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RESEARCH ARTICLE

SHAPE OPERATORS OF A DIRECTIONAL TUBULAR SURFACE IN 4-SPACE

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Abstract

This paper examines a tubular surface, a specific example of a canal surface, in 4-dimensional Euclidean space. In the plane stretched by the quasi-frame vectors $\boldsymbol{B}_q$ and $\boldsymbol{C}_q$, this surface is established by the motion of a circle with a constant radius that uses each point on the curve $\alpha(t)$ as its center. Using the general equation provided in Euclidean 4-space, the first and second partial derivatives are determined. The Gram-Schmidt technique was used to derive the surface's first unit normal vector field $\boldsymbol{U}_1$, and second unit normal vector field $\boldsymbol{U}_2$, using the acquired partial derivatives. Using quasi-vectors, the tubular surface's first and second fundamental form coefficients were found. Furthermore, the shape operator matrices for the tubular surface's the unit normal vector fields $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ were acquired. We have found algebraic invariants of the shape operator, Gaussian curvature, and mean curvature. For a thorough understanding of the obtained theoretical calculations, an example of a directional tubular surface, the equation of the tubular surface has been parametrized using quasi-frame vectors and quasi-frame curvatures for a given space curve in 4-dimensional Euclidean space.

Keywords

Euclidean Space,
Quasi-frame,
Tubular Surface,
Shape Operator

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1. INTRODUCTION

Monge was the first to introduce the canal surfaces [1]. The envelope of a moving sphere with a constant radius function, represented by $\boldsymbol{r}(\boldsymbol{t})$, gives birth to tubular surfaces, which are a specific instance of canal surfaces [2,3]. Relationships between the curvatures and characterizations of these surfaces in various environments have piqued the curiosity of several scholars. Frenet, Bishop, and Darboux frames have also been utilized to parameterize tubular surfaces, which may likewise be employed as pipes [4,5,6]. Apart from these frames, Coquillart presented a different method called the quasi-frame, which is derived from the quasi-normal vector [7]. The quasi-frame adapted over a spatial curve was presented by Dede et al., who also demonstrated how it related to the Frenet frame [8]. In a variety of spaces, this quasi-frame has been used for parametric representations of directional tubular surfaces, also known as directional tubular surfaces [9,10,11]. In Euclidean 4-space, Gezer and Ekici introduced the quasi-frame, its formulae, and the connection with the Frenet frame [12]. Gluck has proposed a straightforward technique for calculating curvatures of curve in Euclidean n-space and Euclidean 4-space that uses a single formula for all curvatures and is based on the Gram-Schmidt orthonormalization process [13]. Frenet elements and derivative equations for unit speed space curves have been studied by academics

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with an emphasis on Euclidean 4-space [14–21]. In both 3-dimensional Euclidean space $\mathbb{E}^{\mathbf{3}}$ and 4-dimensional Euclidean space $\mathbb{E}^{\mathbf{4}}$, the canal surface may be parametrized using Frenet frames and alternative frames [5,22-27]. Coşkun Ekici and Akça [28] have studied the governed surfaces with the quasi-vectors in Euclidean 4-space as well. Mello has looked at the conditions and characteristics of surfaces [29]. Additionally, under parallel transport frame vectors, Kişi has investigated canal surfaces in $\mathbb{E}^{\mathbf{4}}$ [22]. Based on the coefficients of the first and second fundamental forms, Bulca's work provides characterizations of surfaces in $\mathbb{E}^{\mathbf{4}}$ [23]. Furthermore, using the Frenet frame and quasi-frame in $\mathbb{E}^{\mathbf{4}}$, Yağbasan et al. have identified a few algebraic invariants of the parametrization of the tubular surfaces [30,31,32]. First, this paper discusses fundamental concepts and theorems about tubular surfaces and quasi-frames. The parametrization of directed tubular surfaces in Euclidean 4-space is then provided. The directional tubular surface's normals, Gaussian curvature, mean curvature, and shape operator matrices are then provided, in that order. Additionally, the directional tubular surfaces are displayed in the projection spaces, and the directional tubular surface example in $\mathbb{E}^{\mathbf{4}}$ of this research is provided.

2. PRELIMINARIES

Let (x_1, x_2, x_3, x_4), $\boldsymbol{Y} = (y_1, y_2, y_3, y_4)$ and $\boldsymbol{Z} = (z_1, z_2, z_3, z_4)$ be three vectors in $\mathbb{E}^4$, then the dot product is defined $< \boldsymbol{X}, \boldsymbol{Y} >= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. The norm of vector $\boldsymbol{X} \in \mathbb{E}^4$ is given by $\|\boldsymbol{X}\| = \sqrt{< \boldsymbol{X}, \boldsymbol{X} >}$. The vector product of $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ is given by the determinant as follows

$$\boldsymbol{X} \wedge \boldsymbol{Y} \wedge \boldsymbol{Z} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \tag{1}$$

where $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 = \boldsymbol{e}_4, \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_4 = \boldsymbol{e}_1, \boldsymbol{e}_3 \wedge \boldsymbol{e}_4 \wedge \boldsymbol{e}_1 = \boldsymbol{e}_2$ and $\boldsymbol{e}_3 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_1 = -\boldsymbol{e}_4$ [14,15]. Let $\alpha(t) = \alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be any space curve in Euclidean 4-space. The curve is said to be parameterized by arc length s if $< \alpha', \alpha' >= 1$. Let $\{\boldsymbol{T}, \boldsymbol{N}_q, \boldsymbol{B}_q, \boldsymbol{C}_q\}$ be a quasi-frame where $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_q$ and $\boldsymbol{C}_q$ are called the tangent, normal, the first binormal, and the second binormals vector fields, respectively. The quasi-frame is given by

$$\begin{aligned} \boldsymbol{T} &= \frac{\alpha'}{\|\alpha'\|} & \mathbf{N}_q &= \frac{\boldsymbol{T} \wedge \boldsymbol{k}_x \wedge \boldsymbol{k}_y}{\|\boldsymbol{T} \wedge \boldsymbol{k}_x \wedge \boldsymbol{k}_y\|} \\ \mathbf{B}_q &= \boldsymbol{C}_q \wedge \boldsymbol{T} \wedge \mathbf{N}_q & \mathbf{C}_q &= \frac{\alpha' \wedge \boldsymbol{N}_q \wedge \alpha'''}{\|\alpha'\|}, \end{aligned} \tag{2}$$

where $\mathbf{k}_x = (1,0,0,0)$ and $\mathbf{k}_y = (0,1,0,0)$ are the projection vectors [17, 20]. quasi-frame formulas of a unit speed curve $\alpha(t)$ is written as

