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Various types of fixed-point theorems on *S*-metric spaces

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Abstract

Recently, some generalized metric spaces have been studied to obtain new fixed-point theorems. For example, the notion of S-metric space was introduced for this purpose. In this study, some fixed-point results are proved using different contractive conditions on S-metric spaces. Various techniques such as Hard-Rogers type contraction, Khan type contraction, Meir-Keeler-Khan type contraction are used in our theorems to be proved. These fixed-point results extend some known fixed-point theorems on S-metric spaces. Also, to illustrate obtained theoretical results, some examples are given using an S-metric which is not generated by any metric. As an application, a new fixed-circle result is presented using modified C-Khan type contraction on S-metric spaces.

Keywords: *S-metric space, modified Hardy-Rogers type contraction, Khan type contraction, fixed point, fixed circle.*

S-metrik uzaylar üzerinde sabit-nokta teoremlerinin çeşitli türleri

Özet

Son zamanlarda yeni sabit nokta teoremleri elde etmek için bazı genelleştirilmiş metrik uzaylar çalışılmaktadır. Örneğin, S-metrik uzay kavramı bu amaç için tanıtılmıştır. Bu çalışmada, S-metrik uzaylar üzerinde farklı daralma koşulları kullanılarak bazı sabit nokta sonuçları ispatlanmıştır. İspatlanan teoremlerde Hardy-Rogers tipinde daralma, Khan tipinde daralma, Meir-Keeler-Khan tipinde daralma gibi çeşitli teknikler kullanılmıştır. Bu sabit nokta sonuçları S-metrik uzaylar üzerindeki bazı bilinen sabit nokta sonuçlarını genellemektedir. Ayrıca, herhangi bir metrik tarafından üretilemeyen S-metrik örnekleri kullanılarak elde edilen teorik sonuçları gerçekleyecek bazı örnekler

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verilmiştir. S-metrik uzaylar üzerinde bir uygulama olarak değiştirilmiş C-Khan tipinde daralma kavramı kullanılarak yeni bir sabit çember sonucu verilmiştir.

Anahtar kelimeler: *S-metrik uzay, değiştirilmiş Hardy-Rogers tipinde daralma, Khan tipinde daralma, sabit nokta, sabit çember.*

1. Introduction and preliminaries

It is well-known that fixed-point theory was began with the Banach's contraction principle [1]. Using different approaches, this principle has been extended and studied. One of the used approaches is to generalize the studied contractive conditions. For example, it was given various contractions such as Hardy-Rogers type contraction [2], modified Hardy-Rogers type contraction [3], Khan type contraction [4], Meir-Keeler type contraction [5], Meir-Keeler-Khan type contraction [6] etc.

Another approach is to generalize the used metric spaces. For example, the concept of an *S*-metric space was introduced as a generalization of a metric space as follows:

Let X be a nonempty set and $S : X \times X \times X \to [0,\infty)$ be a function satisfying the following conditions for all $u,v,w,t \in X$:

(S1) $S(u,v,w) = 0$ if and only if $u = v = w$,
(S2) $S(u,v,w) \leq S(u,u,t) + S(v,v,t) + S(w,w,t)$.

Then *S* is called an *S*-metric on X and the pair (X,S) is called an *S*-metric space [7].

For an *S*-metric space, the symmetry condition can be considered as follows:

$$S(u,u,v) = S(v,v,u), \qquad (\blacktriangle)$$

for all $u,v \in X$ [7].

In the literature, there exist some examples of *S*-metric which is not generated by any metric (see [8] and [9] for more details). Therefore, it is important to study new fixed-point theorems on *S*-metric spaces. Some fixed-point results have been still investigated using different techniques to generalize some well-known fixed-point theorems (for example, see [10], [11], [12] and [13] for more details).

Recently, a new approach is being studied to do geometric interpretations for fixed points which is called fixed-circle problem [14]. The notions of a circle and a fixed circle were defined on *S*-metric spaces as follows:

Let (X,S) be an *S*-metric space, $C^S_{x_0,r} = \{u \in X : S(u,u,x_0) = r\}$ be a circle centered at x_0 with radius r and $T : X \to X$ be a self-mapping. If $Tu = u$ for all $u \in C^S_{x_0,r}$ then the circle $C^S_{x_0,r}$ is called as the fixed circle of T [15].

