$$

Then we have

$$g(\nabla_\xi Q)X - (\bar{\nabla}_X Q)(\xi,\xi) \tag{4.6}$$

$$= g(\frac{dr(\xi)}{2}(X - \eta(X)\xi,\xi)) = \frac{dr(\xi)}{2}(g(X,\xi) - \eta(X))) = 0.$$

Using (4.6) and (4.5), we obtain

$$g(R(\xi,X)D\psi,\xi) = 0. \tag{4.7}$$

From (2.12)

$$g(\bar{R}(\xi,Y)D\psi,\xi) = (\alpha^2+\beta^2)(g(Y,D\psi) - \eta(Y)\eta(D\psi)).$$

Using (4.7), we get

$$(\alpha^2+\beta^2)(g(Y,D\psi) - \eta(Y)\eta(D\psi)) = 0$$

$$(\alpha^2+\beta^2)(g(Y,D\psi) - \eta(Y)g(D\psi,\xi)) = 0,$$

or

$$(g(Y,D\psi) - g(Y,\xi)g(D\psi,\xi)) = 0,$$

which implies

$$D\psi = (\xi\psi)\xi, \qquad \text{since} \qquad \alpha^2+\beta^2 \neq -\delta(\xi\beta). \tag{4.8}$$

Using (4.8) and (4.4)

$$S(X,Y) + \left(\lambda - \frac{scal}{2}\right) g(X,Y) = g(\nabla_Y D\psi, X) = g(\nabla_Y(\xi\psi)\xi, X)$$

$$= (\xi\psi)g(\bar{\nabla}_Y\xi, X) + Y(\xi\psi)\eta(X)$$

$$= (\xi\psi)g(-\delta\alpha\phi Y - \delta\beta Y - \delta\beta\eta(Y)\xi, X) + Y(\xi\psi)\eta(X)$$

$$S(X,Y) + \left(\lambda - \frac{scal}{2}\right) g(X,Y) = -\delta\alpha(\xi\psi)g(\phi Y,X) - \delta\beta(\xi\psi)g(Y,X) \tag{4.9}$$

$$-\delta\beta(\xi\psi)\eta(Y)\eta(X) + Y(\xi\psi)\eta(X).$$

Putting $X = \xi$ in (4.9) and using (2.15) we get

$$\bar{S}(Y,\xi) + \left(\lambda - \frac{scal}{2}\right)\eta(Y) = Y(\xi\psi) = [\lambda + 2\delta\beta + 2(\alpha^2+\beta^2 - \delta(\xi\beta))]\eta(Y). \tag{4.10}$$

Interchanging X and Y in (4.9), we get

$$S(X,Y)+\left(\lambda-\frac{scal}{2}\right)g(X,Y)=-\delta\alpha(\xi\psi)g(Y,\phi X)-\delta\beta(\xi\psi)g(X,Y) \qquad (4.11)$$

$$-\delta\beta(\xi\psi)\eta(Y)\eta(X)+X(\xi\psi)\eta(Y).$$

Adding (4.9) and (4.11) we get

$$2S(X,Y)+(2\lambda-scal)g(X,Y)=-2\delta\beta(\xi\psi)g(X,Y)+Y(\xi\psi)\eta(X) \qquad (4.12)$$

$$-2\delta\beta(\xi\psi)\eta(X)\eta(Y)+X(\xi\psi)\eta(Y).$$

Using (4.10) in (4.12) we have

$$S(X,Y)+\left(\lambda-\frac{scal}{2}\right)g(X,Y)=-\delta\beta(\xi\psi)[g(X,Y)-\eta(X)\eta(Y)] \qquad (4.13)$$

$$+[\left(\lambda-\frac{scal}{2}\right)+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]\eta(X)\eta(Y).$$

Then using (4.2) we have

$$\nabla_Y D\psi=-\delta\beta(\xi\psi)(Y-\eta(Y)\xi) \qquad (4.14)$$

$$+[\left(\lambda-\frac{scal}{2}\right)+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]\eta(Y)\xi.$$