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_q' \\ \boldsymbol{C}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & 0 \\ -k_1 & 0 & k_3 & 0 \\ -k_2 & -k_3 & 0 & k_4 \\ 0 & 0 & -k_4 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \\ \boldsymbol{C}_q \end{bmatrix} \tag{3}$$

where the functions

$$k_1 = \frac{< \boldsymbol{T}', \mathbf{N}_q >}{\|\alpha'\|}, k_2 = \frac{< \boldsymbol{T}', \mathbf{B}_q >}{\|\alpha'\|}, k_3 = \frac{< \boldsymbol{N}_q', \mathbf{B}_q >}{\|\alpha'\|}, \text{and } k_4 = \frac{< \boldsymbol{B}_q', \mathbf{C}_q >}{\|\alpha'\|} \tag{4}$$

are called the first, the second and the third curvatures, respectively [33, 13]. Let M be a regular surface given with the parameterization $\psi: U \subset \mathbb{R}^2 \to \mathbb{R}^4$, $\psi(t, v)$ in $\mathbb{R}^4$. The tangent space of M at an arbitrary point is spanned by the vectors ψ_t and ψ_v. The coefficients of the first fundamental form of M are defined as

$$E = \langle \psi_t, \psi_t \rangle, F = \langle \psi_t, \psi_v \rangle, G = \langle \psi_v, \psi_v \rangle, \text{and } W = EG - F^2, \tag{5}$$

where $<,>$ is a Euclidean dot product [18,29]. Let $\psi_{tt}, \psi_{tv}, \psi_{vv}$ be the second order partial derivatives and $\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_{n-2}$ be the normal vector fields of M such that the second fundamental form coefficients of M are

$$L_k = \langle \psi_t, \boldsymbol{U}_k \rangle,\ M_k = \langle \psi_t, \boldsymbol{U}_k \rangle \text{ and } N_k = \langle \psi_v, \boldsymbol{U}_k \rangle \text{ for } 1 \leq k \leq n-2 \tag{6}$$

Also, its shape operator matrix with $\boldsymbol{U}_k$ is given

$$S_{\boldsymbol{U}_k} = \begin{bmatrix} \frac{L_k}{E} & \frac{1}{W}(M_k - \frac{F}{E}L_k) \\ \frac{1}{W}(M_k - \frac{F}{E}L_k) & \frac{1}{W^2}(EN_k - 2FM_k + \frac{F^2}{E}L_k) \end{bmatrix} \quad (7)$$

[29, 34]. Therefore, Gaussian curvature and mean curvature vector of the surface can also be written as

$$K = det(S_{\boldsymbol{U}_1}) + det(S_{\boldsymbol{U}_2}) + \cdots + det(S_{\boldsymbol{U}_{n-2}}) \quad (8)$$

and

$$H = \frac{1}{2}\{iz(S_{\boldsymbol{U}_1})\boldsymbol{U}_1 + iz(S_{\boldsymbol{U}_2})\boldsymbol{U}_2 + \cdots + iz(S_{\boldsymbol{U}_{n-2}})\boldsymbol{U}_{n-2}\} \quad (9)$$

respectively [21].

Let $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_1, \boldsymbol{B}_2\}$ be a Frenet frame of a unit speed curve $\alpha(t)$. A canal surface of radius $r(t)$ and centered at a spine curve $\alpha(t)$ is parametrized as

$$\psi(t,v) = \alpha(t) + r(t)(cosv\boldsymbol{B}_1(t) + sinv\boldsymbol{B}_2(t)) \quad (10)$$

where r(t) is a real differentiable function. A parameterization of the tubular surface at a distance r is given as

$$\psi(t,v) = \alpha(t) + r(cosv\boldsymbol{B}_1(t) + sinv\boldsymbol{B}_2(t)) \quad (11)$$

with the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}_1, \boldsymbol{B}_2\}$ in $\mathbb{E}^4$.

3- TUBULAR SURFACES WITH Q-FRAME IN 4-DIMENSIONAL EUCLID SPACE

Consider the unit speed curve $\alpha(t)$ and the parameterization of the tubular surface at a constant r, which may be expressed as

$$\psi(t,v) = \alpha(t) + r(cosv\boldsymbol{B}_q(t) + sinv\boldsymbol{C}_q(t)) \quad (12)$$

with the quasi-frame $\{\boldsymbol{T}, \boldsymbol{N}_q, \boldsymbol{B}_q, \boldsymbol{C}_q\}$ in $\mathbb{E}^4$.

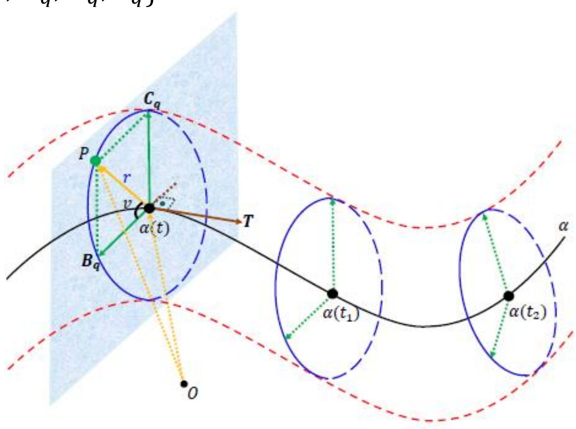

Figure 1. Tubular surface with quasi-frame

Theorem 1: Let $M \subset \mathbb{E}^4$ be a tubular surface at distance r from the principal curve α(t) with respect to the quasi-frame $\{\boldsymbol{T}, \boldsymbol{N}_q, \boldsymbol{B}_q, \boldsymbol{C}_q\}$ with the parameterization $\psi(t,v)$. In 4-dimensional space, the first unit normal vector field of the 2-dimensional tubular surface is

$$\boldsymbol{U}_1 = \frac{rk_3cosv\boldsymbol{T} + (1 - rk_2cosv)\boldsymbol{N_q} + sinv\boldsymbol{B}_q + cosv\boldsymbol{C}_q}{\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} \quad (13)$$

obtained [32].

Proof: The unit normal vector field $\boldsymbol{U}_1$ of the surface has to hold the following concitions

$$\begin{aligned} &< \psi_t, \boldsymbol{U}_1 >= 0, \\ &< \psi_v, \boldsymbol{U}_1 >= 0, \quad \text{and } < \boldsymbol{U}_1, \boldsymbol{U}_2 >= 0, \\ &< \boldsymbol{U}_1, \boldsymbol{U}_1 >= 1, \end{aligned} \quad (14)$$

where ψ_t and ψ_v are the partial derivatives of $\psi(t,v)$, with respect to t and v. Since $<\boldsymbol{U}_1,\boldsymbol{U}_1>=1$, $\boldsymbol{U}_1$ is the unit normal vector field of the tubular surface. In 4-dimensional space, the tubular surface has the unit normal vector fields $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$. If the partial derivatives of the equation (12) are taken and the equations (14) are used, the first unit normal vector field is easily found.

