

On a New Type of q -Baskakov Operators

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(Alınış / Received: 07.09.2017, Kabul / Accepted: 23.01.2018, Online Yayınlanma / Published Online: 29.03.2018)

Keywords
 q -analysis,
Korovkin's theorem,
Baskakov operators

Abstract: In this work, we have introduced a new type of q -analogous of Baskakov Operators. Their respective formulae for central moments are thereby obtained. The approximation properties and the approximation rapid of the sequences of the operators which are defined have been established in terms of the modulus of smoothness.

q -Baskakov Operatörlerinin Yeni bir Tipi Üzerine

Anahtar Kelimeler
 q -analiz,
Korovkin teoremi,
Baskakov operatörleri

Özet: Bu çalışmada, q -Baskakov operatörlerinin yeni bir türü tanıtılmıştır. Merkezi momentler için formüller elde edilmiştir. Tanımlanan q -Baskakov operatörlerinin dizilerinin yaklaşım özellikleri ve yakınsama oranı, süreklilik modülü yardımıyla belirlenmiştir.

1. Introduction

First, let us provide some background information on elements of q -analysis and on their well known formulas which are studied first by Euler in the eighteenth century. Following this, many interesting results in this field were obtained in the beginning of nineteenth century by F. H. Jackson [1] who introduced q -functions. He also developed q -calculus in a systematic way. Below, we give the concepts of q -integers, q -factorials, q -binomial coefficients, and q -derivative. The definitions used in this study are based on terminology and notations as seen in books [2], [3], and [4].

Let $q > 0$. The required q -Calculus theorems and definitions are as outlined below. The q -analogue of the integer $m \in \mathbb{N}$, called q -integer, is given by

$$[m]_q := \frac{1 - q^m}{1 - q}, \quad q \neq 1; \quad [m]_1 := m.$$

Also $[0]_q := 0$. Similarly, the q -analogue of the factorial of m is known by

$$[m]_q! := [m]_q [m-1]_q \cdots [1]_q; \quad [0]_q! := 1$$

Now, let us give the q -version of the Gauss binomial formula. The analogue of $(a+b)^m$ in q -analysis, are given by

$$(a \oplus b)_q^m := \prod_{s=0}^{m-1} (a + q^s b); \quad (a \oplus b)_q^0 := 1.$$

By simple calculations, it follows that

$$(a \oplus b)_q^m := \sum_{v=0}^m \begin{bmatrix} m \\ v \end{bmatrix}_q q^{v(v-1)/2} b^v a^{m-v},$$

where

$$\begin{bmatrix} m \\ v \end{bmatrix}_q := \frac{[m]_q!}{[v]_q! [m-v]_q!}, \quad 0 \leq v \leq m$$

is the q -binomial formula. All the concepts defined above, become their classical cases if q tends to 1.

The derivative of a function f in q calculus, shown by $D_q f$, is given by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0$$

and $(D_q f)(0) = f'(0)$ when $f'(0)$ exists. If f is differentiable at $x \neq 0$, it is clear that

$$\lim_{q \rightarrow 1} (D_q f)(x) = f'(x).$$

Let us define the partial derivatives of a function of two variables in q -calculus, say q -partial derivative:

$$\frac{\partial_q f(x, y)}{\partial_q x} = \frac{f(qx, y) - f(x, y)}{(q-1)x}, \quad x \neq 0.$$

The q -partial derivative of f for the variable y can be given similarly.

The positive linear operators (actually, Bernstein polynomials of degree m) was adapted into the theory of q -analysis by A. Lupas [5] in 1987. In 1997, G. M. Phillips [6] proposed another q -version offers. For interesting properties and different versions this polynomials we refer to [7], [8], [9], [10] [11], [12] and, [13]. Firstly, let us recall the Baskakov Operators [14]: for any function f continuous on $[0, \infty)$,

$$V_m(f; x) = \sum_{v=0}^{\infty} \binom{m+v-1}{v} x^v (1+x)^{-m-v} f\left(\frac{v}{m}\right) \quad (1)$$