Using the above definitions, some fixed-circle results were obtained on *S*-metric spaces (see [15] and [16] for more details).

Motivated by the above studies, the aim of this paper is to prove some fixed-point theorems using different contractive conditions on *S*-metric spaces. In Section 2, a new fixed-point result is obtained using the modified Hardy-Rogers type contraction and some relationships between this result and well-known corollaries are established. In Section 3, two fixed-point theorems are presented using the Khan type contraction and the Meir-Keeler-Khan type contraction with some illustrative examples. In Section 4, as an application of the obtained results, a new fixed-circle result is given on *S*-metric spaces.

2. A fixed-point theorem using modified Hardy-Rogers type Contractive condition

In this section, a new fixed-point theorem is obtained using the modified Hardy-Rogers type contractive condition on a complete *S*-metric space.

At first, we recall the following definition:

2.1. Definition

Let (X,S) be an *S*-metric space and $\{u_n\}$ be a sequence in X.

(1) A sequence $\{u_n\}$ converges to $u \in X$ if $S(u_n,u_n,u) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(u_n,u_n,u) < \varepsilon$.

(2) A sequence $\{u_n\}$ is a Cauchy sequence if $S(u_n,u_n,u_m) \to 0$ as $n,m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \geq n_0$ we have $S(u_n,u_n,u_m) < \varepsilon$.

(3) (X,S) is complete if every Cauchy sequence is a convergent sequence [7].

We give the following definition.

2.2. Definition

Let (X,S) be an *S*-metric space and $T : X \to X$ be a self-mapping. If T satisfies the following condition

$$S(Tu,Tu,Tv) \leq \alpha S(u,u,v) + \beta S(u,u,Tv) + \gamma S(v,v,Tu) + \eta \frac{S(v,v,Tv)\left[1+S(u,u,Tu)\right]}{1+S(u,u,v)}$$

$$+ \lambda \frac{S(v,v,Tv)+S(v,v,Tu)}{1+S(v,v,Tv)S(v,v,Tu)} + \mu \frac{S(u,u,Tu)\left[1+S(v,v,Tu)\right]}{1+S(u,u,v)+S(v,v,Tv)},$$

where $\alpha, \beta, \gamma, \eta, \lambda, \mu \geq 0$ with $\alpha + 3\beta + \gamma + \eta + \lambda + \mu < 1$ for all $u,v \in X$, then T is called a modified Hardy-Rogers type contraction.

2.3. Theorem

Let (X,S) be a complete S-metric space and $T:X\to X$ be a self-mapping. If T satisfies the modified Hardy-Rogers type contractive condition, then T has a unique fixed point in X .

Proof. Let $u_0\in X$ and define the sequence $\{u_n\}$ as follows:

$$Tu_n=u_{n+1}.$$

Assume that $u_n\neq u_{n+1}$ for all n. Using the modified Hardy-Rogers type contractive condition, we have

$$\begin{aligned}S(u_n,u_n,u_{n+1})=S(Tu_{n-1},Tu_{n-1},Tu_n)&\leq \alpha S(u_{n-1},u_{n-1},u_n)+\beta S(u_{n-1},u_{n-1},u_{n+1})+\gamma S(u_n,u_n,u_n)\\&+\eta\frac{S(u_n,u_n,u_{n+1})\left[1+S(u_{n-1},u_{n-1},u_n)\right]}{1+S(u_{n-1},u_{n-1},u_n)}+\lambda\frac{S(u_n,u_n,u_{n+1})+S(u_n,u_n,u_n)}{1+S(u_n,u_n,u_{n+1})S(u_n,u_n,u_n)}\\&+\mu\frac{S(u_{n-1},u_{n-1},u_n)\left[1+S(u_n,u_n,u_n)\right]}{1+S(u_{n-1},u_{n-1},u_n)+S(u_n,u_n,u_{n+1})}\\&\leq(\alpha+2\beta+\mu)S(u_{n-1},u_{n-1},u_n)+(\beta+\eta+\lambda)S(u_n,u_n,u_{n+1})\end{aligned}$$

