Aksaray University
Journal of Science and Engineering

e-ISSN: 2587-1277
http://asujse.aksaray.edu.tr

Aksaray J. Sci. Eng.
Volume **9**, Issue **2**, pp. **78-83.**

Available online at **DergiPark**

Research Article

Statistical Convergence in Function Spaces with Semi-Uniform Topologies

İsmail Osmanoğlu[1*], Serdar Minaz[2]

[1]*Department of Computer Technologies, Sandıklı Vocational School, Afyon Kocatepe University, Afyonkarahisar, Turkey.*
[2]*Department of Mathematics, Graduate School of Natural and Applied Sciences, Afyon Kocatepe University, Afyonkarahisar, Turkey.*

Keywords
Statistical convergence, Topology of semi-uniform convergence, Function spaces

Article information
Received: Apr 24, 2025
Revised: Oct 23, 2025
Accepted: Oct 27, 2025
Online: Nov 3, 2025

***doi:* 10.29002/asujse.1682667**

Abstract

In this work, we focus on investigating statistical convergence in function spaces equipped with semi-uniform convergence topologies. We provide a comparative analysis of pointwise and uniform statistical convergence under these topological structures and investigate the conditions that allow transitions between these types of convergence.

1. Introduction

Let Y^X denote the set of all functions from a topological space X to a semi-uniform space Y, and let $C(X,Y) \subseteq Y^X$ represent the subcollection of all continuous functions. In the case where Y is endowed with a uniform structure $\mathfrak{U}$, the corresponding topologies of uniform convergence have been the subject of extensive study, most notably in the foundational work of Kelley [1]. Additional foundational insights are offered by Bourbaki [2], who highlights the critical role that uniform structures play in defining topologies on spaces of functions.

The concept of a semi-uniform space, first introduced by M. Hušek [3, 4] in 1964 , extends the classical notion of uniform spaces by relaxing some of their axioms. This idea was further developed by M. Katě tov and Z. Frolík in their contributions to the revised edition by E. Čech [5]. A comprehensive treatment of both uniform and semi-uniform structures can also be found in the work of W. Page [6], which serves as a fundamental reference on the subject.

The authors previously studied various topologies of semi-uniform approximation defined on Y^X, analyzing their properties, comparing them, and investigating the conditions under which they can be equivalent.

Statistical convergence was originally introduced for sequences of real numbers by H. Fast [7] and H. Steinhaus [8], both inspired by the notion of asymptotic density of subsets of $\mathbb{N}$. However, the initial traces of statistical convergence can be found in the first edition of Zygmund's classic monograph [9], published in 1935. Since then, this type of convergence has found numerous applications in mathematical analysis and topology, leading to a substantial body of research in recent years [10-12].

When dealing with sequences of functions, the topological nature of the codomain plays a crucial role in determining the mode of convergence. With this in mind, we further specialize the target space to be uniform, thus improving and extending the previous notion. Pointwise and uniform statistical convergence for sequences of functions taking values in uniform spaces was obtained in [13].

In this study, we undertake a detailed investigation of statistical convergence within function spaces equipped with semi-uniform convergence topologies, which extend the classical uniform convergence framework by relaxing certain axioms. Within this setting, we examine and compare both pointwise and uniform types of statistical convergence, emphasizing how the semi-uniform structure affects their behavior. Our analysis focuses on determining the exact conditions under which a statistically pointwise convergent sequence also satisfies statistical uniform convergence, thereby illuminating the relationships and transitions between these convergence types in generalized topological environments. This approach provides a natural framework for understanding how weaker forms of uniformity can still preserve essential convergence properties, offering a bridge between classical and generalized convergence theories.

---

[*]***Corresponding Author:*** ▪ismailosmanoglu@yahoo.com 0000-0002-1005-4075

2. Preliminaries

In this section, we will give basic information about semi-uniform spaces and statistical convergence of sequences of functions.

The basic definitions and theorems in this chapter, and even more, can be found in [5] and [6].