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for $x \in [0, \infty)$ and $m \in \mathbb{N}$. Recently, it has been introduced some various q -versions of the Baskakov operators. In [15], A. Aral and V. Gupta introduced the following two generalisations: For $0 < q < 1$,

$$P_{m,q}(f; x) = \left(\frac{qx}{1+x}; q \right)_{m} \sum_{v=0}^{\infty} \begin{bmatrix} m+v-1 \\ k \end{bmatrix}_q \left(\frac{qx}{1+x} \right)^v f \left(\frac{[v]_q}{q^v[m]_q} \right) \tag{2}$$

and

$$P_{m,q}^*(f; x) = \left(\frac{q^2x}{1+x}; q \right)_{m} \sum_{v=0}^{\infty} \begin{bmatrix} m+v+1 \\ v \end{bmatrix}_q \left(\frac{q^2x}{1+x} \right)^v f \left(\frac{[v]_q}{q^{v+1}[m]_q} \right). \tag{3}$$

They also gave the following in [16]:

$$\mathcal{B}_{m,q}(f; x) = \sum_{v=0}^{\infty} \begin{bmatrix} m+v-1 \\ v \end{bmatrix}_q q^{\frac{v(v-1)}{2}} x^v (-x; q)_{m+v}^{-1} f \left(\frac{[v]_q}{q^{v-1}[m]_q} \right). \tag{4}$$

where $q > 0$. C. Radu gave the another version for $q \in (0, 1)$ in [17]:

$$V_{m,q}^*(f; x) = \sum_{v=0}^{\infty} \begin{bmatrix} m+v-1 \\ v \end{bmatrix}_q q^{\frac{v(v-1)}{2}} (qx)^v \frac{1}{(1+qx)_q^{m+v}} f \left(\frac{[v]_q}{q^{v-1}[m]_q} \right); \tag{5}$$

where $(x; q)_m = (1-x)(1-qx) \cdots (1-q^{m-1}x)$. For the case $q = 1$, these operators return to the usual Baskakov operators.

In this study, we have introduced a new type of q -analogue of the classical Baskakov operators which are produced from general discrete type operators that are defined in the next section. Then, the approximation properties and the rate of convergence of the sequences of this operators have been established in terms of the modulus of smoothness of order 1.

2. Material and Method

Let $q > 0$ and $m \in \mathbb{N}$. For $f \in C(I)$ ($I = [0, 1]$ or $[0, \infty)$), the authors introduced the following operators on $C(I)$ in [19] and the approximation properties and the rate of convergence of these sequences of q -discrete type is established by means of the modulus of continuity in [20]:

$$E_{m,q}(f; x) = \sum_{v=0}^{\infty} \frac{1}{[v]_q!} \frac{\partial_q^v \Phi_{m,q}(x, u)}{\partial_q u^v} \Big|_{u=0} f \left(\frac{[v]_q}{\alpha_{m,v,q}} \right), \tag{6}$$

where $\alpha_{m,v,q}$ are positive numbers and $\{\Phi_{m,q}(x, u)\}$ generating real functions defined on $I \times [0, \infty)$, have the following conditions:

- (i) $\Phi_{m,q}(x, 0) \neq 0$ and $\Phi_{m,q}(x, 1) = 1$ for all $m \in \mathbb{N}$ and $x \in I$.
- (ii) $\frac{\partial_q^v \Phi_{m,q}(x, u)}{\partial_q u^v} \Big|_{u=0}$ exist and are continuous functions of x for all $v \in \mathbb{N}_0$ and $m \in \mathbb{N}$.

- (iii) For all $v \in \mathbb{N}_0, x, u \geq 0$,

$$\frac{\partial_q^v \Phi_{m,q}(x, u)}{\partial_q u^v} \geq 0, \quad m \in \mathbb{N}.$$

It is clear that the operators are positive and linear, thanks to (iii) on the space of bounded functions on $I, B(I)$. The test functions $e_{r,i}$ are given by

$$e_{r,i}(t) = \left(\frac{t}{1+(1-i)t} \right)^r, \quad r \in \mathbb{N}_0, i = 0, 1, 2.$$

The functions of $e_{r,0}$ are used as test functions for q -Butzer-Bleimann-Hahn Operators, the functions $e_{r,1}$ are used as test functions for q -Bernstein, q -Szász-Mirakyan, q -Lupas and q -Baskakov Operators and the functions of $e_{r,2}$ for q -Meyer-König and Zeller Operators.

In continuation of the relation for the numbers $\alpha_{n,v,q}$ indicated in (10), we assume the following:

$$e_{r,i} \left(\frac{[v]_q}{\alpha_{m,v,q}} \right) = \frac{[v]_q^r}{\alpha_{m,v,q}^r}, \quad r \in \mathbb{N}_0,$$

where $\alpha_{m,q}$ are positive numbers independent of v .