Theorem 2: Let $M \subset \mathbb{E}^4$ be a tubular surface at as distance r from the spine curve $\alpha(t)$ according to the quasi-frame $\{\boldsymbol{T},\boldsymbol{N}_q,\boldsymbol{B}_q,\boldsymbol{C}_q\}$ with parametrization $\psi(t,v)$, given by $\psi: U \subset \mathbb{E}^2 \to \mathbb{E}^4$, $(t,v)\in U$ and let tangent space to M at a point $p\in\psi(t,v)$ be spanned by $\{\psi_t,\psi_v\}$. The shape operator S_{U_1} of tubular surface with the unit normal vector field $\boldsymbol{U}_1$ in $\mathbb{E}^4$ is obtained as

$$S_{U_1}=\begin{bmatrix} \dfrac{\mathcal{A}_1((1-rk_2cosv)^2+r^2k_3^2cos^2v+r^2k_4^2)^{-1}}{\sqrt{(1-rk_2cosv)^2+1+r^2k_3^2cos^2v}} & \dfrac{\mathcal{C}_1((1-rk_2cosv)^2+r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_1\sqrt{(1-rk_2cosv)^2+1+r^2k_3^2cos^2v}} \\ \dfrac{\mathcal{C}_1((1-rk_2cosv)^2+r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_1\sqrt{(1-rk_2cosv)^2+1+r^2k_3^2cos^2v}} & \dfrac{-\mathcal{D}_1}{r\mathcal{E}_1\sqrt{(1-rk_2cosv)^2+1+r^2k_3^2cos^2v}} \end{bmatrix}, \tag{15}$$

where

$$\begin{aligned}
\mathcal{A}_1 ={}& rcos^2v(rk_3'k_2-rk_2'k_3+rk_1k_3^2-k_2^2-k_3^2+rk_2^2k_1)+rk_3k_4sinv+cosv(k_2+2rk_1k_2-rk_3')-rk_4^2+k_1,\\
\mathcal{B}_1 ={}& ((1-rk_2cosv)^2+r^2k_3^2cos^2v+r^2k_4^2),\\
\mathcal{C}_1 ={}& r^2cos^2vsinv(k_2^2k_3+k_3^2)+r^3k_4cos^2v(k_2'k_3-k_1k_3^2-k_1k_2^2-k_2k_3')-k_4-2rk_2k_3cosvsinv-rk_1k_4\\
&+rk_4cosv(2rk_1k_2+k_2+rk_3')+k_3sinv,\\
\mathcal{D}_1 ={}& r^2k_3k_4sinv(2+r^2k_4^2)+r^4cos^4v(k_2^2+k_3^2)^2+cosv(2r^4k_1k_2k_4^2-r^3k_2k_4^2-4rk_2+r^4k_3'k_4^2)+1\\
&+2r^3cosvsinv(rk_3^3k_4cosv+rk_2^2k_3k_4cosv-2k_2k_3k_4)-r^3k_1k_4^2-4r^3cos^3v(k_2k_3^2+k_2^3)+2r^4k_2^2k_3^2cos^4v\\
&+cos^2v\Big(2r^2(k_3^2+3k_2^2)-r^5k_4^2(k_1k_2^2+k_2k_3'-k_2'k_3+k_3^2k_1)+r^4k_4^2(k_2^2+k_3^2)\Big),\\
\mathcal{E}_1 ={}& 1-4rk_2cosv-4r^3cos^3v(k_2^3+k_2k_3^2)+r^2cos^2v(r^2k_2^2k_4^2-2rk_2k_4^2+r^2k_3^2k_4^2+6k_2^2+2k_3^2)\\
&+r^4cos^4v(k_2^2+k_3^2)^2+r^2k_4^2,
\end{aligned}$$

respectively.

Proof: Tubular surface, at a distance r from the spine curve $\alpha(t)$ with the quasi-frame $\{\boldsymbol{T},\boldsymbol{N}_q,\boldsymbol{B}_q,\boldsymbol{C}_q\}$ are parametrized by

$$\psi(t,v)=\alpha(t)+r(cosv\boldsymbol{B}_q+sinv\boldsymbol{C}_q).$$

The partial derivatives of $\psi(t,v)$, with respect to t and v, are determined by

$$\psi_t=(1-rk_2cosv)\boldsymbol{T}-rk_3cosv\boldsymbol{N}_q-rk_4sinv\boldsymbol{B}_q+rk_4cosv\boldsymbol{C}_q, \tag{16}$$

and

$$\psi_v=r\big(-sinv\boldsymbol{B}_q+cosv\boldsymbol{C}_q\big). \tag{17}$$

Then, the second order partial derivatives of $\psi(t,v)$, with respect to t and v, are given as

$$\begin{aligned}
\psi_{tt} ={}& (-rk_2'cosv+rk_2k_4sinv+rk_1k_3cosv)\boldsymbol{T}+(k_1-rk_1k_2cosv-rk_3'cosv+rk_3k_4sinv)\boldsymbol{N}_q\\
&+(k_2-rk_2^2cosv-rk_3^2cosv-rk_4^2cosv-rk_4'sinv)\boldsymbol{B}_q+(rk_4'cosv-rk_4^2sinv)\boldsymbol{C}_q,
\end{aligned} \tag{18}$$

$$\psi_{tv}=rk_2sinv\boldsymbol{T}+rk_3sinv\boldsymbol{N}_q-rk_4cosv\boldsymbol{B}_q-rk_4sinv\boldsymbol{C}_q, \tag{19}$$

and

$$\psi_{vv}=-rcosv\boldsymbol{B}_q-rsinv\boldsymbol{C}_q. \tag{20}$$

The equations (16) and (17) can be substituted into the equation (5) to yield the coefficients of the first fundamental form of tubular surfaces

$$\begin{aligned}
E&=(1-rk_2cosv)^2+r^2k_3^2cos^2v+r^2k_4^2,\\
F&=r^2k_4,\\
G&=r^2,
\end{aligned} \tag{21}$$

and

$$W = r^2((1 - rk_2 cosv)^2 + r^2 k_3^2 cos^2 v) \tag{22}$$

are obtained. The equations (13), (16), and (17) can be substituted into the equation (6) to yield the coefficients of the second fundamental form of the tubular surface with the unit vector field $\boldsymbol{U}_1$ in $\mathbb{E}^4$ obtained as,

$$\begin{aligned} L_1 = &\ \frac{r^2 cos^2 v(k_1 k_2^2 - k_2' k_3 + k_1 k_3^2 + k_3' k_2) - rk_4^2 - rcos^2 v(k_3^2 + k_2^2) + k_1}{\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}} \\ &+ \frac{cosv(k_2 - 2rk_1 k_2 sinv - rk_3') + rk_3 k_4 sinv}{\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}}, \\ M_1 = &\ \frac{r(k_3 sinv - k_4)}{\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}}, \\ N_1 = &\ \frac{-r}{\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}}. \end{aligned} \tag{23}$$

