


## Some Identities for $q$ -Leonardo Tessarines

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### Abstract

In this paper, we introduce the concept of  $q$ -Leonardo tessarines, which are defined using the  $q$ -integers. We then proceed to investigate the algebraic properties of these numbers, encompassing recurrence relations, generating functions, and several noteworthy identities. The study offers a thorough examination of the  $q$ -Leonardo tessarines and their implications within algebraic frameworks.

**Keywords:** Tessarine Number,  $q$ -Integer,  $q$ -Leonardo Number,  $q$ -Leonardo Tessarine.

### 1. Introduction

The study of mathematical structures that extend the concept of complex numbers has been a subject of extensive research, with the aim of generalising algebraic and geometric formulations. One such extension is constituted by the tessarine numbers, first introduced by Cockle [1-3]. These numbers belong to a larger class of hypercomplex numbers and exhibit interesting algebraic properties that make them useful in various applications, including physics and engineering.

Tessarines and bicomplex numbers are considered to be members of the family of complex number systems [8,14,19,20]. Both systems exhibit distinctive multiplication rules and physical readings, thereby classifying them as members of the family of hypercomplex number systems. The incorporation of a hyperbolic component within the Tessarine number system offers distinct advantages in domains such as geometric transformations and wave equations. The bicomplex number system has been shown to offer distinct advantages in the domains of complex analysis and quantum physics.

**Definition 1.1.** The tessarine  $T$  is defined as follows:

$$T = a + ib + jc + kd, \quad (1)$$

where  $a, b, c, d$  are real numbers and

$$i^2 = -1, j^2 = 1, ij = ji = k. \quad (2)$$

The multiplication process is executed through the utilization of tessarine units, as delineated in Table 1. It is noteworthy that this product is characterized by commutativity [1,3,10].

**Table 1.** The multiplication (denoted by juxtaposition) scheme for tessarine units

	1	$i$	$j$	$k$
1	1	$i$	$j$	$k$
$i$	$i$	-1	$k$	$-j$
$j$	$j$	$k$	1	$i$
$k$	$k$	$-j$	$i$	-1

The initial description of Leonardo tessarines is as follows:

$$T\mathcal{L}_n = Le_n + Le_{n+1}i + Le_{n+2}j + Le_{n+3}k \quad (3)$$

where  $Le_n, Le_{n+1}, Le_{n+2}, Le_{n+3}$  are Leonardo numbers.

The study of special number sequences has long been an area of interest in both theoretical and applied mathematics. Among these, the Leonardo numbers and their extension to Leonardo tessarines have garnered considerable attention due to their interesting algebraic properties and applications across various scientific domains. For a more thorough examination of the Leonardo numbers [21-24] and Leonardo tessarines, readers are directed to consult the references listed in the following section [4-10]. The  $q$ -calculus has been

studied a lot in many areas of mathematics, as well as in other subjects

**Definition 1.2.** For  $n \in \mathbb{N}_0$ ,  $q$ -integer is denoted by  $[n]_q$  and it is defined as

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots + q^{n-1}, q \neq 1. \quad (4)$$

By (3) for all  $m, n \in \mathbb{Z}$  can be written with ease

$$[m + n]_q = [m]_q + q^m [n]_q.$$

For further details pertaining to the quantum calculus, readers are directed to consult [11-13,15].

**Definition 1.3.** The Leonardo sequence [7], denoted by  $\{Le_n\}_{n=0}^\infty$ , is characterized by the following recurrence relation for  $n \geq 2$

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \quad (5)$$

and

$$Le_0 = Le_1 = 1.$$

The sequence can also be expressed as follows:

$$Le_{n+1} = 2Le_n - Le_{n-2}, n \geq 2. \quad (6)$$

Catarino and Borges [7] established various properties for Leonardo numbers, such as Cassini's identity, Catalan's identity, and d'Ocagne's identity. These properties highlight the deep connections between Leonardo numbers, Fibonacci numbers, and Lucas numbers. Among these relationships, two key connections are especially relevant to this study and will be discussed in detail later.

