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Quintic B-spline Differential Quadrature for Burgers' Equation

Alper Korkmaz

Department of Mathematics, Çankırı Karatekin University, Turkey,
e-mail: alperkorkmaz7@gmail.com

Abstract: In this study differential quadrature method based on quintic B-spline functions is setup for numerical solutions for nonlinear viscous Burgers' equation. After space discretization with differential quadrature and application of boundary conditions, the resultant ordinary differential equation system is integrated in time by using Runge-Kutta method of order four. The method is validated by solving two initial value problems for the Burgers' equation. The errors of the numerical solutions are measured by using discrete maximum norm. A comparison with some earlier works also given for the problem modeling fadeout of an initial shock.

Keywords: quintic B-spline; Burgers' Equation; differential quadrature method, shock wave, sinusoidal disturbance.

1. Introduction

One dimensional nonlinear Burgers' equation (NBE)

$$\frac{\partial U(x,t)}{\partial t} + U(x,t)\frac{\partial U(x,t)}{\partial x} - \beta\frac{\partial^2 U(x,t)}{\partial x^2} = 0 \tag{1}$$

where $\beta > 0$ denotes the constant viscosity coefficient was first proposed by Bateman with its steady state solutions [1, 2]. Burgers[3] used the NBE as a simple mathematical model to illustrate the theory of turbulence due to its analogy to Navier-Stokes equations[4]. In the equation, the terms UU_x and βU_{xx} represent simple nonlinear advection and linear diffusion, respectively[5]. The NBE models a competition of wave steepening UU_x and diffusion βU_{xx}[6].

Different types of the NBE is used as a model in various areas covering physics, engineering, etc. The NBE was used as a model in gas dynamics and continuous stochastic process[2]. Formation, propagation and decay of shock waves can be modeled by the NBE[7]. Kachroo et al. showed that the NBE is a model for traffic flow problem[8].

In astrophysics, the evolution of density inhomogenities and the velocity field in an expanding continuous medium was studied based on the NBE[9]. Katz and Green considered the NBE as a very simple model of interstellar dynamics with the assumption of isobaric gases and impulse giving random supernova explosions[10]. Kofman and Raga also used one dimensional viscous NBE to model structures of knots in jet flows[11].

According to Wazwaz, the NBE is a completely integrable and an approximation of lowest order for one-dimensional weak shock wave motion in a fluid, description of a highway traffic phenomena, and a model for acoustic transmission[12]. Cole used the same equation for heat conduction problems[13]. In the same study, he solved the NBE analytically after reducing it linear diffusion equation by using a nonlinear transformation based upon logarithmic functions[13]. Hopf also solved the NBE simultaneously by using the same reduction with Cole and showed that it can be solved for arbitrary initial conditions[14]. Some more exact solutions were reported by Benton and Platzman[15].

Having analytical solutions for various initial boundary value problems attracts many researchers to validate the developed numerical algorithms. So far, various methods included in finite element, finite difference, and spectral method families have been proposed for solutions of the NBE. Implicit fourth-order compact finite difference method was developed by Liao to solve the NBE, which is converted to the linear heat equation by Cole-Hopf transformation[16]. Sari and Gürarslan used a compact finite difference method of order six to discretize the NBE in space and then integrated the obtained ordinary differential equation system in time with third-order Runge-Kutta method. They compared their results with some earlier results for two initial boundary value problems containing trigonometric and polynomial initial conditions and homogenous Dirichlet boundary conditions at both ends of the finite problem interval[17]. Zhu and Wang' s study aimed to obtain the solutions of the NBE by implementation of cubic B-spline quasi-interpolation[19].

Many finite element methods covering Galerkin approaches [18, 21, 22] or B-spline based finite elements[23] were also constructed for the numerical solutions the NBE. Moreover some collocation methods[24, 25, 20], and differential quadrature methods based upon various basis functions such as Lagrange polynomials with nonuniform grid distrubiton, cubic and quartic B-splines were constructed for different initial boundary value problems for the NBE[26, 27, 28].