With the help of the equations (21), (22) and (23), the components of the shape operator matrix are as follows

$$\frac{L_1}{E} = \frac{\mathcal{A}_1((1 - rk_2 cosv)^2 + r^2 k_3^2 cos^2 v + r^2 k_4^2)^{-1}}{\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}}, \tag{24}$$

where

$$\begin{aligned} \mathcal{A}_1 = &\ rcos^2 v(rk_3' k_2 - rk_2' k_3 + rk_1 k_3^2 - k_2^2 - k_3^2 + rk_2^2 k_1) + rk_3 k_4 sinv \\ &+ cosv(k_2 + 2rk_1 k_2 - rk_3') - rk_4^2 + k_1, \end{aligned}$$

and

$$\frac{1}{W}\left(M_1 - \frac{F}{E} L_1\right) = \frac{\mathcal{C}_1((1 - rk_2 cosv)^2 + r^2 k_3^2 cos^2 v)^{-1/2}}{\mathcal{B}_1\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}}, \tag{25}$$

where

$$\mathcal{B}_1 = (1 - rk_2 cosv)^2 + r^2 k_3^2 cos^2 v + r^2 k_4^2,$$

$$\begin{aligned} \mathcal{C}_1 = &\ r^2 cos^2 v sinv(k_2^2 k_3 + k_3^2) + r^3 k_4 cos^2 v(k_2' k_3 - k_1 k_3^2 - k_1 k_2^2 - k_2 k_3') - k_4 \\ &+ rk_4 cosv(2rk_1 k_2 + k_2 + rk_3') + k_3 sinv - 2rk_2 k_3 cosvsinv - rk_1 k_4, \end{aligned}$$

and

$$\frac{1}{W^2}(EN_1 - 2FM_1 + \frac{F^2}{E} L_1) = \frac{-\mathcal{D}_1}{r\mathcal{E}_1\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}}, \tag{26}$$

where

$$\begin{aligned} \mathcal{D}_1 = &\ r^2 k_3 k_4 sinv(2 + r^2 k_4^2) + r^4 cos^4 v(k_2^2 + k_3^2)^2 + cosv(2r^4 k_1 k_2 k_4^2 - r^3 k_2 k_4^2 - 4rk_2 + r^4 k_3' k_4^2) + 1 \\ &+ 2r^3 cosvsinv(rk_3^3 k_4 cosv + rk_2^2 k_3 k_4 cosv - 2k_2 k_3 k_4) - 4r^3 cos^3 v(k_2 k_3^2 + k_3^3) + 2r^4 k_2^2 k_3^2 cos^4 v \\ &+ cos^2 v\big(2r^2(k_3^2 + 3k_2^2) - r^5 k_4^2(k_1 k_2^2 + k_2 k_3' - k_2' k_3 + k_3^2 k_1) + r^4 k_4^2(k_2^2 + k_3^2)\big) - r^3 k_1 k_4^2, \end{aligned}$$

$$\begin{aligned} \mathcal{E}_1 = &\ 1 - 4rk_2 cosv - 4r^3 cos^3 v(k_3^3 + k_2 k_3^2) + r^2 cos^2 v(r^2 k_2^2 k_4^2 - 2rk_2 k_4^2 + r^2 k_3^2 k_4^2 + 6k_2^2 + 2k_3^2) \\ &+ r^4 cos^4 v(k_2^2 + k_3^2)^2 + r^2 k_4^2, \end{aligned}$$

Substituting the equations (24), (25) and (26) into the equation (7) implies that the shape operator with respect to $\boldsymbol{U}_1$ following as

$$S_{U_1} = \begin{bmatrix} \frac{\mathcal{A}_1((1 - rk_2 cosv)^2 + r^2 k_3^2 cos^2 v + r^2 k_4^2)^{-1}}{\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}} & \frac{\mathcal{C}_1((1 - rk_2 cosv)^2 + r^2 k_3^2 cos^2 v)^{-1/2}}{\mathcal{B}_1\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}} \\ \frac{\mathcal{C}_1((1 - rk_2 cosv)^2 + r^2 k_3^2 cos^2 v)^{-1/2}}{\mathcal{B}_1\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}} & \frac{-\mathcal{D}_1}{r\mathcal{E}_1\sqrt{(1 - rk_2 cosv)^2 + 1 + r^2 k_3^2 cos^2 v}} \end{bmatrix}.$$

Theorem 3: Let $M \subset \mathbb{E}^4$ be a tubular surface at distance r from the principal curve α(t) with respect to the quasi-frame $\{\boldsymbol{T}, \boldsymbol{N}_q, \boldsymbol{B}_q, \boldsymbol{C}_q\}$ with the parameterization $\psi(t,v)$. In 4-dimensional space, the second unit normal vector field of the 2-dimensional tubular surface

$$\boldsymbol{U}_2 = \frac{rk_3 cosv\boldsymbol{T} + (rk_2cosv - 1)\boldsymbol{N_q} + cosv((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)\boldsymbol{B}_q}{((1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v)^{3/2}\sqrt{(1 - rk_2cosv)^2 + r^2k_3^2cos^2v}} - \frac{sinv((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)\boldsymbol{C}_q}{((1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v)^{3/2}\sqrt{(1 - rk_2cosv)^2 + r^2k_3^2cos^2v}} \tag{27}$$

is obtained [32].

Proof: The unit normal vector fields $\boldsymbol{U}_2$ of the surface has to be provided with the following conditions

$$\begin{aligned} &< \psi_t, \boldsymbol{U}_2 >= 0 \\ &< \psi_v, \boldsymbol{U}_2 >= 0 \quad \text{and} \quad < \boldsymbol{U}_1, \boldsymbol{U}_2 >= 0 \\ &< \boldsymbol{U}_2, \boldsymbol{U}_2 >= 1 \end{aligned} \tag{28}$$

where ψ_t and ψ_v are the partial derivatives of $\psi(t,v)$, with respect to t and v. Since $< \boldsymbol{U}_2, \boldsymbol{U}_2 >= 1$, $\boldsymbol{U}_2$ is the unit normal vector field of the tubular surface. If the partial derivatives of the equation (12) are taken and the following equations (28) are used, the second unit normal vector field is easily found.