and so

$$(1-\beta-\eta-\lambda)S(u_n,u_n,u_{n+1})\leq(\alpha+2\beta+\mu)S(u_{n-1},u_{n-1},u_n),$$

which implies

$$S(u_n,u_n,u_{n+1})\leq\frac{\alpha+2\beta+\mu}{1-\beta-\eta-\lambda}S(u_{n-1},u_{n-1},u_n). \tag{1}$$

If we take $h=\dfrac{\alpha+2\beta+\mu}{1-\beta-\eta-\lambda}$, then we have $h<1$ since $\alpha+3\beta+\eta+\lambda+\mu<1$. Also we note that $1-\beta-\eta-\lambda\neq 0$ since $\beta+\eta+\lambda<1$. Using the inequality (1), we obtain

$$S(u_n,u_n,u_{n+1})\leq h^nS(u_0,u_0,u_1). \tag{2}$$

For all $n,m\in\mathbb{N}$, $n<m$, using the inequality (2) and the condition (S2), we get

$$S(u_n,u_n,u_m)\leq\frac{2h^n}{1-h}S(u_0,u_0,u_1)$$

and

$$\lim_{n,m\to\infty}S(u_n,u_n,u_m)=0 \text{ since } \lim_{n,m\to\infty}\frac{2h^n}{1-h}S(u_0,u_0,u_1)=0.$$

Hence $\{u_n\}$ is a Cauchy sequence in X. From the completeness hypothesis, there exists $u \in X$ such that $u_n \to u$. Let us consider $Tu \neq u$, that is, the point u is not a fixed point of T. Therefore, we have

$$\begin{aligned} S(u_n,u_n,Tu) = S(Tu_{n-1},Tu_{n-1},Tu) &\leq \alpha S(u_{n-1},u_{n-1},u) + \beta S(u_{n-1},u_{n-1},Tu) + \gamma S(u,u,u_n) \\ &+ \eta \frac{S(u,u,Tu)\left[1+S(u_{n-1},u_{n-1},u_n)\right]}{1+S(u_{n-1},u_{n-1},u)} + \lambda \frac{S(u,u,Tu)+S(u,u,u_n)}{1+S(u,u,Tu)S(u,u,u_n)} \\ &+ \mu \frac{S(u_{n-1},u_{n-1},u_n)\left[1+S(u,u,u_n)\right]}{1+S(u_{n-1},u_{n-1},u)+S(u,u,Tu)} \end{aligned}$$

and so taking limit for $n \to \infty$, we obtain

$$S(u,u,Tu) \leq (\beta + \eta + \lambda) S(u,u,Tu),$$

which implies that $Tu = u$ since $\beta + \eta + \lambda < 1$. Finally, we show that u is a unique fixed point of T. On the contrary, suppose that v is another fixed point of T such that $u \neq v$. Using the modified Hardy-Rogers type contractive condition and symmetry condition, we get

$$\begin{aligned} S(Tu,Tu,Tv) = S(u,u,v) &\leq \alpha S(u,u,v) + \beta S(u,u,Tv) + \gamma S(v,v,Tu) \\ &+ \eta \frac{S(v,v,Tv)\left[1+S(u,u,Tu)\right]}{1+S(u,u,v)} + \lambda \frac{S(v,v,Tv)+S(v,v,Tu)}{1+S(v,v,Tv)S(v,v,Tu)} \\ &+ \mu \frac{S(u,u,Tu)\left[1+S(v,v,Tu)\right]}{1+S(u,u,v)S(v,v,Tv)} = (\alpha + \beta + \gamma + \lambda) S(u,u,v), \end{aligned}$$

which implies that $u = v$ since $\alpha + \beta + \gamma + \lambda < 1$. Consequently, u is a unique fixed point of T. □

Now we give the following illustrative example of Theorem 2.3.