This study aims to solve initial boundary value problems for the NBE by setting a differential quadrature algorithm based upon quintic B-spline functions. The accuracy and validity of the proposed algorithm will be checked by comparison with the exact solutions and some studies in literature. In the study, the general form of an initial boundary value problem for the NBE defined by

$$\begin{aligned} \frac{\partial U(x,t)}{\partial t} + U(x,t)\frac{\partial U(x,t)}{\partial x} - \beta\frac{\partial^2 U(x,t)}{\partial x^2} &= 0,\ a<x<b, t>t_0 \\ U(x,t_0) &= U_0(x) \\ U(a,t) &= s_1(t), \\ U(b,t) &= s_2(t) \end{aligned} \tag{2}$$

over a finite problem interval $[a,b]$ will be considered. Two different initial boundary value problems modeling fadeout of an initial shock and sinusoidal disturbance will be solved by application of the proposed method.

2. Quintic B-spline Differential Quadrature Method (QBSDQ)

Differential quadrature method (DQM) is a direct derivative approximate technique and is used to solve differential equations (possible for both ordinary and partial) numerically[29]. Since the derivatives of the functions at nodes is approximated by the weighted sum of nodal functional values at the whole domain, the main idea of the approximation can be analogous to finite difference. However, the application of the method has two fundamental steps. Following the determination of the weights of nodal functional values by using basis functions spanning the problem interval, the derivative approximations are substituted into the related derivative terms in the equation.

Let $P : a = x_1 < x_2 < \ldots < x_N = b$ be a uniform node distribution of the finite problem interval $[a,b]$. The approximation to $\frac{\partial U^r(x,t)}{\partial x^r}$ at x_i is written as

$$\frac{\partial U^r(x_i,t)}{\partial x^r} = \sum_{j=1}^{N} w_{i,j}^{(r)} U(x_j,t), i = 1,2,...,N, r = 1,2 \tag{3}$$

where $w_{i,j}^{(r)}$ is the weights the function $U(x,t)$ at the grid x_i. So far, the determination of the weights $w_{i,j}^{(1)}$ and $w_{i,j}^{(2)}$ has been accomplished by using different basis function sets such as spline functions, radial basis functions, harmonic functions and Lagrange polynomials[29, 30, 31, 32, 33, 34, 35, 36, 37]. In this study, we will calculate the weights by substituting the quintic B-spline functions spanning the problem interval $[a,b]$.

Let $L_m(x), m = -1,0,\ldots,N+2$ be a quintic B-spline functions defined as

$$L_m(x) = \frac{1}{h^5}\begin{cases} \tau_1 & , \quad [x_{m-3}, x_{m-2}] \\ \tau_1 - 6\tau_2 & , \quad [x_{m-2}, x_{m-1}] \\ \tau_1 - 6\tau_2 + 15\tau_3 & , \quad [x_{m-1}, x_m] \\ \tau_1 - 6\tau_2 + 15\tau_3 - 20\tau_4 & , \quad [x_m, x_{m+1}] \\ \tau_1 - 6\tau_2 + 15\tau_3 - 20\tau_4 + 15\tau_5 & , \quad [x_{m+1}, x_{m+2}] \\ \tau_1 - 6\tau_2 + 15\tau_3 - 20\tau_4 + 15\tau_5 - 6\tau_6 & , \quad [x_{m+2}, x_{m+3}] \end{cases} \tag{4}$$

where $\tau_1 = (x-x_{m-3})^5, \tau_2 = (x-x_{m-2})^5, \tau_3 = (x-x_{m-1})^5, \tau_4 = (x-x_m)^5, \tau_5 = (x-x_{m+1})^5, \tau_6 = (x-x_{m+2})^5$[38]. Each quintic B-spline $L_m(x)$ has zero value out of the interval $[x_{m-3}, x_{m+3}]$ and the set $\{L_m(x)\}_{m=-1}^{N+2}$ spans $[a,b]$ and forms a basis for the functions defined in this interval. Substitution of each of the basis functions into Eq.(3) for fixed x_i and r leads to

$$\frac{d^r L_m(x_i)}{dx^r} = \sum_{j=m-2}^{m+2} w_{ij}^{(r)} L_m(x_j), \; m = -1,0,...,N+2, \tag{5}$$