A semi-uniformity or a semi-uniform structure on the set Y is a family $\mathfrak{U}$ of subsets of the set $Y \times Y$ with the following properties:

(SU1) $\Delta_Y \subseteq V$ for all $V \in \mathfrak{U}$;

(SU2) If $V \in \mathfrak{U}$, then $V^{-1} \in \mathfrak{U}$;

(SU3) If $V \in \mathfrak{U}$ and $V \subseteq W$, then $W \in \mathfrak{U}$;

(SU4) If $V, W \in \mathfrak{U}$, then $V \cap W \in \mathfrak{U}$.

The pair $(Y, \mathfrak{U})$ is called a semi-uniform space, where $\Delta_Y = \{(y, y) : y \in Y\}$ is the diagonal of Y and $V^{-1} = \{(y, x) : (x, y) \in V\}$ is the inverse of V.

(U5) If $V \in \mathfrak{U}$, there is some $W \in \mathfrak{U}$ such that $W^2 \subseteq V$, where $W^2 = W \circ W = \{(x, y) \in Y \times Y : \exists z \in Y$ such that $(x, z) \in W$ and $(z, y) \in W\}$.

The pair $(Y, \mathfrak{U})$ is called a uniform space if $\mathfrak{U}$ is a semi-uniform structure and (U5) is satisfied.

The elements of semi-uniformity are called entourages of $Y \times Y$ or entourages on Y.

The non-emptiness of $\mathfrak{U}$, taken together with (SU3) and (SU4), states that $\mathfrak{U}$ is a filter on $Y \times Y$.

Let $\mathfrak{U}$ be a semi-uniformity on a set Y. A base for $\mathfrak{U}$ is a subcollection $\mathfrak{B} \subseteq \mathfrak{U}$ such that every element of $\mathfrak{U}$ contains some element of $\mathfrak{B}$. Clearly, any base for a semi-uniformity is also a filter base for the corresponding filter $\mathfrak{U}$.

If $V = V^{-1}$, then V is called symmetric. The set of all symmetric elements of a semi-uniformity $\mathfrak{U}$ forms a base for $\mathfrak{U}$.

For any $x \in Y$, and $V \subseteq Y \times Y$, define

$$V[y] = \{z \in Y : (y, z) \in V\}, \quad \text{and} \quad V[A] = \bigcup_{y \in A} V[y],$$

for $A \subseteq Y$. If $\mathfrak{U}$ is a semi-uniformity, then the collection $\{V[y] : V \in \mathfrak{U}\}$ forms a filter on Y for each $y \in Y$.

Now, suppose $\mathfrak{U}$ is a semi-uniformity on Y. Then there exists a unique topology $\tau_{\mathfrak{U}}$ on Y, such that $\{V[y_0] : V \in \mathfrak{U}\}$ is a neighborhood base at $y_0 \in Y$. Equivalently, $\tau_{\mathfrak{U}}$ is given by:

$$\tau_{\mathfrak{U}} = \{O \subseteq Y : \text{for each } y \in O, \quad \text{there exists } V \in \mathfrak{U} \text{ such that } V[y] \subseteq O\}.$$

This topology $\tau_{\mathfrak{U}}$ is referred to as the topology generated by the semi-uniformity $\mathfrak{U}$.

Finally, let X be a topological space, and let $(Y, \mathfrak{U})$ be a semi-uniform space. These settings will be assumed throughout the article.

The authors have previously studied topologies of semi-uniform convergence, the definitions of which are given below.

Consider a collection $\mathfrak{A}$ of subsets of X that covers X. For each $A \in \mathfrak{A}$ and $V \in \mathfrak{U}$, define the set:

$$V_A = \{(f, g) \in Y^X \times Y^X : (f(x), g(x)) \in V \text{ for all } x \in A\}.$$