$$Le_n = 2F_{n+1} - 1, n \geq 0, \quad (7)$$

$$Le_n = L_{n+2} - F_{n+2} - 1, n \geq 0. \quad (8)$$

In recent years, various extensions of classical number sequences have been explored, opening up new avenues for research and applications. One such sequence is constituted by the Leonardo numbers, which have been studied in depth due to their interesting algebraic and combinatorial properties. In [7], the authors introduced the notion of incomplete Leonardo numbers, a sequence of integers with distinct properties and associated generating functions. This work has paved the way for a deeper understanding of how these numbers can be applied in different mathematical contexts. Further extending the concept of Leonardo numbers, Shannon defined generalized Leonardo numbers in [17].

In [4], additional properties of Leonardo numbers were investigated, enhancing the foundational understanding of their behavior in different mathematical settings.

Building on these classical studies, in [18], Aydın Torunbalcı introduced  $q$ -Fibonacci bicomplex numbers and  $q$ -Lucas bicomplex numbers, extending the well-known Fibonacci and Lucas sequences into the realm of bicomplex numbers. In [21], Beites and Catarino introduced and studied a new sequence of quaternions related to the Leonardo numbers, namely the Leonardo quaternion sequence. In [22], Moreira et al. investigated a generalization of Leonardo numbers. A historical overview of Leonardo numbers was presented in [23]. In [24], Shannon and Horadam explored the matrix representation of Leonardo numbers.

## 2. Materials and Methods

In this section, we introduce the concept of  $q$ -Leonardo tessarines, a specialized extension of the traditional tessarine numbers. These numbers are defined using the basis  $\{1, i, j, k\}$ , where the imaginary units  $i, j$ , and  $k$  satisfy the conditions (2) and identities (3), (4) and (7). The structure of  $q$ -Leonardo tessarines is a refinement of the classical tessarines.

**Definition 2.1.** The  $q$ -Leonardo tessarine  $TLe_n$  is defined as follows

$$TLe_n = (2\alpha^n [n+1]_q - 1) + (2\alpha^{n+1} [n+2]_q - 1)i \\ + (2\alpha^{n+2} [n+3]_q - 1)j + (2\alpha^{n+3} [n+4]_q - 1)k.$$

It can be demonstrated that

$$q = \frac{\beta}{\alpha} = \frac{-1}{\alpha^2},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the characteristic equation

$$x^2 - x - 1 = 0$$

for the Fibonacci number sequence. The following can be written:

$$TLe_n = 2\alpha^{n+1} \left( \frac{1-q^{n+1}}{\alpha-\alpha q} \right) \\ + 2\alpha^{n+2} \left( \frac{1-q^{n+2}}{\alpha-\alpha q} \right) i \\ + 2\alpha^{n+3} \left( \frac{1-q^{n+3}}{\alpha-\alpha q} \right) j \\ + 2\alpha^{n+4} \left( \frac{1-q^{n+4}}{\alpha-\alpha q} \right) k - A \quad (9)$$

where  $A = 1 + i + j + k$ .

The following recurrence relation corresponding to expressions (5) and (7) is presented herewith.

Subsequent theorems analogous to those presented in [14] can be posited.

**Theorem 2.1.** For  $n \in \mathbb{N}$ , the following equation could be write:

$$TLe_n = 2TF_{n+1} - A,$$

where  $TF_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$  is a Fibonacci tessarine and  $A = 1 + i + j + k$ .

**Proof.** Using relations (5) and (7) we obtain

$$\begin{aligned} TLe_n &= (2\alpha^n[n+1]_q - 1) \\ &\quad + (2\alpha^{n+1}[n+2]_q - 1)i \\ &\quad + (2\alpha^{n+2}[n+3]_q - 1)j \\ &\quad + (2\alpha^{n+3}[n+4]_q - 1)k \\ &= 2 \left( \frac{\alpha^{n+1}}{\alpha - (\alpha q)} (1 + \alpha i + \alpha^2 j + \alpha^3 k) \right) \\ &\quad - 2 \left( \frac{(\alpha q)^{n+1}}{\alpha - (\alpha q)} (1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k) \right) - A \\ &= \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} - A \end{aligned}$$

where  $\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$  and  $\hat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k$ .