Theorem 2.1 ([19]). *If the sequence $\{\Phi_{m,q}(x, u)\}$ satisfies the conditions (i)-(iii) for all $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$, then the following relation is true:*

$$E_{m,q}(e_{r,i}; x) = \frac{1}{\alpha_{m,q}^r} \sum_{v=0}^r q^{\binom{v}{2}} \mathbb{S}_q(r, v) \frac{\partial_q^v \Phi_{m,q}(x, u)}{\partial_q u^v} \Big|_{u=1}, \tag{7}$$

$\mathbb{S}_q(r, v)$ appeared in are the q -Stirling numbers of the second kind for detail, see [22].

Corollary 2.2. *By virtue of equality (7), we have*

$$\begin{aligned} E_{m,q}(e_{0,i}; x) &= 1; \\ E_{m,q}(e_{1,i}; x) &= \frac{1}{\alpha_{m,q}} \frac{\partial_q \Phi_{m,q}(x, u)}{\partial_q u} \Big|_{u=1}; \\ E_{m,q}(e_{2,i}; x) &= \frac{1}{\alpha_{m,q}^2} \left\{ q \frac{\partial_q^2 \Phi_{m,q}(x, u)}{\partial_q u^2} + \frac{\partial_q \Phi_{m,q}(x, u)}{\partial_q u} \right\} \Big|_{u=1}; \end{aligned}$$

Now, let us construct the new type of q -analogue of Baskakov operators. For $m \in \mathbb{N}$ and $q \in (0, 1)$, we consider the function

$$\Phi_{m,q}(x, u) = \frac{((1+x) \ominus x)_q^m}{((1+x) \ominus ux)_q^m}, \quad x \in [0, \infty). \tag{8}$$

$\Phi_{m,q}(x, 0) \neq 0$ and $\Phi_{m,q}(x, 1) = 1$ for every $m \in \mathbb{N}$ and $x \in [0, \infty)$. By the definition of q -partial derivatives, we obtain

$$\begin{aligned} \frac{\partial_q \Phi_{m,q}(x, u)}{\partial_q u} &= \frac{\Phi_{m,q}(x, qu) - \Phi_{m,q}(x, u)}{(q-1)u} \\ &= \frac{\frac{((1+x) \ominus x)_q^m}{((1+x) \ominus qux)_q^m} - \frac{((1+x) \ominus x)_q^m}{((1+x) \ominus ux)_q^m}}{(q-1)u} \\ &= \frac{[m]_q x ((1+x) \ominus x)_q^m}{\prod_{s=0}^m (1+x - q^s xu)} = \frac{[m]_q x ((1+x) \ominus x)_q^m}{((1+x) \ominus ux)_q^{m+1}}. \end{aligned}$$

So we have,

$$\frac{\partial_q \varphi_{m,q}(x, u)}{\partial_q u} = \frac{[m]_q x ((1+x) \ominus x)_q^m}{((1+x) \ominus ux)_q^{m+1}}.$$

By induction, we obtain that

$$\frac{\partial_q^v \varphi_{m,q}(x, u)}{\partial_q u^v} = \frac{[m]_q [m+1]_q \cdots [m+v-1]_q x^v ((1+x) \ominus x)_q^m}{((1+x) \ominus ux)_q^{m+v}}.$$

If we write $u = 0$ in the last equality, then we get

$$\begin{aligned} \left. \frac{\partial_q^v \varphi_{m,q}(x, u)}{\partial_q u^v} \right|_{u=0} &= \frac{[m]_q \cdots [m+v-1]_q x^v ((1+x) \ominus x)_q^m}{(1+x)^{m+v}} \\ \frac{1}{[v]_q!} \left. \frac{\partial_q^v \varphi_{m,q}(x, u)}{\partial_q u^v} \right|_{u=0} &= \\ ((1+x) \ominus x)_q^m \left[\begin{matrix} m+v-1 \\ v \end{matrix} \right]_q x^v (1+x)^{-m-v}. \end{aligned} \tag{9}$$

Since the right hand of the equality (9) is a rational function of x which does not have any singular points in $[0, \infty)$, then the condition (ii) holds and since $q < 1$ and $x \in [0, \infty)$, then the condition (iii) is satisfied too, there by the functions $\varphi_{m,q}(x, u)$ defined by (8) generate some positive and linear operators.