Let $L_{m,j}$ be $L_m(x_j)$. Then, this equation system can be written in matrix form as

$$\mathbf{A}\mathbf{w} = \mathbf{C} \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix} L_{-1,-3} & L_{-1,-2} & L_{-1,-1} & L_{-1,0} & L_{-1,1} & & & \\ & L_{0,-2} & L_{0,-1} & L_{0,0} & L_{0,1} & L_{0,2} & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & & \\ & & & & & & & \\ & & L_{N+1,N-1} & L_{N+1,N} & L_{N+1,N+1} & L_{N+1,N+2} & L_{N+1,N+3} & \\ & & & L_{N+2,N} & L_{N+2,N+1} & L_{N+2,N+2} & L_{N+2,N+3} & L_{N+2,N+4} \end{bmatrix}$$

,

$$\mathbf{w} = \left[w_{i,-3}^{(r)}, w_{i,-1}^{(r)}, \ldots, w_{i,N+4}^{(r)} \right]^T$$

and

$$\mathbf{C} = \left[\frac{d^r L_{-1}(x_i)}{dx^r}, \frac{d^r L_0(x_i)}{dx^r}, \ldots, \frac{d^r L_{N+2}(x_i)}{dx^r} \right]^T$$

Since the linear equation system (7) has $N+4$ equations with $N+8$ unknowns, it is not uniquely solvable in its present form. Adjoining the equations

$$\frac{d^{r+1}L_{-1}(x_i)}{dx^{r+1}} = \sum_{j=-3}^{1} w_{i,j}^{(r)} \frac{dL_{-1}(x_j)}{dx}$$

$$\frac{d^{r+1}L_{0}(x_i)}{dx^{r+1}} = \sum_{j=-2}^{2} w_{i,j}^{(r)} \frac{dL_{0}(x_j)}{dx}$$

$$\frac{d^{r+1}L_{N+1}(x_i)}{dx^{r+1}} = \sum_{j=N-1}^{N+3} w_{i,j}^{(r)} \frac{dL_{N+1}(x_j)}{dx}$$

$$\frac{d^{r+1}L_{N+2}(x_i)}{dx^{r+1}} = \sum_{j=N}^{N+4} w_{i,j}^{(r)} \frac{dL_{N+2}(x_j)}{dx}$$

to the system (7), converts it to a solvable system

$$\tilde{\mathbf{A}}\mathbf{w} = \tilde{\mathbf{C}} \tag{7}$$

where

$$\tilde{\mathbf{A}}=\begin{bmatrix}
L_{-1,-3} & L_{-1,-2} & L_{-1,-1} & L_{-1,0} & L_{-1,1} & & & \\
L'_{-1,-3} & L'_{-1,-2} & L'_{-1,-1} & L'_{-1,0} & L'_{-1,1} & & & \\
 & L_{0,-2} & L_{0,-1} & L_{0,0} & L_{0,1} & L_{0,2} & & \\
 & L'_{0,-2} & L'_{0,-1} & L'_{0,0} & L'_{0,1} & L'_{0,2} & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & & & \\
 & & L_{N+1,N-1} & L_{N+1,N} & L_{N+1,N+1} & L_{N+1,N+2} & L_{N+1,N+3} & \\
 & & L'_{N+1,N-1} & L'_{N+1,N} & L'_{N+1,N+1} & L'_{N+1,N+2} & L'_{N+1,N+3} & \\
 & & & L_{N+2,N} & L_{N+2,N+1} & L_{N+2,N+2} & L_{N+2,N+3} & L_{N+2,N+4} \\
 & & & L'_{N+2,N} & L'_{N+2,N+1} & L'_{N+2,N+2} & L'_{N+2,N+3} & L'_{N+2,N+4}
\end{bmatrix}$$