So, the proof is completed.

## 2. Results and Discussion

In this section, we derive a Binet-like formula, establish exponential generating functions, and present key identities associated with the  $q$ -Fibonacci tessarine sequence. These results provide a deeper analytical framework for studying the sequence and its properties.

**Proposition 2.1.** The equality

$$\hat{\gamma}\hat{\delta} = \delta\hat{\gamma},$$

where  $\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$  and  $\hat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k$ , holds.

**Proof.** It can be easily verified.

**Theorem 2.2.** (Binet-like formula)

Let  $TLe_n$  denote the  $q$ -Leonardo tessarine. For  $n \geq 1$ , a Binet-like formula for these numbers, is given by:

$$TLe_n = \left( \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} \right) - A, \quad (10)$$

where

$$\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k, \quad \alpha = \frac{1 + \sqrt{5}}{2}$$

$$\hat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k, \quad \alpha q = \frac{-1}{\alpha}$$

$$A = 1 + i + j + k.$$

**Proof.** Substituting (4) and (9) into (3) results in the following conclusion:

$$\begin{aligned} TLe_n &= (2\alpha^n[n+1]_q - 1) \\ &\quad + i(2\alpha^{n+1}[n+2]_q - 1) \\ &\quad + j(2\alpha^{n+2}[n+3]_q - 1) \\ &\quad + ij(2\alpha^{n+3}[n+4]_q - 1) \\ &= \left( \frac{2\alpha^{n+1} - 2(\alpha q)^{n+1}}{\alpha - \alpha q} - 1 \right) \\ &\quad + i \left( \frac{2\alpha^{n+2} - 2(\alpha q)^{n+2}}{\alpha - \alpha q} - 1 \right) \\ &\quad + j \left( \frac{2\alpha^{n+3} - 2(\alpha q)^{n+3}}{\alpha - \alpha q} - 1 \right) \\ &\quad + k \left( \frac{2\alpha^{n+4} - 2(\alpha q)^{n+4}}{\alpha - \alpha q} - 1 \right) \\ &= \frac{2\alpha^{n+1}[1 + \alpha i + \alpha^2 j + \alpha^3 k]}{\alpha - (\alpha q)} \\ &\quad - \frac{[1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k]}{\alpha - (\alpha q)} \\ &\quad - (1 + i + j + k) \\ &= \left( \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} \right) - A. \end{aligned}$$

Thus, the proof is completed.

**Theorem 2.3.** (Exponential generating function)

$TLe_n$  is employed to denote the  $q$ -Leonardo tessarine. The exponential generating functions for these numbers can be expressed as follows:

$$g_{TLe_n} \left( \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} TLe_n \frac{t^n}{n!}$$

$$= 2 \left( \frac{\hat{\gamma} e^{\alpha t} - q \hat{\delta} e^{(\alpha q)t}}{1 - q} \right) - A e^t$$

where  $\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ ,  $\hat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k$  and  $A = 1 + i + j + k$ .

**Proof.** Substituting (10) into the equation yields

$$\sum_{n=0}^{\infty} TLe_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[ \left( \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} - A \right) \frac{t^n}{n!} \right]$$

$$= 2 \frac{\alpha \hat{\gamma}}{\alpha - \alpha q} \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!} - 2 \frac{(\alpha q) \hat{\delta}}{\alpha - \alpha q} \sum_{n=0}^{\infty} \frac{(\alpha q t)^n}{n!}$$

$$- A \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$= 2 \left( \frac{\hat{\gamma} e^{\alpha t} - q \hat{\delta} e^{(\alpha q)t}}{1 - q} \right) - A e^t$$

Consequently, the proof is completed.