Writing (9) and considering $\alpha_{m,v,q} = [m]_q$ in the operators $E_{m,q}$ given by (6), we have, for $f \in C[0, \infty)$, $q \in (0, 1)$, $x \in [0, \infty)$ and $m \in \mathbb{N}$,

$$\begin{aligned} E_{m,q}(f; x) &= \\ ((1+x) \ominus x)_q^m \sum_{v=0}^{\infty} \left[\begin{matrix} m+v-1 \\ v \end{matrix} \right]_q x^v (1+x)^{-m-v} f \left(\frac{[v]_q}{[m]_q} \right). \end{aligned} \tag{10}$$

3. Results

In this section we give some classical approximation properties of the operators $E_{m,q}$. Let $q_m \in (0, 1)$ and $1 - q_m = o(\frac{1}{m})$ when $m \rightarrow \infty$. In the sequel for $j \in \mathbb{N}_0$, $m \in \mathbb{N}$, we use notations:

$$\begin{aligned} e_r(t) &:= t^r, \quad r \in \mathbb{N}_0, \\ \mu_{m,j}(x, q) &:= E_{m,q}((e_1(\cdot) - e_1(x))^j; x), \\ I_A &:= [0, A], \quad A > 0. \end{aligned}$$

By simple calculations, we get the following lemmas. First, we get the following lemma from Corollary 2.2.

Lemma 3.1.

$$\begin{aligned} E_{m,q}(e_0; x) &= 1; \\ E_{m,q}(e_1; x) &= \frac{x}{1+x(1-q^m)}; \\ E_{m,q}(e_2; x) &= \frac{q[m+1]_q x^2}{[m]_q (1+x(1-q^m))(1+x(1-q^{m+1}))} \\ &\quad + \frac{1}{[m]_q} \frac{x}{(1+x(1-q^m))}; \end{aligned}$$

And, we obtain the following results using Lemma 3.1.

Lemma 3.2.

$$\begin{aligned} \mu_{m,1}(x, q) &= \frac{x}{1+x(1-q^m)} - x; \\ \mu_{m,2}(x, q) &= \frac{q[m+1]_q x^2}{[m]_q (1+x(1-q^m))(1+x(1-q^{m+1}))} \\ &\quad + \frac{1}{[m]_q} \frac{x}{(1+x(1-q^m))} - \frac{2x^2}{(1+x(1-q^m))} + x^2; \end{aligned}$$

Lemma 3.3. If $A > 0$, then

$$|E_{m,q}(e_r; x) - e_r(x)| \leq rA^{r-1} \sqrt{\mu_{m,2}(x, q)}, \quad r \in \mathbb{N}_0,$$

for all $t, x \in I_A$.

Proof. For the case $r = 0$ the assertion is obvious. We assume that $r \in \mathbb{N}$. For $t, x \in I_A$ with $A > 0$,

$$\begin{aligned} |e_r(t) - e_r(x)| &= |t-x| \cdot |t^{r-1} + \cdots + x^{r-1}| \\ &\leq |e_1(t) - e_1(x)| \cdot |A^{r-1} + \cdots + A^{r-1}| \\ &= rA^{r-1} |e_1(t) - e_1(x)|. \end{aligned}$$

By monotonicity of the operators $E_{m,q}$ and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |E_{m,q}(e_r; x) - e_r(x)| &\leq rA^{r-1} E_{m,q}(|e_1(\cdot) - e_1(x)|; x) \\ &\leq rA^{r-1} \sqrt{E_{m,q}((e_1(\cdot) - e_1(x))^2; x)} \end{aligned}$$

for all $m \in \mathbb{N}_0$, thus we obtain

$$|E_{m,q}(e_r; x) - e_r(x)| \leq rA^{r-1} \sqrt{\mu_{m,2}(x, q)}$$

what we wanted to prove. \square

Lemma 3.4. For each $x \in I_A$, we have

$$\lim_{m \rightarrow \infty} E_{m,q_m}(e_r; x) = e_r(x), \quad r = 0, 1, 2$$

Proof. Indeed,

$$\begin{aligned} \mu_{m,2}(x, q_m) &= \frac{q_m [m+1]_{q_m} x^2}{[m]_{q_m} (1+x(1-q_m^m))(1+x(1-q_m^{m+1}))} \\ &\quad + \frac{1}{[m]_{q_m}} \frac{x}{(1+x(1-q_m^m))} - \frac{2x^2}{(1+x(1-q_m^m))} + x^2 \end{aligned}$$