,

$$\tilde{\mathbf{C}}=\left[\frac{d^rL_{-1}(x_i)}{dx^r},\frac{d^{r+1}L_{-1}(x_i)}{dx^{r+1}},\frac{d^rL_0(x_i)}{dx^r},\frac{d^{r+1}L_0(x_i)}{dx^{r+1}},\ldots,\frac{d^rL_{N+2}(x_i)}{dx^r},\frac{d^{r+1}L_{N+2}(x_i)}{dx^{r+1}}\right]^T$$

for $w_{ij}^{(r)}$ with equal numbers of unknowns and equations. The system (7) can be modified to a system with a five banded coefficient matrix and then can be solved by using compatible Thomas algorithm. Solving the last system for $r=1$ and substitutions of each x_i gives the weights $w_{ij}^{(1)}$ ($N\times N$ coefficients for $i,j=1,2,\ldots,N$, the other coefficients are not used in the approximation) of the first order derivative approximation. Similarly, when $r=2$, this system generates the weights of the second order derivative term approximation $w_{ij}^{(2)}$.

3. Discretization of the NBE

Substitution of the approximations given in Eq.(3) into the NBE (1) instead of the derivative terms U_x and U_{xx} and the boundary conditions leads to the ODE system

$$\frac{\partial U(x_i,t)}{\partial t}=-U(x_i,t)\sum_{j=2}^{N-1}w_{ij}^{(1)}U(x_j,t)+\beta\sum_{j=2}^{N-1}w_{ij}^{(2)}U(x_j,t)+\kappa_i, i=2,3,...,N-1 \tag{8}$$

where $\kappa_i=-s_1(t)\left[w_{i,1}^{(1)}s_1(t)+w_{i,N}^{(1)}s_2(t)\right]+\beta\left[w_{i,1}^{(2)}s_1(t)+w_{i,N}^{(2)}s_2(t)\right]$. Then, the time integration of (8) is accomplished by Runge-Kutta method of order four owing to its high accuracy with low memory usage properties.

4. Problems

The designed algorithm is used for two initial boundary value problems modeling fadeout of an initial shock and sinusoidal disturbance. The accuracy of the method is calculated by measuring the error between the numerical solutions and the analytical solutions via discrete norm L_∞ The approximate numerical values of vector norms L_∞ for discrete nodes are computed using

$$L_\infty[U(x_i,t)] = \max_{2\leq i\leq N-1} \left|U^{\text{analytical}}(x_i,t) - U^{\text{dqm}}(x_i,t)\right|$$

where $U^{\text{analytical}}(x_i,t)$ and $U^{\text{dqm}}(x_i,t)$ are analytical and computed solutions at the node x_i at a fixed time t , respectively.

4.1. Fadeout of an initial Shock

The shock-like behaviors of the solutions of the NBE originate from the solutions of inviscid Burgers' equation($\beta = 0$ in Eqn.(1)). In many cases, those solutions both become steeper and fade out as time goes. In fact, the NBE takes the form of inviscid Burgers' equation as $\beta \to 0$.

Consider the fadeout of a shock solution for the NBE represented by [6, 39]:

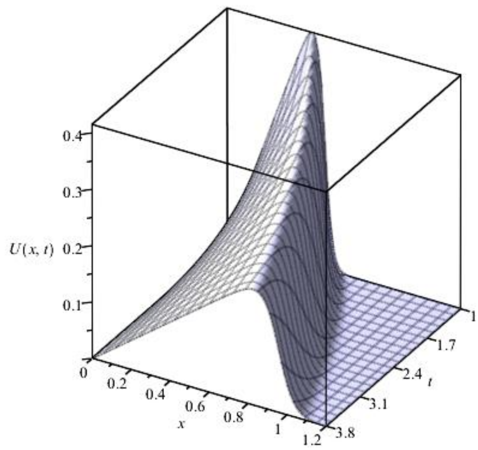
$$U(x,t) = \frac{\frac{x}{t}}{1+\exp(\frac{x^2}{4\beta t})\sqrt{\frac{t}{\exp(\frac{1}{8\beta})}}}, t \geq 1, 0 \leq x \leq 1.2$$