2.4. Example
Let $\mathbb{R}$ be the *S*-metric space with

$$S(u,v,w) = |u - w| + |u + w - 2v|,$$

for all $u,v,w \in \mathbb{R}$ [9]. Notice that this *S*-metric is not generated by any metric. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ by

$$Tu = \begin{cases} u + 130 & \text{if} \quad u \in \{0,4\} \\ 125 & \text{otherwise} \end{cases},$$

for all $u \in \mathbb{R}$. Then T satisfies the modified Hardy-Rogers type contractive condition for $\alpha = \gamma = \frac{1}{3}$, $\beta = \frac{1}{9}$ and $\eta = \lambda = \mu = 0$. Therefore T has a unique fixed point $u = 125$

in $\mathbb{R}$. But T does not satisfy the condition of Corollary 2.7 (see page 118 in [10]) which is called the Banach's contraction principle on *S*-metric spaces. Indeed, if we take $u=0$ and $v=4$ then we obtain

$$S(Tu,Tu,Tv)=8\leq LS(u,u,v)=8L,$$

which is a contradiction since $L\in[0,1)$.

2.5. Remark

(1) If we take $\alpha=L\in[0,1)$ and $\beta=\gamma=\eta=\lambda=\mu=0$ in Theorem 2.3, then we get Corollary 2.7 on page 118 in [10].

(2) If we take $\beta=\gamma=a\in\left[0,\frac{1}{3}\right)$ and $\alpha=\eta=\lambda=\mu=0$ in Theorem 2.3, then we get Corollary 2.15 on page 121 in [10].

(3) If we take $\alpha=a$, $\beta=b$, $\gamma=c$, $\alpha+\beta+\gamma<1$, $\alpha+3\beta<1$ and $\eta=\lambda=\mu=0$ in Theorem 2.3, then we get Corollary 2.17 on page 122 in [10].

If we consider a complex valued *S*-metric space defined in [17], then we obtain the following theorem.

2.6. Theorem

Let (X,S_C) be a complete complex valued *S*-metric space and $T:X\to X$ be a self-mapping. If T satisfies the following condition

$$\begin{aligned}S_C(Tu,Tu,Tv)\precsim\ &\alpha S_C(u,u,v)+\beta S_C(u,u,Tv)+\gamma S_C(v,v,Tu)\\ &+\eta\frac{S_C(v,v,Tv)\left[1+S_C(u,u,Tu)\right]}{1+S_C(u,u,v)}+\lambda\frac{S_C(v,v,Tv)+S_C(v,v,Tu)}{1+S_C(v,v,Tv)S_C(v,v,Tu)}\\ &+\mu\frac{S_C(u,u,Tu)\left[1+S_C(v,v,Tu)\right]}{1+S_C(u,u,v)+S_C(v,v,Tv)},\end{aligned}$$

where $\alpha,\beta,\gamma,\eta,\lambda,\mu\geq 0$ with $\alpha+3\beta+\gamma+\eta+\lambda+\mu<1$ for all $u,v\in X$, then T has a unique fixed point in X.

Proof. It follows easily by the similar arguments used in the proof of Theorem 2.3 and using some properties of complex numbers such as convergence of sequences, modulus of complex numbers etc.□

3. Some fixed-point results with Khan type contractive conditions

In this section, we define the Khan type and Meir-Keeler-Khan type contractive conditions. Using these new conditions, some new fixed-point theorems are proved on *S*-metric spaces.

3.1. Definition

Let (X,S) be an *S*-metric space and $T:X\to X$ be a self-mapping. If T satisfies the following condition

$$S(Tu,Tu,Tv)\leq\begin{cases} h\dfrac{S(u,u,Tu)S(u,u,Tv)+S(v,v,Tv)S(v,v,Tu)}{S(u,u,Tv)+S(Tu,Tu,v)} & \text{if} \quad S(u,u,Tv)+S(Tu,Tu,v)\neq 0 \\ 0 & \text{if} \quad S(u,u,Tv)+S(Tu,Tu,v)=0 \end{cases},$$

where $h\in[0,1)$ for all $u,v\in X$, then T is called a Khan type contraction.