Theorem 4: Let $M \subset \mathbb{E}^4$ be a tubular surface at a distance r from the spine curve $\alpha(t)$ according to the quasi-frame $\{\boldsymbol{T}, \boldsymbol{N}_q, \boldsymbol{B}_q, \boldsymbol{C}_q\}$ with parametrization $\psi(t,v)$, given by $\psi: U \subset \mathbb{E}^2 \to \mathbb{E}^4$, $(t,v) \in U$ and let tangent space to M at a point $p \in \psi(t,v)$ be spanned by $\{\psi_t, \psi_v\}$. The shape operator $S_{\boldsymbol{U}_2}$ of tubular surface with the unit normal vector field $\boldsymbol{U}_2$ in $\mathbb{E}^4$ is obtained as

$$S_{U_2} = \begin{bmatrix} \dfrac{\mathcal{A}_2((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_2\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} & \dfrac{-\mathcal{C}_2((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_2\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} \\ \dfrac{-\mathcal{C}_2((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_2\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} & \dfrac{\mathcal{D}_2\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}{\mathcal{B}_2 r((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)^{3/2}} \end{bmatrix}, \tag{29}$$

where

$$\begin{aligned} \mathcal{A}_2 = {} & k_1 + rk_4^2 + cos^2v(3rk_2^2 + rk_3^2 + r^2(k_1k_3^2 + rk_2^2k_4^2 + rk_3^2k_4^2 - k_2'k_3 + k_1k_2^2 - 2k_2k_4^2 + k_2k_3')) \\ & + rk_3k_4sinv - 3r^2k_2cos^3v(k_3^2 + k_2^2) + r^3cos^4v(k_2^2 + k_3^2)^2 - cosv[k_2 + rk_3' + 2rk_1k_2], \end{aligned}$$

$$\begin{aligned} \mathcal{B}_2 = {} & 2 + r^2cos^2v(7k_2^2 + r^2k_2^2k_4^2 + r^2k_3^2k_4^2 + 3k_3^2) - 4r^3k_2cos^3v(k_2^2 + k_3^2) + r^4k_2^2cos^4v(2k_3^2 + k_2^2) \\ & + r^4k_3^4cos^4v + 2r^2k_4^2 - 6rk_2cosv - 2r^3k_2k_4^2cosv, \end{aligned}$$

$$\begin{aligned} \mathcal{C}_2 = {} & r^2cos^2v(rk_1k_3^2k_4 + rk_1k_2^2k_4 + rk_2k_3'k_4 - 3k_2^2k_4 - k_3^2k_4 - k_2^2k_3sinv - rk_2'k_3k_4 - k_3^3sinv) - k_4 \\ & + rk_2cosv(2k_3sinv + 3k_4) - k_3sinv + r^2k_4cosv(rk_2k_3^2 - 2k_1k_2 - k_3' + rk_2^3k_4cos^2v) + rk_1k_4, \end{aligned}$$

$$\begin{aligned} \mathcal{D}_2 = {} & 1 + r^4cos^4v(18k_2^2k_3^2 + 15k_2^4 + 3k_3^4 + 2r^2k_2^2k_3^2k_4^2 + r^2k_3^4k_4^2 + r^2k_2^4k_4^2) + r^3k_1k_4^2 \\ & + cosv(-2r^4k_1k_2k_4^2 - 6rk_2 - r^3k_2k_4^2 - r^4k_3'k_4^2 + 4r^3k_2k_3k_4sinv) - r^2k_3k_4sinv(2 + r^2k_4^2) \\ & + r^2cos^2v(3k_3^2 + 15k_2^2 + r^2k_4^2(3k_2^2 + k_3^2 - rk_2'k_3 + rk_1k_3^2 + rk_3'k_3) - 2r^2k_4sinv(k_2^2k_3 + k_3^3)) \\ & - r^3cos^3v(20k_2^3 + 12k_2k_3^2 + 3r^2k_2k_3^2k_4^2 + 3r^2k_2^3k_4^2) - 6r^5cos^5v(2k_2^3k_3^2 + k_2k_3^4 + k_2^5) \\ & + r^6cos^6v(3k_2^4k_3^2 + 3k_2^2k_3^4 + k_3^6 + k_2^6), \end{aligned}$$

respectively.

Proof: The equations (27), (16), and (17) are substituted into the equation (6) to yield the coefficients of the second fundamental form of the tubular surface with the unit vector field $\boldsymbol{U}_2$ in $\mathbb{E}^4$ obtained as,

$$L_2 = \frac{cos^2v(3rk_2^2 + rk_3^2 - r^2k_2'k_3 + r^2k_1k_3^2 + r^2k_1k_2^2 + r^2k_2k_3' + r^3k_2^2k_4^2 + r^3k_3^2k_4^2)}{\sqrt{(1-rk_2cosv)^2 + r^2k_3^2cos^2v}\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}$$
$$+\frac{r^3cos^4v(k_3^4 + k_2^4 + 2k_2^2k_3^2) - 3r^2cos^3v(k_2k_3^2 + k_2^3) + rk_4^2 + k_1}{\sqrt{(1-rk_2cosv)^2 + r^2k_3^2cos^2v}\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}$$
$$+\frac{cosv(rk_3k_4sinv - k_2 - 2rk_1k_2 - rk_3' - 2r^2k_2k_4^2)}{\sqrt{(1-rk_2cosv)^2 + r^2k_3^2cos^2v}\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}, \tag{30}$$

$$M_2 = \frac{r(k_3sinv + k_4 - 2rk_2k_4cosv + r^2k_2^2k_4cos^2v + r^2k_3^2k_4cos^2v)}{\sqrt{(1-rk_2cosv)^2 + r^2k_3^2cos^2v}\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}},$$

$$N_2 = \frac{r\sqrt{(1-rk_2cosv)^2 + r^2k_3^2cos^2v}}{\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}.$$