**Theorem 2.4.** (Honsberger-like identity)

Let  $n$  and  $m$  be both greater than or equal to zero. A Honsberger-like identity for the  $q$ -Leonardo tessarines,  $TLe_n$  and  $TLe_m$ , is as follows:

$$TLe_n TLe_m + TLe_{n+1} TLe_{m+1}$$

$$= 4 \frac{\alpha^{n+m+2}}{(\alpha - \alpha q)^2} (1 + \alpha^2) \hat{\gamma}^2$$

$$+ 4 \frac{\alpha^{n+m+2}}{(\alpha - \alpha q)^2} (1 + (\alpha q)^2) (q^{n+m+2}) \hat{\delta}^2$$

$$- 4 \frac{\alpha^{n+m+2}}{(\alpha - \alpha q)^2} \hat{\gamma} \hat{\delta} (1 + \alpha^2 q) (q^{n+1} + q^{m+1})$$

$$- [TLe_n + TLe_{n+1}]A - [TLe_m + TLe_{m+1}]A - 2A^2$$

where  $\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ ,  $\hat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k$  and  $A = 1 + i + j + k$ .

**Proof.** The utilization of equation (10) yields the following result:

$$TLe_n TLe_m + TLe_{n+1} TLe_{m+1}$$

$$= (2TF_{n+1} - A)(2TF_{m+1} - A)$$

$$+ (2TF_{n+2} - A)(2TF_{m+2} - A)$$

$$= 4 \left( \frac{\alpha^{n+1}\hat{\gamma} - (\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} \right) \left( \frac{\alpha^{m+1}\hat{\gamma} - (\alpha q)^{m+1}\hat{\delta}}{\alpha - \alpha q} \right)$$

$$+ 4 \left( \frac{\alpha^{n+2}\hat{\gamma} - (\alpha q)^{n+2}\hat{\delta}}{\alpha - \alpha q} \right) \left( \frac{\alpha^{m+2}\hat{\gamma} - (\alpha q)^{m+2}\hat{\delta}}{\alpha - \alpha q} \right)$$

$$- A \left( \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} \right)$$

$$- A \left( \frac{2\alpha^{m+1}\hat{\gamma} - 2(\alpha q)^{m+1}\hat{\delta}}{\alpha - \alpha q} \right)$$

$$- A \left( \frac{2\alpha^{n+2}\hat{\gamma} - 2(\alpha q)^{n+2}\hat{\delta}}{\alpha - \alpha q} \right)$$

$$- A \left( \frac{2\alpha^{m+2}\hat{\gamma} - 2(\alpha q)^{m+2}\hat{\delta}}{\alpha - \alpha q} \right)$$

$$= 4 \frac{\alpha^{n+m+2}}{(\alpha - \alpha q)^2} (1 + \alpha^2) \hat{\gamma}^2$$

$$+ 4 \frac{\alpha^{n+m+2}}{(\alpha - \alpha q)^2} (1 + (\alpha q)^2) (q^{n+m+2}) \hat{\delta}^2$$

$$- 4 \frac{\alpha^{n+m+2}}{(\alpha - \alpha q)^2} \hat{\gamma} \hat{\delta} (1 + \alpha^2 q) (q^{n+1} + q^{m+1})$$

$$- [TLe_n + TLe_{n+1}]A - [TLe_m + TLe_{m+1}]A - 2A^2$$

In this particular instance, the notation  $\hat{\gamma} \hat{\delta} = \hat{\delta} \hat{\gamma}$  is utilized. Consequently, the proof is completed.

**Theorem 2.5.** (d'Ocagne-like identity)

Let  $n$  and  $m$  be both greater than or equal to zero. A d'Ocagne-like identity for the  $q$ -Leonardo tessarines, denoted by  $TLe_n$ , is as follows:

$$TLe_m TLe_{n+1} - TLe_{m+1} TLe_n$$

$$= 4 \left[ \frac{\alpha^{n+m+1} (q^{n+1} - q^{m+1}) \hat{\gamma} \hat{\delta}}{1 - q} \right]$$

$$+ A(TLe_{m+1} - TLe_m) + A(TLe_n - TLe_{n+1})$$

where  $\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ ,  $\hat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k$  and  $A = 1 + i + j + k$ .