This anti-symmetric solution in fact is determined by using a particular solution of the one dimensional heat equation and the reduction of the NBE to the one dimensional heat equation with Cole-Hopf transformation[6]. The fadeout simulations of this solution are generated by using the compatible initial condition, derived by substitution of $t = 1$ in the analytical solution,

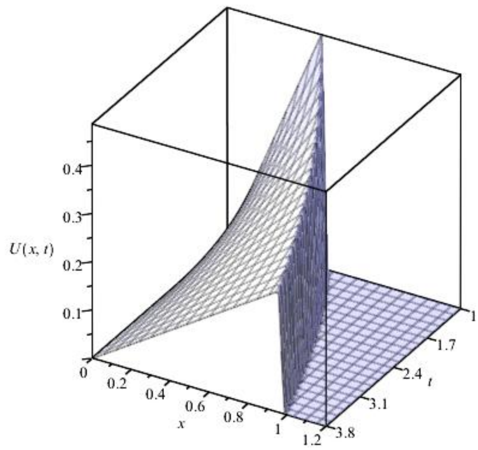
$$U(x,t) = \frac{x}{1+\exp(\frac{x^2}{4\beta})\sqrt{\frac{1}{\exp(\frac{1}{8\beta})}}}$$

and the homogenous boundary conditions at both ends of the problem interval. The fadeout of the shock is simulated to the terminating time $t = 3.8$ by the designed routine with the viscosity coefficients $\beta = 0.005$, Fig 1(a), and $\beta = 0.0005$, Fig 1(b). It is clear that when the viscosity coefficient is reduced to 0.0005 from 0.005, the right side of the shock becomes steeper but the velocity of diffusion to the right decreases. The initial shock fades out while moving to the right along the horizontal axis as time goes.

The accuracy of the proposed method is determined by the calculation of the discrete maximum error norm for $\beta = 0.005$, $\Delta t = 0.001$, and $N = 101$. A comparison with some earlier studies in literature is also tabulated in Table 1. According to the comparison, the results generated by the QBSDQ have four decimal digits accuracy at the time $t = 2.4$ almost similar to the accuracy of the results generated by QBCM2 and QRTDQ, as the BSQI and the CBSFEM are accurate to three decimal digits, the QBCM1 is five decimal digits. Here the results of Galerkin method has only two decimal digits accuracy. At the time $t = 3.1$, the results obtained by the BSQI, the CBSFEM,



(a) For $\beta = 0.005$



(b) For $\beta = 0.0005$

FIGURE 1. The simulation of fadeout of an initial shock

the QBCM1 and the QBCM2 are accurate to three decimal digits as the QRTDQ and the QBSDQ generate four decimal digits accurate results. In fact the main reason of this error is the forced right boundary condition. In this case, the accuracy of the Galerkin method is computed in two decimal digits. The comparison of the L_∞ at the simulation terminating time $t = 3.6$ shows that all the results obtained by the QBSDQ, the MCBC, the CBSDQ-I, the CBSDQ-II, the CBSDQ-III and the QRTDQ are accurate to four decimal digits.

TABLE 1. A comparison of the error with some earlier works for $\beta = 0.005$

			$L_\infty \times 10^3$		
Method	N	Δt	$t = 2.4$	$t = 3.1$	$t = 3.6$
QBSDQ(present)	101	0.001	0.67	0.54	0.39
Galerkin[18]			11.66	15.87	
BSQI[19]	51	0.01	6.31	6.85 ($t = 3.2$)	
MCBC[20]	101	0.001			0.17
CBSFEM[21]	101	0.01	1.68	1.30	
QBCM1[24]	200	0.01	0.06	4.43	
QBCM2[24]	200	0.01	0.80	4.79	
CBSDQ-I[26]	101	0.001			0.59
CBSDQ-II[26]	101	0.001			0.64
CBSDQ-III[26]	101	0.001			0.63
QRTDQ[27]	101	0.001	0.34	0.27	0.23