With the help of the equations (21), (22) and (30), the components of the shape operator matrix are as follows

$$\frac{L_2}{E} = \frac{\mathcal{A}_2((1-rk_2cosv)^2 + r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_2\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}, \tag{31}$$

where

$$\mathcal{A}_2 = cos^2v(3rk_2^2 + rk_3^2 + r^2(k_1k_3^2 + rk_2^2k_4^2 + rk_3^2k_4^2 - k_2'k_3 + k_1k_2^2 - 2k_2k_4^2 + k_2k_3')) + rk_4^2$$
$$-3r^2k_2cos^3v(k_3^2 + k_2^2) + r^3cos^4v(k_2^2 + k_3^2)^2 - cosv(k_2 + rk_3' + 2rk_1k_2) + rk_3k_4sinv + k_1,$$

$$\mathcal{B}_2 = 2 + r^2cos^2v(7k_2^2 + r^2k_2^2k_4^2 + r^2k_3^2k_4^2 + 3k_3^2) - 4r^3k_2cos^3v[k_2^2 + k_3^2] + r^4k_2^2cos^4v[2k_3^2 + k_2^2]$$
$$+r^4k_3^4cos^4v + 2r^2k_4^2 - 6rk_2cosv - 2r^3k_2k_4^2cosv,$$

and

$$\frac{1}{W}\left(M_2 - \frac{F}{E}L_2\right) = \frac{-\mathcal{C}_2((1-rk_2cosv)^2 + r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_2\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}, \tag{32}$$

where

$$\mathcal{C}_2 = r^2cos^2v(rk_1k_3^2k_4 + rk_1k_2^2k_4 + rk_2k_3'k_4 - 3k_2^2k_4 - k_3^2k_4 - k_2^2k_3sinv - rk_2'k_3k_4 - k_3^3sinv)$$
$$-k_3sinv + r^2k_4cosv(rk_2k_3^2 - 2k_1k_2 - k_3' + rk_2^3k_4cos^2v) + rk_1k_4$$
$$+rk_2cosv(2k_3sinv + 3k_4) - k_4,$$

and

$$\frac{1}{W^2}(EN_2 - 2FM_2 + \frac{F^2}{E}L_2) = \frac{\mathcal{D}_2((1-rk_2cosv)^2 + r^2k_3^2cos^2v)^{-3/2}}{\mathcal{B}_2r\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}, \tag{33}$$

where

$$\mathcal{D}_2 = 1 + r^4cos^4v(18k_2^2k_3^2 + 15k_2^4 + 3k_3^4 + 2r^2k_2^2k_3^2k_4^2 + r^2k_3^4k_4^2 + r^2k_2^4k_4^2) + r^3k_1k_4^2$$
$$+cosv(-2r^4k_1k_2k_4^2 - 6rk_2 - r^3k_2k_4^2 - r^4k_3'k_4^2 + 4r^3k_2k_3k_4sinv) - r^2k_3k_4sinv(2 + r^2k_4^2)$$
$$+r^2cos^2v(3k_3^2 + 15k_2^2 + r^2k_4^2(3k_2^2 + k_3^2 - rk_2'k_3 + rk_1k_3^2 + rk_3'k_3) - 2r^2k_4sinv(k_2^2k_3 + k_3^3))$$
$$-r^3cos^3v(20k_2^3 + 12k_2k_3^2 + 3r^2k_2k_3^2k_4^2 + 3r^2k_2^3k_4^2) - 6r^5cos^5v(2k_2^3k_3^2 + k_2k_3^4 + k_2^5)$$
$$+r^6cos^6v(3k_2^4k_3^2 + 3k_2^2k_3^4 + k_3^6 + k_2^6)$$

Substituting equations (31), (32) and (33) into the equation (7) implies that the shape operator matrix with respect to $\boldsymbol{U}_2$ following as

$$S_{\boldsymbol{U}_2} = \begin{bmatrix} \frac{\mathcal{A}_2((1-rk_2cosv)^2 + r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_2\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} & \frac{-\mathcal{C}_2((1-rk_2cosv)^2 + r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_2\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} \\ \frac{-\mathcal{C}_2((1-rk_2cosv)^2 + r^2k_3^2cos^2v)^{-1/2}}{\mathcal{B}_2\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} & \frac{\mathcal{D}_2\sqrt{(1-rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}}{\mathcal{B}_2r((1-rk_2cosv)^2 + r^2k_3^2cos^2v)^{3/2}} \end{bmatrix}.$$

Theorem 5: Let $M \subset \mathbb{E}^4$ be a tubular surface at a distance r from the spine curve $\alpha(t)$ according to the quasi-frame $\{\boldsymbol{T}, \boldsymbol{N}_q, \boldsymbol{B}_q, \boldsymbol{C}_q\}$ with the parametrization $\psi(t,v)$, given by $\psi: U \subset \mathbb{E}^2 \rightarrow \mathbb{E}^4$, $(t,v) \in$

U, and let tangent space to M at a point $p \in \psi(t,v)$ be spanned by $\{\psi_t, \psi_v\}$. The Gaussian and mean curvatures are obtained as

$$
\begin{aligned}
K = \; & -\frac{r^2cos^2v(k_2'k_3 - k_1k_3^2 - k_1k_2^2 - k_2k_3') - rcos^2v(k_2^2 + 2k_3^2) + k_1}{r(2 - 6rk_2cosv + r^2cos^2v(7k_2^2 + 3k_3^2) - 4r^3cos^3v(k_2^3 + k_2k_3^3) + r^4cos^4v(k_2^2 + k_3^2)^2)} \\
& + \frac{cosv(k_2 - 2rk_1k_2 - rk_3') + rk_3^2 - rk_3k_4sinv}{r(2 - 6rk_2cosv + r^2cos^2v(7k_2^2 + 3k_3^2) - 4r^3cos^3v(k_2^3 + k_2k_3^3) + r^4cos^4v(k_2^2 + k_3^2)^2)} \\
& + \frac{r^3cos^4v(k_2^4 + k_3^4 + 2k_2^2k_3^2) - cosv(k_2 + rk_3' + 2r^2k_2k_4^2 + 2rk_1k_2)}{r((1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v)} \\
& + \frac{cos^2v(3rk_2^2 + rk_3^2 + r^2k_2k_3' - r^2k_2'k_3 + r^2k_1(k_3^2 + k_2^2) + r^3k_4^2(k_2^2 + k_3^2))}{r((1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v)} \\
& + \frac{k_1 + rk_4^2 + rk_3k_4sinv - 3r^2k_2cos^3v(k_3^2 + k_2^2)}{r((1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v)},
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
H = \; & \frac{rk_1 + rcosv(3k_2 - 2rk_1k_2 - rk_3') + r^3cos^2v(k_1k_2^2 - k_2'k_3 + k_1k_3^2 + k_2k_3')}{2((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} \\
& - \frac{2r^2cos^2v(k_2^2 + k_3^2) + 1 + r^2k_3k_4sinv}{2((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}} \\
& + \frac{r^3cos^2v(k_1k_3^2 + k_1k_2^2 + k_2k_3' - 7k_2k_3^2cosv - 7k_2^3cosv - k_2'k_3) + rk_1 - 5rk_2cosv}{2r\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)^{3/2}} \\
& + \frac{1 + r^2(cosv(3k_3^2cosv + 9k_2^2cosv - k_3' - 2k_1k_2) - k_3k_4sinv) + 2r^4cos^4v(k_2^2 + k_3^2)^2}{2r\sqrt{(1 - rk_2cosv)^2 + 1 + r^2k_3^2cos^2v}((1 - rk_2cosv)^2 + r^2k_3^2cos^2v)^{3/2}}.
\end{aligned}
\tag{35}
$$

Proof: If the equations (15) and (29) are substituted into the equations given by (8) and (9), the Gaussian and mean curvatures are found to be as in (34) and (35), respectively.