3.2. Theorem

Let (X,S) be a complete *S*-metric space and $T:X\to X$ be a self-mapping. If T satisfies the Khan type contractive condition, then T has a unique fixed point in X.

Proof. Let $u_0\in X$ and the sequence $\{u_n\}$ be defined as in the proof of Theorem 2.3. Assume that $u_n\neq u_{n+1}$ for all $n\in\mathbb{N}$. Using the Khan type contractive condition, we have the following cases:

Case 1. Suppose that

$$S(u_m,u_m,Tu_n)+S(Tu_m,Tu_m,u_n)\neq 0,$$

for all $m\in\mathbb{N}-\{0\}$ and $n\in\mathbb{N}$. Then we get

$$\begin{aligned} S(u_n,u_n,u_{n+1}) &= S(Tu_{n-1},Tu_{n-1},Tu_n) \\ &\leq h\frac{S(u_{n-1},u_{n-1},u_n)S(u_{n-1},u_{n-1},u_{n+1})+S(u_n,u_n,u_{n+1})S(u_n,u_n,u_n)}{S(u_{n-1},u_{n-1},u_{n+1})+S(u_n,u_n,u_n)} \\ &= hS(u_{n-1},u_{n-1},u_n). \end{aligned}$$

From the definition of the Khan type contractive condition, we have

$$S(u_n,u_n,u_{n+1})\leq h^n S(u_0,u_0,u_1). \tag{3}$$

For all $n,m\in\mathbb{N}$, $n<m$, using the inequality (3) and the condition (S2), we get

$$S(u_n,u_n,u_m)\leq\frac{2h^n}{1-h}S(u_0,u_0,u_1)$$

and

$$\lim_{n,m\to\infty}S(u_n,u_n,u_m)=0 \text{ since } \lim_{n,m\to\infty}\frac{2h^n}{1-h}S(u_0,u_0,u_1)=0.$$

Hence $\{u_n\}$ is a Cauchy sequence in X. By the completeness hypothesis, there exists $u \in X$ such that $u_n \to u$. Let us consider $Tu \neq u$, that is, the point u is not a fixed point of T. Therefore, we have

$$S(u_n, u_n, Tu) = S(Tu_{n-1}, Tu_{n-1}, Tu)$$
$$\leq h \frac{S(u_{n-1}, u_{n-1}, u_n) S(u_{n-1}, u_{n-1}, Tu) + S(u, u, Tu) S(u, u, u_n)}{S(u_{n-1}, u_{n-1}, Tu) + S(u_n, u_n, u)}$$

and so taking the limit for $n \to \infty$, we obtain

$$S(u, u, Tu) \leq 0,$$

which implies that $Tu = u$ from the condition (S1). Finally, we show that u is a unique fixed point of T. On the contrary, suppose that v is another fixed point of T such that $u \neq v$. Using the Khan type contractive condition, we get

$$S(Tu, Tu, Tv) = S(u, u, v)$$
$$\leq h \frac{S(u, u, Tu) S(u, u, Tv) + S(v, v, Tv) S(v, v, Tu)}{S(u, u, Tv) + S(Tu, Tu, v)} = 0,$$

which implies that $u = v$ from the condition (S1). Consequently, u is a unique fixed point of T.

Case 2: Suppose that

$$S(u_m, u_m, Tu_n) + S(Tu_m, Tu_m, u_n) = 0,$$

for all $m \in \mathbb{N} - \{0\}$ and $n \in \mathbb{N}$. Then it can be easily seen that T has a unique fixed point in X. □

In the following, we see an example of a self-mapping satisfying the Khan type contractive condition.