4.2. Sinusoidal Disturbance

Consider the initial boundary value problem

$$\begin{aligned} \frac{\partial U(x,t)}{\partial t} + U(x,t)\frac{\partial U(x,t)}{\partial x} - \beta\frac{\partial^2 U(x,t)}{\partial x^2} &= 0,\, t > 0,\, 0 < x < 1 \\ U(x,0) &= \sin 2\pi x \\ U(0,t) &= 0 \\ U(1,t) &= 0 \end{aligned} \tag{9}$$

The solution of this problem is a sinusoidal disturbance of an initial sine wave in the interval $[0,1]$ as the time increases. The analytic solution of this problem is given by

$$U(x,t) = \frac{\int_{-\infty}^{\infty} (x-\xi)A(x,t,\xi)d\xi}{t\int_{-\infty}^{\infty} A(x,t,\xi)d\xi} \tag{10}$$

where

$$A(x,t,\xi) = e^{-\left[\frac{(x-\xi)^2}{4\beta t} + \frac{1}{2\beta}\int_0^{\xi} U(0,\eta)d\eta\right]}$$

using the Cole-Hopf transformation[40]. The solution of the problem is simulated with $\beta = \pi/100$ for $t \in [0, 0.5]$, Fig 2. The solutions are obtained by using different space step sizes with a fixed time increment size $\Delta t = 0.0025$. The error between the numerical and analytical solutions are calculated by the maximum error norm L_∞, Table 2. The calculation of the analytical solution (10) is accomplished by using Gauss-Hermite quadrature rule. When number of nodes is chosen as $N = 16$, the results obtained by the proposed method are accurate to two decimal digits at $t = 0.14$ and only one decimal digit at $t = 0.26, 0.38$ and the simulation terminating time $t = 0.50$. The

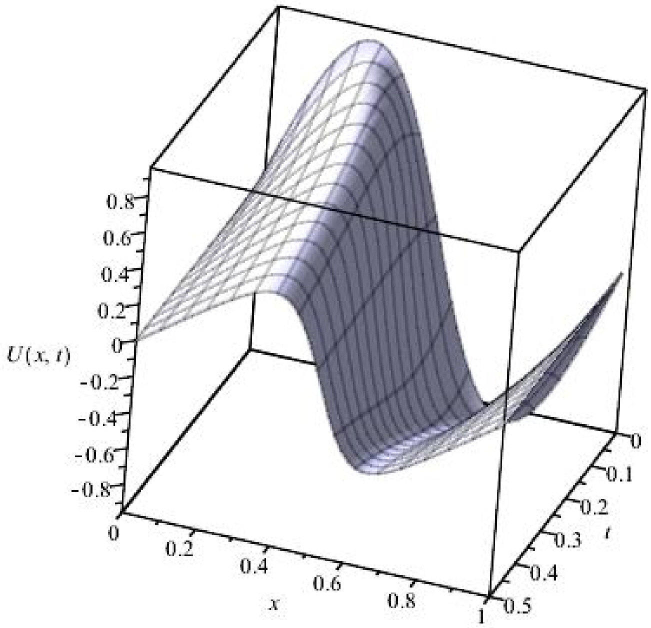


FIGURE 2. The simulation of sinusoidal disturbance for $\beta = \frac{\pi}{100}$

TABLE 2. A comparison with analytical solutions for $\beta = \pi/100$

		$L_\infty \times 10^2$			
N	Δt	$t = 0.14$	$t = 0.26$	$t = 0.38$	$t = 0.50$
16	0.0025	5.55	23.90	22.22	21.09
32	0.0025	1.33	19.53	19.08	15.57
64	0.0025	0.66	10.15	8.14	4.79
128	0.0025	0.19	1.33	0.72	0.22

increase of the number of nodes to $N = 32$ does not affect the accuracy in decimal digits, but the choice of $N = 64$ generates three decimal digits, one decimal digit, two decimal digits and two decimal digits accurate results at the times 0.14, 0.26, 0.38 and 0.50, respectively. When the number of nodes are taken as 128, the results are improved in one digit decimal more at all calculation times except $t = 0.14$.