Corollary: It is easily seen that the Gaussian and mean curvature calculations obtained using the shape operator are the same as those obtained with the fundamental form coefficients in [32].

Example Let $\alpha(t)$ be a centre curve with quasi-frame of tubular surface in $\mathbb{E}^4$ such as

$$\alpha(t) = (\frac{1}{3\sqrt{2}}cos3t, \frac{1}{3\sqrt{2}}sin3t, \frac{1}{\sqrt{2}}cost, \frac{1}{\sqrt{2}}sint) \tag{36}$$

From $||\alpha(t)|| = 1$, it is easy to see that quasi-vectors are given as

$$
\begin{aligned}
&\boldsymbol{T} = (-\frac{1}{\sqrt{2}}sin3t, \frac{1}{\sqrt{2}}cos3t, -\frac{1}{\sqrt{2}}sint, \frac{1}{\sqrt{2}}cost) \\
&\boldsymbol{N_q} = (0,0,cost,sint) \\
&\boldsymbol{B}_q = (-\frac{1}{\sqrt{2}}(4cos^2v - 1)sint, \frac{1}{\sqrt{2}}(4cos^2v - 3)cost, \frac{1}{\sqrt{2}}sint, -\frac{1}{\sqrt{2}}cost) \\
&\boldsymbol{C}_q = ((4cos^2v - 3)cost, (4cos^2v - 1)sint, 0,0)
\end{aligned}
\tag{37}
$$

and from equation (8), quasi-curvatures are given as

$$k_1 = -\frac{1}{\sqrt{2}}, \qquad k_2 = 0, \qquad k_3 = -\frac{1}{\sqrt{2}} \quad and \qquad k_4 = -\frac{3}{\sqrt{2}}$$

Substituting equations (36) and (37) into equation (10), the tubular surface is parametrized as

$$
\begin{aligned}
\psi(t,v) = \; & \Big(\frac{1}{3\sqrt{2}}cos3t + r\Big(\frac{-1}{\sqrt{2}}(4cos^2v - 1)cosvsint + (4cos^2v - 3)sinvcost\Big), \\
& \frac{1}{3\sqrt{2}}sin3t + r\Big(\frac{1}{\sqrt{2}}(4cos^2v - 3)cosvcost + (4cos^2v - 1)sinvsint\Big), \\
& \frac{1}{\sqrt{2}}cost + \frac{1}{\sqrt{2}}rcosvsint, \frac{1}{\sqrt{2}}sint - \frac{1}{\sqrt{2}}rcosvcost\Big).
\end{aligned}
$$

Hence for $r = \sqrt{2}$, it is straight forward by found that

$$\psi(t,v) = \Big(\frac{1}{3\sqrt{2}}cos3t - (4cos^2v-1)cosvsint + \sqrt{2}(4cos^2v-3)sinvcost,$$
$$\frac{1}{3\sqrt{2}}sin3t + (4cos^2v-3)cosvcost + \sqrt{2}(4cos^2v-1)sinvsint,$$
$$\frac{1}{\sqrt{2}}cost + cosvsint, \frac{1}{\sqrt{2}}sint - cosvcost\Big).$$

Then for $r=\sqrt{2}$, the unit normal vector fields in the equation (38) of tubular surface are given as

$$\begin{aligned} \boldsymbol{U}_1 &= \frac{1}{\sqrt{2+2cos^2v}}\left(-\sqrt{2}cosv, 1, cosv, sinv\right), \\ \boldsymbol{U}_2 &= \frac{1}{\sqrt{2+2cos^2v}}\left(\frac{-\sqrt{2}cosv}{\sqrt{1+2cos^2v}}, \frac{1}{\sqrt{1+2cos^2v}}, -cosv\sqrt{1+2cos^2v}, -sinv\sqrt{1+2cos^2v}\right). \end{aligned} \tag{38}$$

The shape operators of tubular surface are given as

$$S_{U_1} = \frac{(19+2cos^2v)^{-1}}{\sqrt{2+2cos^2v}}\begin{bmatrix} \frac{\mathcal{A}_1}{2} & \frac{\mathcal{B}_1}{2\sqrt{1+2cos^2v}} \\ \frac{\mathcal{B}_1}{2\sqrt{1+2cos^2v}} & \frac{36\mathcal{A}_1 - 2(19+2cos^2v) - 24(sinv-3)}{4+8cos^2v} \end{bmatrix}, \tag{39}$$

and

$$S_{U_2} = \frac{(19+2cos^2v)^{-1}}{\sqrt{2+2cos^2v}}\begin{bmatrix} \frac{\mathcal{A}_2}{2\sqrt{1+2cos^2v}} & \frac{-6-\sqrt{2}\,(3+sinv)}{2} \\ \frac{-6-\sqrt{2}\,(3+sinv)}{2} & \frac{\mathcal{B}_2}{2(1+2cos^2v)^{3/2}} \end{bmatrix}, \tag{40}$$

where

$$\begin{aligned} \mathcal{A}_1 &= 6(sinv-3) - \sqrt{2} - \left(2\sqrt{2}+2\right)cos^2v, \\ \mathcal{B}_1 &= 3\sqrt{2}(1-sinv) - 6 + \left(2\sqrt{2}sinv + 12\right)cos^2v, \\ \mathcal{A}_2 &= 6(sinv+3) - \sqrt{2} - \left(2\sqrt{2}-38\right)cos^2v + 4cos^4v, \\ \mathcal{B}_2 &= 1 + 18\sqrt{2} - 120sinv + 4cos^4v(21+2cos^2v) + \left(42 - 36\sqrt{2} - 24sinv\right)cos^2v, \end{aligned}$$

respectively. The Gaussian and mean curvatures of the tubular surface are given as

$$K = \det\left(S_{U_1}\right) + \det\left(S_{U_2}\right), \quad and \quad H = iz\left(S_{U_1}\right) + iz\left(S_{U_2}\right), \tag{40}$$

respectively, where

$$\begin{aligned} \det\left(S_{U_1}\right) &= \frac{6sinv + \sqrt{2} - 2 + (2\sqrt{2}+4)cos^2v}{8(1+3cos^2v+2cos^4v)}, \\ iz(S_{U_1}) &= \frac{-(6sinv + \sqrt{2} + 1 + (2\sqrt{2}+4)cos^2v)}{2(1+2cos^2v)\sqrt{2+2cos^2v}}, \\ \det\left(S_{U_2}\right) &= \frac{-6sinv - \sqrt{2} - 2 + \left(4 - 12sinv - 4\sqrt{2}\right)cos^2v + \left(8-4\sqrt{2}\right)cos^4v + 8cos^6v}{8(1+5cos^2v+8cos^4v+4cos^6v)}, \\ iz(S_{U_2}) &= \frac{-6sinv - \sqrt{2} + 1 + (6-2\sqrt{2})cos^2v + 8cos^4v}{2(1+3cos^2v)^{3/2}\sqrt{2+2cos^2v}}. \end{aligned}$$