**Proof.** The utilization of equation (10) yields the following result:

$$TLe_m TLe_{n+1} - TLe_{m+1} TLe_n$$

$$= (2TF_{m+1} - A)(2TF_{n+2} - A)$$

$$- (2TF_{m+2} - A)(2TF_{n+1} - A)$$

$$\begin{aligned}
 &= 4 \left( \frac{\alpha^{m+1}\hat{\gamma} - (\alpha q)^{m+1}\hat{\delta}}{\alpha - \alpha q} \right) \left( \frac{\alpha^{n+2}\hat{\gamma} - (\alpha q)^{n+2}\hat{\delta}}{\alpha - \alpha q} \right) \\
 &\quad - 4 \left( \frac{\alpha^{m+2}\hat{\gamma} - (\alpha q)^{m+2}\hat{\delta}}{\alpha - \alpha q} \right) \left( \frac{\alpha^{n+1}\hat{\gamma} - (\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} \right) \\
 &+ A \left[ \frac{2\alpha^{m+2}\hat{\gamma} - 2(\alpha q)^{m+2}\hat{\delta}}{\alpha - \alpha q} - \frac{2\alpha^{m+1}\hat{\gamma} - 2(\alpha q)^{m+1}\hat{\delta}}{\alpha - \alpha q} \right] \\
 &+ A \left[ \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} - \frac{2\alpha^{n+2}\hat{\gamma} - 2(\alpha q)^{n+2}\hat{\delta}}{\alpha - \alpha q} \right] \\
 &= 4 \left[ \frac{\alpha^{n+m+1}(q^{n+1} - q^{m+1})\hat{\gamma}\hat{\delta}}{(1-q)} \right] + A(TLe_{m+1} - TLe_m) \\
 &\quad + A(TLe_n - TLe_{n+1}).
 \end{aligned}$$

The proof is concluded.

**Theorem 2.6.** (Cassini-like identity)

For the  $q$ -Leonardo tessarines  $TLe_n$ , a Cassini-like identity is as follows:

$$\begin{aligned}
 &TLe_{n+1}TLe_{n-1} - TLe_n^2 \\
 &= 4 \left( \frac{\alpha^{2n}q^{n+1}(1 - q^{-1})\hat{\gamma}\hat{\delta}}{(1-q)} \right) \\
 &\quad + A(TLe_n - TLe_{n+1}) + A(TLe_n - TLe_{n-1})
 \end{aligned}$$

where  $n \geq 1$ ,  $\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ ,  $\hat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k$  and  $A = 1 + i + j + k$ .

**Proof.** The utilization of equation (10) results in the following conclusion:

$$\begin{aligned}
 &TLe_{n+1}TLe_{n-1} - TLe_n^2 = (2TF_{n+2} - A)(2TF_n - A) \\
 &\quad - (2TF_{n+1} - A)(2TF_{n+1} - A) \\
 &= 4 \left( \frac{\alpha^{n+2}\hat{\gamma} - (\alpha q)^{n+2}\hat{\delta}}{\alpha - \alpha q} \right) \left( \frac{\alpha^n\hat{\gamma} - (\alpha q)^\delta}{\alpha - \alpha q} \right) \\
 &\quad - 4 \left( \frac{\alpha^{n+1}\hat{\gamma} - (\alpha q)^{n+1}\hat{\delta}}{(\alpha - \alpha q)} \right)^2 \\
 &\quad + A \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}}{(\alpha - \alpha q)} \\
 &\quad - A \frac{2\alpha^{n+2}\hat{\gamma} - 2(\alpha q)^{n+2}\hat{\delta}}{(\alpha - \alpha q)} \\
 &\quad + A \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}\hat{\delta}}{(\alpha - \alpha q)}
 \end{aligned}$$

$$\begin{aligned}
 &-A \frac{2\alpha^n\hat{\gamma} - 2(\alpha q)^n\hat{\delta}}{(\alpha - \alpha q)} \\
 &= 4 \left[ \frac{\alpha^{2n}q^{n+1}(1 - q^{-1})\hat{\gamma}\hat{\delta}}{(1-q)} \right] \\
 &\quad + A(TLe_n - TLe_{n+1}) \\
 &\quad + A(TLe_n - TLe_{n-1})
 \end{aligned}$$

We utilize the fundamental property that  $\hat{\gamma}\hat{\delta} = \hat{\delta}\hat{\gamma}$ . This completes the proof.