3.3. Example

Let $X = \{1, 2, 3\}$ be the *S*-metric space with

$$S(1,1,2) = S(2,2,1) = 5,$$
$$S(2,2,3) = S(3,3,2) = S(1,1,3) = S(3,3,1) = 2,$$
$$S(u,v,w) = 0 \text{ if } u = v = w,$$
$$S(u,v,w) = 1 \text{ otherwise},$$

for all $u, v, w \in X$ in [9]. We note that this *S*-metric is not generated by any metric. Let us define the self-mapping $T : X \to X$ by

$$Tu = \begin{cases} 2 & \text{if} \quad u = 1 \\ 3 & \text{if} \quad u \neq 1 \end{cases},$$

for all $u \in X$. Then T satisfies the Khan type contractive condition for $h = \frac{1}{2}$. Therefore T has a unique fixed point $u = 3$ in X.

If we consider a complex valued *S*-metric space, then we obtain the following theorem.

3.4. Theorem

Let (X, S_C) be a complete complex valued *S*-metric space and $T : X \to X$ be a self-mapping. If T satisfies the following condition

$$S_C(Tu, Tu, Tv) \precsim \begin{cases} h\dfrac{S_c S(u,u,Tu) S_c(u,u,Tv) + S_c(v,v,Tv) S_c(v,v,Tu)}{S_c(u,u,Tv) + S_c(Tu,Tu,v)} & \text{if} \quad S_c(u,u,Tv) + S_c(Tu,Tu,v) \neq 0 \\ 0 & \text{if} \quad S_c(u,u,Tv) + S_c(Tu,Tu,v) = 0 \end{cases},$$

where $h \in [0,1)$ for all $u, v \in X$, then T has a unique fixed point in X.

Proof. It follows easily by the similar arguments used in the proof of Theorem 3.2 and using some properties of complex numbers such as convergence of sequences, modulus of complex numbers etc. □

Now we consider the condition that if $T : X \to X$ is a self-mapping, then for all $u, v \in X$,

$$u \neq v \Rightarrow S(u,u,Tv) + S(v,v,Tu) \neq 0. \qquad (\blacklozenge)$$

3.5. Definition

Let (X, S) be an *S*-metric space and $T : X \to X$ be a self-mapping. Then T is called a Meir-Keeler-Khan type contraction whenever for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq h\frac{S(u,u,Tu)S(u,u,Tv) + S(v,v,Tv)S(v,v,Tu)}{S(u,u,Tv) + S(Tu,Tu,v)} < \varepsilon + \delta(\varepsilon) \Rightarrow S(Tu,Tu,Tv) < \varepsilon,$$

where $h \in [0,1)$.

Notice that if T is a Meir-Keeler-Khan type contraction on X, then we get

$$S(Tu,Tu,Tv) \leq h\frac{S(u,u,Tu)S(u,u,Tv) + S(v,v,Tv)S(v,v,Tu)}{S(u,u,Tv) + S(Tu,Tu,v)}. \qquad (4)$$

3.6. Theorem

Let (X,S) be a complete *S*-metric space and $T:X\to X$ be a self-mapping. If T satisfies the Meir-Keeler-Khan type contractive condition, then T has a unique fixed point in X.

Proof. By the similar arguments used in the proof of Theorem 3.2, it can be easily seen that T has a unique fixed point in X. Indeed, if we consider the condition (♦) and the inequality (4), then we can use the techniques given in Case 1 in the proof of Theorem 3.2.□

If we consider a complex valued *S*-metric space, then we obtain the following theorem.

3.7. Theorem

Let (X,S_C) be a complete complex valued *S*-metric space and $T:X\to X$ be a self-mapping. If T satisfies the following condition for each $\varepsilon>0$, there exists $\delta>0$ such that

$$\varepsilon \precsim h\frac{S_C(u,u,Tu)S_C(u,u,Tv)+S_C(v,v,Tv)S_C(v,v,Tu)}{S_C(u,u,Tv)+S_C(Tu,Tu,v)} \prec \varepsilon+\delta(\varepsilon) \Rightarrow S_C(Tu,Tu,Tv)\prec\varepsilon ,$$

where $h\in[0,1)$, then T has a unique fixed point in X.

Proof. It follows easily by the similar arguments used in the proof of Theorem 3.2 and using some properties of complex numbers such as convergence of sequences, modulus of complex numbers etc. □

4. An application to fixed-circle problem on *S*-metric spaces

In this section, we obtain a fixed-circle result using modified C-Khan type contractive condition on *S*-metric spaces.