5. Conclusion

In the present study, the quintic B-spline differential quadrature method on uniform grid distribution is setup for solutions of the one dimensional NBE. The weight coefficients required for the derivative approximation in differential quarature method are determined by solving algebraic equation systems with five-banded coefficient matrices. The time integration of the space discretized system is completed using classical Runge-Kutta method of order four. The validity of the proposed method is checked by solving two initial boundary value problems modeling fadeout of an initial shock and sinusoidal disturbance for the NBE. The accuracy of the method is measured by discrete maximum error norm. The error of the solution of shock fadeout problem is compared with some results given in the literature. The results show that the proposed method generates acceptable accurate solutions for both problems.

References

[1] H. Bateman, Some Recent Researches on the Motion of Fluids, Mon.Weather Rev., 43, 163-170, 1915.

[2] M. Seydaoglu, U. Erdoğan, T. Öziş, Numerical solution of Burgers' equation with high order splitting methods, Journal of Computational and Applied Mathematics, 291, 410-421, 2016.

[3] J. M. Burgers, A Mathematical Model Illustrating the Theory of Turbulence, Adv. in App. Mech. I, 171-199, 1948.

[4] D. C. Leslie, Developments in the Theory of Turbulence, Clarendon Press Oxford, 1973.

[5] P. J. Olver, C. Shakiban, Applied Mathematics, 2004.

[6] J. Billingham, A. C. King, Wave Motion, Cambridge University Press, Cambridge, 2000.

[7] M.J., Lighthill, Viscocity effects in sound waves of finite amplitude, in Batchlor, G. K. and Davies, R. M. (Eds), Survey in Mechanics, Cambridge University Press, Cambridge, 250-351, 1956.

[8] P. Kachroo, K. Özbay, S. Kang, J. A. Burns, System Dynamics and Feedback Control Problem Formulations for Real Time Dynamic Traffic Routing, Mathl. Comput. Modelling, 27, Q-11, 27-49, 1998.

[9] S. N. Gurbatov, A. I. Saichev, S. F. Shandarin, The large-scale structure of the Universe in the frame of the model equation og non-linear diffusion, Mon. Not. R. Astr. Soc., 236, 385-402, 1989.

[10] J. L. Katz, M. L. Green, A Burgers model of interstellar dynamics, Astron. Astrophys., 161, 139-141, 1986.

[11] L. Kofman, A. C. Raga, Modeling Structures of Knots in Jet Flows with the Burgers Equation, The Astrophysical Journal, 390, 359-364, 1992.

[12] A. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2009.

[13] J. D. Cole, On a Quasi-linear Parabolic Equation in Aerodynamics, Quarterly of Applied Math., 9, 225-236, 1951.

[14] E. Hopf, The Partial Differential Equation $U_t + UU_x = \mu U_{xx}$, Comm. Pure App. Math., 3, 201-230, 1950.

[15] E. Benton, G.W. Platzman, A table of solutions of the one-dimensional Burgers equations, Quart. Appl. Math., 30, 195-212, 1972.

[16] W. Liao, An implicit fourth-order compact finite difference scheme for one-dimensional Burgers' equation, Applied Mathematics and Computation, 206, 755-764, 206.

[17] M. Sari, G. Gürarslan, A sixth-order compact finite difference scheme to the numerical solutions of Burgers' equation, Applied Mathematics and Computation, 208, 475-483, 2009.

[18] A. Doğan, A Galerkin finite element approach to Burgers' equation, Applied Mathematics and Computation, 157, 331- 346, 2004.

[19] C. G. Zhu, R. H. Wang, Numerical solution of Burgers' equation by cubic B-spline quasi-interpolation, Applied Mathematics and Computation, 208, 260-272, 2009.