In this case, the curvatures are obtained as

$$\begin{aligned} K = {} & \frac{6sinv + \sqrt{2} - 2 + (2\sqrt{2}+4)cos^2v}{8(1+3cos^2v+2cos^4v)} + \frac{\left(4 - 12sinv - 4\sqrt{2}\right)cos^2v + \left(8-4\sqrt{2}\right)cos^4v}{8(1+5cos^2v+8cos^4v+4cos^6v)} \\ & + \frac{8cos^6v - 6sinv - \sqrt{2} - 2}{8(1+5cos^2v+8cos^4v+4cos^6v)}, \end{aligned}$$

$$H = \frac{-(6sinv + \sqrt{2} + 1 + (2\sqrt{2}+4)cos^2v)}{2(1+2cos^2v)\sqrt{2+2cos^2v}} + \frac{8cos^4v + (6-2\sqrt{2})cos^2v - 6sinv - \sqrt{2} + 1}{2(1+3cos^2v)^{3/2}\sqrt{2+2cos^2v}}.$$

Finally, the tubular surface shown in Figure 2.a. a nd Figure 2.b. are parametrized as

$$\psi(t,v) = \Big(\frac{1}{3\sqrt{2}}cos3t - (4cos^2v-1)cosvsint + \sqrt{2}(4cos^2v-3)sinvcost\,,$$
$$\frac{1}{3\sqrt{2}}sin3t + (4cos^2v-3)cosvcost + \sqrt{2}(4cos^2v-1)sinvsint, \frac{1}{\sqrt{2}}cost + cosvsint\Big),$$
$$\psi(t,v) = \Big(\frac{1}{3\sqrt{2}}cos3t - (4cos^2v-1)cosvsint + \sqrt{2}(4cos^2v-3)sinvcost\,,$$
$$\frac{1}{3\sqrt{2}}sin3t + (4cos^2v-3)cosvcost + \sqrt{2}(4cos^2v-1)sinvsint, \frac{1}{\sqrt{2}}sint - cosvcost\Big)$$

for $r = \sqrt{2}$ in projection spaces xyz and xyt, respectively.

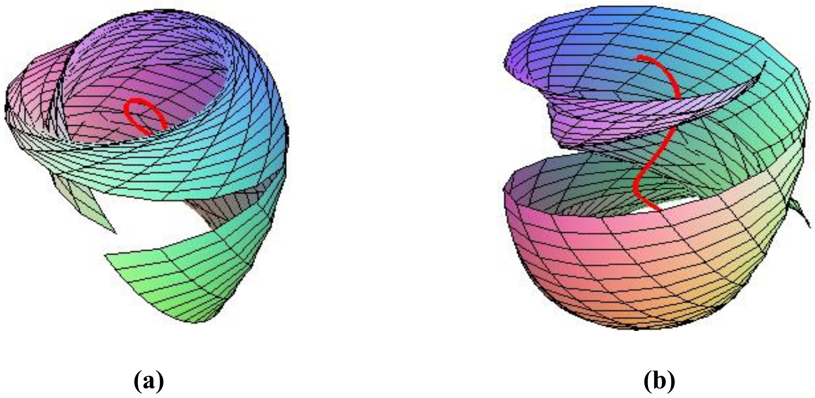

Figure 2. (a) Tubular surface in space xyz**; (b)** Tubular surface in space xyt.

Similarly, the tubular surface shown in Figure 3.a. and Figure 3.b. are parametrized as

$$\psi(t,v) = \Big(\frac{1}{3\sqrt{2}}cos3t - (4cos^2v-1)cosvsint + \sqrt{2}(4cos^2v-3)sinvcost\,,$$
$$\frac{1}{\sqrt{2}}cost + cosvsint, \frac{1}{\sqrt{2}}sint - cosvcost\Big),$$
$$\psi(t,v) = \Big(\frac{1}{3\sqrt{2}}sin3t + (4cos^2v-3)cosvcost + \sqrt{2}(4cos^2v-1)sinvsint,$$
$$\frac{1}{\sqrt{2}}cost + cosvsint, \frac{1}{\sqrt{2}}sint - cosvcost\Big)$$

for $r = \sqrt{2}$ in projection spaces xzt and yzt, respectively.

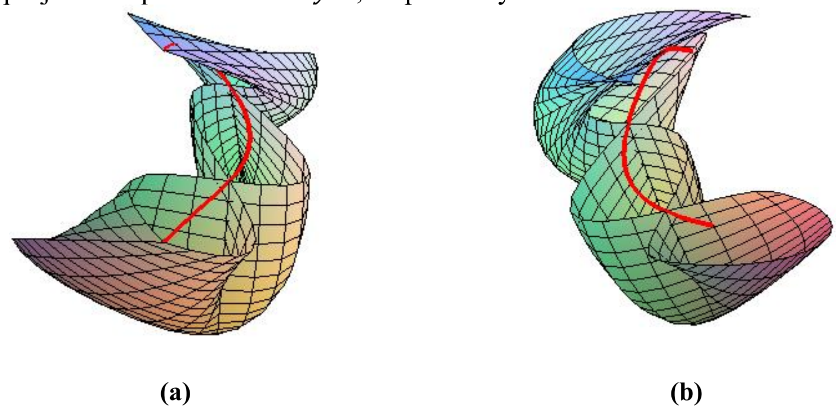

Figure 3. (a) Tubular surface in space xzt**; (b)** Tubular surface in space yzt .

The figures were made using the Maple program.

4- CONCLUSIONS

In this study, the tubular surface is parameterized using a quasi-frame. By introducing directional tubular surfaces in Euclidean 4-space, their unit normal vectors, fundamental form coefficients, shape operator matrices, Gaussian curvatures, and mean curvatures are calculated. Then, a center curve is taken in Euclidean 4-space and calculations are provided with an example. Finally, directional tubular surfaces are visualized in projection spaces.

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CONFLICT OF INTEREST

The author(s) stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Başak Yağbasan: Formal analysis, Writing - original draft, Investigation, Conceptualization. **Cumali Ekici**: Supervision, Writing – Review & Editing, Formal analysis, Visualization

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