**Theorem 2.7.** (Catalan-like identity)

For the  $q$ -Leonardo tessarines  $TL_n$ , Catalan's identity is written as follows:

$$\begin{aligned}
 &TL_{n+r}TL_{n-r} - TL_n^2 \\
 &= 4 \left[ \frac{\alpha^{2n}q^{n+1}(1 - q^r)(1 - q^{-r})\hat{\gamma}\hat{\delta}}{(1-q)^2} \right] \\
 &\quad + A(TL_{n+1} - TL_{n+r+1}) \\
 &\quad + A(TL_{n+1} - TL_{n-r+1})
 \end{aligned}$$

where  $n \geq r$ ,  $\hat{\gamma} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ ,  $\hat{\delta} = 1 + (\alpha q)i + (\alpha q)^2 j + (\alpha q)^3 k$  and  $A = 1 + i + j + k$ .

**Proof.** We get this by using (10):

$$\begin{aligned}
 &TL_{n+r}TL_{n-r} - TL_n^2 \\
 &= (2TF_{n+r+1} - A)(2TF_{n-r+1} - A) \\
 &\quad - (2TF_{n+1} - A)(2TF_{n+1} - A) \\
 &= 4 \left( \frac{\alpha^{n+r+1}\hat{\gamma} - (\alpha q)^{n+r+1}\hat{\delta}}{\alpha - \alpha q} \right) \\
 &\quad \cdot \left( \frac{\alpha^{n-r+1}\hat{\gamma} - (\alpha q)^{n-r+1}\hat{\delta}}{\alpha - \alpha q} \right) \\
 &\quad - 4 \left( \frac{\alpha^{n+1}\hat{\gamma} - (\alpha q)^{n+1}\hat{\delta}}{(\alpha - \alpha q)} \right)^2 \\
 &\quad + A \left( \frac{2\alpha^{n+1}\hat{\gamma} - 2(\alpha q)^{n+1}\hat{\delta}}{\alpha - \alpha q} \right) \\
 &\quad - A \left( \frac{2\alpha^{n-r+1}\hat{\gamma} - 2(\alpha q)^{n-r+1}}{\alpha - \alpha q} \right) \\
 &= \frac{4}{(1-q)} \hat{\gamma}\hat{\delta} [\alpha^{2n}q^{n+1}(1 - q^{-r})(1 - q^r)] \\
 &\quad + A(TLe_{n+1} - TLe_{n+r+1}) + A(TLe_{n+1} - TLe_{n-r+1}).
 \end{aligned}$$



This completes the proof.

### 3. Conclusion

In this study, the classical theory of Leonardo numbers has been extended to the tessarine (hyperbolic quaternionic) framework through the incorporation of  $q$ -calculus. By utilizing the algebraic properties of tessarines and the analytical tools of  $q$ -calculus, we have derived recurrence relations, generating functions, and closed-form expressions for these sequences. These findings contribute to the broader development of  $q$ -special functions and provide a novel perspective in hypercomplex analysis.

The results presented herein not only enhance the theoretical understanding of tessarine-based number sequences but also offer a foundation for future research in non-commutative algebra, quantum mathematics, and discrete dynamical systems. Potential applications include signal processing, cryptographic algorithms, and the modeling of complex physical phenomena. Future studies may further explore the interplay between tessarine structures and other hypercomplex algebras, leading to new mathematical insights and practical implementations.

### Author's Contributions

**Ali Atasoy:** The author carried out all aspects of this research, including the conceptualization, formal analysis, and the writing and editing of the manuscript.

### Ethics

There are no ethical issues after the publication of this manuscript.

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