Using following inequality

$$\frac{1}{1+x(1-q_m^m)} \leq 1$$

we have

$$\begin{aligned} \mu_{m,2}(x, q_m) &\leq \frac{[m+1]_{q_m} x^2}{[m]_{q_m}} + \frac{x}{[m]_{q_m}} - \frac{2x^2}{(1+x(1-q_m^m))} + x^2 \\ &= \frac{2x^3(1-q_m^m)}{(1+x(1-q_m^m))} + \frac{x(1+x)}{[m]_{q_m}} \\ &\leq 2(1-q_m^m)A^3 + \frac{A(1+A)}{[m]_{q_m}}, \end{aligned}$$

hence

$$\mu_{m,2}(x, q_m) \leq 2(1 - q_m^m)A^3 + \frac{A(1+A)}{[m]_{q_m}}.$$

Consequently, we have

$$\lim_{m \rightarrow \infty} \mu_{m,2}(x, q_m) = 0.$$

From Lemma 3.3 we obtain Lemma 3.4. \square

Corollary 3.5. *If $f \in C(I_A)$, then we have*

$$\lim_{m \rightarrow \infty} E_{m,q_m}(f; x) = f(x), \quad x \in I_A.$$

For $f \in C(I_A)$ and $\delta > 0$, the modulus of smoothness of order 1 (it is also called modulus of continuity) of f with step $\delta > 0$ given by

$$\omega(f; \delta) := \sup_{u,v \in I_A, |u-v| \leq \delta} |f(u) - f(v)|.$$

For any $\delta > 0$, we get

$$|f(u) - f(v)| \leq \left(1 + \frac{|u-v|}{\delta}\right) \omega(f, \delta). \quad (11)$$

Theorem 3.6. *Let $E_{m,q}$ be given by (10) and $f \in C(I_A)$, then the inequality*

$$|E_{m,q}(f; x) - f(x)| \leq \left(1 + \frac{1}{\delta} \sqrt{\mu_{m,2}(x, q)}\right) \omega(f, \delta).$$

holds for any $\delta > 0$. *Proof.* Since $E_{m,q}(1; x) = 1$, then we have

$$|E_{m,q}(f; x) - f(x)| \leq E_{m,q}(|f(\cdot) - f(x)|; x) \quad (12)$$

for all $n \in \mathbb{N}$. Now using (11) in inequality (12) we obtain

$$|f(t) - f(x)| \leq \omega(|t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega(f, \delta), \quad (13)$$

for all $\delta > 0$. Using the Cauchy-Schwartz Inequality and (13) it follows that

$$|E_m(f; x) - f(x)| \leq \left(1 + \frac{1}{\delta} \sqrt{E_{m,q}((e_1(\cdot) - e_1(x))^2; x)}\right) \omega(f, \delta),$$

thus for any $\delta > 0$

$$|E_{m,q}(f; x) - f(x)| \leq \left(1 + \frac{1}{\delta} \sqrt{\mu_{m,2}(x, q)}\right) \omega(f, \delta). \quad \square$$

As always, we write $f \in Lip_M \alpha$, ($M > 0, 0 < \alpha \leq 1$), if the relation

$$|f(u) - f(v)| \leq M|u - v|^\alpha \quad (14)$$

is satisfied for all $u, v \in I_A$.

Theorem 3.7. *For all $f \in Lip_M \alpha$ and $x \in I_A$, we have*

$$|E_{m,q}(f; x) - f(x)| \leq M(\mu_{m,2}(x, q))^{\alpha/2}.$$

Proof. Applying $E_{m,q}$ to the inequality (14), we have

$$\begin{aligned} |E_{m,q}(f; x) - f(x)| &\leq E_{m,q}(|f(\cdot) - f(x)|; x) \\ &\leq M E_{m,q}(|e_1(\cdot) - e_1(x)|^\alpha; x). \end{aligned}$$

If we consider the Hölder Inequality for $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$ we get,

$$\begin{aligned} |E_{m,q}(f; x) - f(x)| &\leq M \left(E_{m,q}((e_1(\cdot) - e_1(x))^2; x) \right)^{\alpha/2} \\ &\leq M(\mu_{m,2}(x, q))^{\alpha/2} \end{aligned}$$

thus

$$|E_{m,q}(f; x) - f(x)| \leq M(\mu_{m,2}(x, q))^{\alpha/2}. \quad \square$$

4. Discussion and Conclusion

The type of q -Baskakov Operators which we constructed above is new and different from ones exists in literature. The results which are obtained enrich the literate of convergence in q -calculus in operator theory. Consequently, the operators so established can be found fruitful in several situation appearing in the literature on Approximation Theory and Operator Theory.

Acknowledgment

The author is very much thankful to Assoc. Prof. Tuncay Tunç for his valuable guidance in preparing this paper.

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