Now we consider the condition (♦).

4.1. Definition

Let (X,S) be an *S*-metric space and $T:X\to X$ be a self-mapping. Then T is called a modified C-Khan type contraction if there exists $x_0\in X$ such that

$$S(Tu,Tu,u)\leq h\frac{S(u,u,Tu)S(u,u,Tx_0)+S(x_0,x_0,Tx_0)S(x_0,x_0,Tu)}{S(u,u,Tx_0)+S(Tu,Tu,x_0)},$$

where $h\in[0,1)$ for all $u\in X$.

4.2. Theorem

Let (X,S) be an *S*-metric space, $T:X\to X$ be a self-mapping and $C^S_{x_0,r}$ be a circle on X. If T is a modified C-Khan type contraction for all $u\in C^S_{x_0,r}$ with $Tx_0=x_0$, then T fixes the circle $C^S_{x_0,r}$.

Proof. Let $u\in C^S_{x_0,r}$. Assume that $u\neq Tu$. Hence using the modified C-Khan type contractive condition with $Tx_0=x_0$, we get

$$\begin{aligned} S(Tu,Tu,u) &\leq h\frac{S(u,u,Tu)S(u,u,Tx_0)+S(x_0,x_0,Tx_0)S(x_0,x_0,Tu)}{S(u,u,Tx_0)+S(Tu,Tu,x_0)} \\ &= \frac{hrS(u,u,Tu)}{r+S(Tu,Tu,x_0)}. \end{aligned}$$

Then using the above inequality and the symmetry condition (▲), we get the following cases:

Case 1. If $S(Tu,Tu,x_0)=r$ then we have

$$S(Tu,Tu,u)\leq\frac{hrS(u,u,Tu)}{2r}=\frac{h}{2}S(u,u,Tu)=\frac{h}{2}S(Tu,Tu,u),$$

which implies $Tu=u$ since $h\in[0,1)$.

Case 2. $S(Tu,Tu,x_0)>r$ then we have

$$S(Tu,Tu,u)\leq\frac{hrS(u,u,Tu)}{2r}=\frac{h}{2}S(u,u,Tu)=\frac{h}{2}S(Tu,Tu,u),$$

which implies $Tu=u$ since $h\in[0,1)$. This is a contradiction since $u\in C^S_{x_0,r}$.

Case 3. $S(Tu,Tu,x_0)<r$ then we have

$$S(Tu,Tu,u)\leq hS(u,u,Tu)=hS(Tu,Tu,u),$$

which implies $Tu=u$ since $h\in[0,1)$. This is a contradiction since $u\in C^S_{x_0,r}$.

Consequently, $C^S_{x_0,r}$ is a fixed circle of T. □

Finally, we give an example of a self-mapping which has a fixed circle.

4.3. Example

Let $\mathbb{R}$ be the *S*-metric space given in Example 2.4 and consider the circle $C^S_{0,4}$. Let us define the self-mapping $T:\mathbb{R}\to\mathbb{R}$ by

$$Tu = \begin{cases} u & \text{if} \quad u \in \{-2, 0, 2\} \\ 0 & \quad \text{otherwise} \end{cases},$$

for all $u \in \mathbb{R}$. Then T satisfies the modified C-Khan type contractive condition and $T0 = 0$. From Theorem 4.2, it is clear that $C_{0,4}^S$ is a fixed circle of T.

5. Conclusion

In this paper, new fixed-point theorems are proved using various types of contractive conditions such as modified Hardy-Rogers type, Khan type etc. These obtained results generalize some well-known fixed-point theorems on *S*-metric spaces. Also, as an application of our results, a fixed-circle theorem is given with modified C-Khan type contraction. We expect that this study will help to generate some new researches and applications about fixed-point or fixed-circle theorems on different generalized metric spaces. For example, it is possible to study similar results on an S_b-metric space (see [18] and [19] for more details about S_b-metric spaces).

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