Using (4.14) we calculate

$$\begin{aligned} R(X,Y)D\psi &= \nabla_X\nabla_Y D\psi-\nabla_Y\nabla_X D\psi-\nabla_{[X,Y]}D\psi \\ &= -\delta\beta X(\xi\psi)Y+\delta\beta Y(\xi\psi)X \qquad (4.15) \\ &\quad -\delta\beta Y(\xi\psi)\eta(X)\xi+\delta\beta X(\xi\psi)\eta(Y)\xi \\ &+[\left(\lambda-\frac{scal}{2}\right)+\delta\beta+2(\alpha^2+\beta^2)-\delta(\xi\beta)](\nabla_X\eta)(Y)\xi-(\nabla_Y\eta)(X)\xi \\ &+[\left(\lambda-\frac{scal}{2}\right)+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]((\nabla_X\xi)\eta(Y)\xi-(\nabla_Y\xi)\eta(X)). \end{aligned}$$

Taking inner product with ξ in (4.15), we get

$$0=g((X,Y)D\psi,\xi)=2\delta\alpha[\left(\lambda-\frac{scal}{2}\right)+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]g(\phi Y,X). \qquad (4.16)$$

Thus we have $2\delta\alpha[\left(\lambda-\frac{scal}{2}\right)+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]=0$.

Now we consider the following cases:
Case (i) $\delta\alpha=0$, or
Case (ii) $[\left(\lambda-\frac{scal}{2}\right)+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]=0$,
Case (iii) $\alpha=0$ and $[\left(\lambda-\frac{scal}{2}\right)\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]=0$.

Case (i) If $\alpha=0$, the manifold reduces to a δ-Lorentzian β-Kenmotsu manifold.
Case (ii) Let $[\left(\lambda-\frac{scal}{2}\right)+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]=0$. If we use this in (4.10) we get $Y(\xi\psi)=-\delta\beta(\xi\psi)\eta(Y)$. Substitute this value in (4.12) we obtain

$$S(X,Y)+\left(\lambda-\frac{scal}{2}\right)g(X,Y)=-\delta\beta(\xi\psi)g(X,Y)-2\delta\beta\eta(X)\eta(Y). \qquad (4.17)$$

Now, contracting (4.17), we get

$$r+3\left(\lambda-\frac{scal}{2}\right)=-3\delta\beta(\xi\psi)-2\delta\beta, \qquad (4.18)$$

which implies

$$(\xi\psi)=\frac{r}{-3\delta\beta}+\frac{(2\lambda-scal)}{-2\delta\beta}+\frac{2}{-3}. \qquad (4.19)$$

If $r = constant$, then $(\xi\psi) = constant = k(say)$. Therefore from (4.8) we have $D\psi = (\xi\psi)\xi = k\xi$. Thus we can write this equation as

$$g(D\psi, X) = k\eta(X), \tag{4.20}$$

which means that $d\psi(X) = k\eta(X)$. Applying d to this, we get $kd\eta = 0$. Since $d\eta \neq 0$, we have $k = 0$. Hence we get $D\psi = 0$. This means that $\psi = constant$. Therefore equation (4.1) reduces to

$$S(X,Y) = 2(\alpha^2 + \beta^2 - \delta(\xi\beta))g(X,Y),$$

that is M is an *Einstein* manifold.
Case (iii) Using $\alpha = 0$ and $[(\lambda - \frac{scal}{2}) + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$. in (4.10) we obtain $Y(\xi\psi) = -\delta\beta(\xi\psi)\eta(Y)$. Now as in *Case (ii)* we conclude that the manifold is an *Einstein* manifold.
Thus we have the following :

Theorem 4.1. *If a 3-dimensional δ-Lorentzian Trans Sasakian manifold with constant scalar curvature admits gradient Einstein soliton, then the manifold is either a δ-Lorentzian-Kenmotsu manifold or an Einstein manifold provided $\alpha, \beta = constant$.*

In [38] it was proved that if a 3-dimensional compact connected Trans-Sasakian manifold is of constant curvature, then it is either α-Sasakian or β-Kenmotsu. Since for a 3-dimensional Riemannian manifold constant curvature and Einstein manifold are equivalent, therefore from the Theorem 3 in [38] we state the following:

Corollary 4.2. *If a compact 3-dimensional δ-Lorentzian Trans-Sasakian manifold with constant scalar curvature admits Ricci soliton, then the manifold is either δ-Lorentzian α-Sasakian or δ-Lorentzian β-Kenmotsu.*

Also in [38], authors proved that a 3-dimensional connected Trans-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants. Hence from Theorem 3 in [38] we obtain the following:

Corollary 4.3. *If a locally ϕ-symmetric 3-dimensional connected δ-Lorentzian Trans-Sasakian manifold its admits gradient Einstein soliton, then manifold is either δ-Lorentzian β-Kenmotsu or Einstein manifold provided $\alpha, \beta = constant$.*

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