[20] R. C. Mittal, R. K. Jain, Numerical solutions of nonlinear Burgers' equation with modified cubic B-splines collocation method, Applied Mathematics and Computation, 218, 7839-7855, 2012.

[21] A. A. Soliman, A Galerkin Solution for Burgers' Equation Using Cubic B-Spline Finite Elements, Abstract and Applied Analysis, 2012, Article ID 527467, 1-15, 2012.

[22] A. H. A. Ali, L. R. T. Gardner, G. A. Gardner, A Galerkin Approach to the Solution of Burgers' Equation, Maths Preprint Series, no. 90.04, University College of North Wales, Bangor, 1990.

[23] L. R. T. Gardner, G. A. Gardner, B-spline Finite Elements, U.C.N.W. Maths Preprint, 91.10.

[24] B. Saka, İ. Dağ, Quartic B-spline collocation method to the numerical solutions of the Burgers' equation, Chaos, Solitons and Fractals, 32, 1125-1137, 2007.

[25] İ. Dağ, D. Irk, A. Şahin, B-spline collocation methods for numerical solutions of the Burgers' equation, Mathematical Problems in Engineering, 5, 521-538, 2005.

[26] A. Korkmaz, İ. Dağ, Cubic B-spline differential quadrature methods and stability for Burgers' equation, Engineering Computations, 30, 3, 320-344, 2013.

[27] A. Korkmaz, A. M. Aksoy, İ. Dağ, Quartic B-spline differential quadrature method, International Journal of Nolinear Science, 11, 4, 403-411, 2011.

[28] A. Korkmaz, İ. Dağ, Polynomial based differential quadrature method for numerical solution of nonlinear Burgers' equation, Journal of the Franklin Institute, 348, 10, 2863-2875, 2011.

[29] R. Bellman, B. G. Kashef, J. Casti, Differential Quadrature: A Tecnique for the Rapid Solution of Nonlinear Differential Equations, Journal of Computational Physics 10, 40-52, 1972.

[30] R. Bellman , Kashef Bayesteh, Lee E. S., Vasudevan R., Differential quadrature and splines, Computers and mathematics with applications, pp. 371-376. Pergamon, Oxford, 1976.

[31] C. Shu, Y.L. Wu, Integrated radial basis functions-based differential quadrature method and its performance, Int. J. Numer. Meth. Fluids 2007; 53:969–984.

[32] J. R. Quan, C. T. Chang, New sightings in involving distributed system equations by the quadrature methods-I, Comput. Chem. Engrg., Vol 13, 779-788, 1989.

[33] J. R. Quan, C. T. Chang, New sightings in involving distributed system equations by the quadrature methods-II, Comput. Chem. Engrg., Vol 13, 1017-1024, 1989.

[34] C. Shu, H. Xue, Explicit Computation of Weighting Coefficients in the Harmonic Differential Quadrature, Journal of Sound and Vibration,204, 3, 549-555, 1997.

[35] Q. Guo, H. Zhong, Non-linear vibration analysis of beams by a spline-based differential quadrature method, Journal of Sound and Vibration, 269, 413-420, 2004.

[36] H. Zhong, Spline-based differential quadrature for fourth order differential equations and its application to Kirchhoff plates, Applied Mathematical Modelling, 28, 353-366, 2004.

[37] H. Zhong, M. Lan, Solution of nonlinear initial-value problems by the spline-based differential quadrature method, Journal of Sound and Vibration, 296, 908-918, 2006.

[38] P. M. Prenter, Splines and Variational Methods, John Wiley & Sons, New York, NY, USA, 1989.

[39] H. Nguyen, J. Reynen, A space-time finite element approach to Burgers equation, in Taylor, C., Hinton, E., Owen, D.R.J. and Onate, E. (Eds), Numerical Methods for Non-linear Problems, Vol. 2, Pineridge Publisher, Swansea, 718-728,1982.

[40] B. V. R. Kumar, M. Mehra, Wavelet-Taylor Galerkin Method for the Burgers Equation, BIT Numerical Mathematics, 45, 543-560, 2005.