This collection $\{V_A : A \in \mathfrak{A}, V \in \mathfrak{U}\}$ forms a subbase for a semi-uniformity on Y^X, which we refer to as the semi-uniformity of semi-uniform convergence on the sets in $\mathfrak{A}$. The topology generated by this uniformity, denoted as $\tau_{\mathfrak{A}}$, is called the semi-uniform topology on $\mathfrak{A}$ or the topology of semi-uniform convergence on $\mathfrak{A}$. A base for this topology consists of the following sets:

$$V_A[f] = \{g \in Y^X : (g, f) \in V_A\}.$$

If $\mathcal{A} = \mathcal{F}(X) = \{A \subseteq X : A \text{ is finite}\}$, $\mathcal{A} = \mathcal{K}(X) = \{A \subseteq X : A \text{ is compact}\}$, or $\mathcal{A} = \{X\}$, then the corresponding semi-uniformities $\mathfrak{U}_{\mathcal{A}}$ are called the semi-uniformities of pointwise, compact, and semi-uniform convergence, respectively. The topologies induced by these structures are referred to as the topologies of pointwise, compact, and semi-uniform convergence, and denoted by τ_{sp}, τ_{sk}, and τ_{su}, respectively.

Let f be a function from a topological space X into a semi-uniform space $(Y, \mathfrak{U})$. Then f is said to be continuous at a point $x_0 \in X$ if, for every entourage $V \in \mathfrak{U}$, there exists an open set $O_{x_0} \subseteq X$ containing x_0 such that $(f(x_0), f(x)) \in V$ for all $x \in O_{x_0}$.

The function f is said to be continuous on X if it is continuous at each point $x \in X$ [14].

If $A \subset \mathbb{N}$, then $A(n)$ denotes the set

$$A(n) = \{k \in A: k \leq n\}.$$

The natural (or asymptotic) density of A is given by

$$\delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n},$$

if it exists. Here the density is in $[0,1]$. Clearly, finite subsets have natural density zero. A subset A of $\mathbb{N}$ is said to be statistically dense if $\delta(A) = 1$ [15]. We recall also that

$$\delta(\mathbb{N}\backslash A) = 1 - \delta(A),$$

for $A \subset \mathbb{N}$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to converge statistically to $x \in X$, if for every neighborhood O of x,

$$\delta(\{n \in \mathbb{N}: x_n \notin O\}) = 0.$$

For $X = \mathbb{R}$ or for a first countable space X, equivalently this definition says that there exists a subset $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that the sequence $(x_n)_{n\in A}$ converges to x, i.e., for every neighborhood O of x, there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ and $n \in A$ imply $x_n \in O$ [15].

This will be denoted by $x_n \overset{st-\tau}{\rightarrow} x$, where τ is a topology on X.

The following definitions for a topological space X and a uniform space Y are given in [13]. In the present work, we shall adopt these definitions specifically in the context of semi-uniform convergence on Y^X, generalizing the classical notions of statistical convergence.

Definition 1. A sequence $(f_n)_{n\in\mathbb{N}}$ *in* Y^X is said to be *(pointwise) statistically convergent* to f if for each $x \in X$ and for each $V \in \mathfrak{U}$,

$$\delta\big(\{n \in \mathbb{N}: \big(f_n(x), f(x)\big) \notin V\}\big) = 0$$

and is denoted by $f_n \overset{st}{\rightarrow} f$.

Definition 2. A sequence $(f_n)_{n\in\mathbb{N}}$ *in* Y^X is said to be *uniformly statistically convergent* to f on X if for each $V \in \mathfrak{U}$,

$$\delta(\{n \in \mathbb{N}: (f_n(x), f(x)) \notin V \text{ for all } x \in X\}) = 0$$

and is denoted by $f_n \overset{st-u}{\rightarrow} f$.s.

3. Main Results

Definition 3. A sequence $(f_n)_{n\in\mathbb{N}}$ in $(Y^X, \tau_{\mathfrak{A}})$ is said to be *statistically convergent* to f if for each $A \in \mathfrak{A}$ and for each $V \in \mathfrak{U}$,

$$\delta(\{n \in \mathbb{N}: (f_n, f) \notin V_A\}) = 0.$$

that is,

$$\delta(\{n \in \mathbb{N}: (f_n(x), f(x)) \notin V, \text{ for all } x \in A\}) = 0.$$

We call f the statistical limit of (f_n).

By Definition 1, Definition 2 and Definition 3, we easily obtain the following propositions.

Proposition 4. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in Y^X, $f \in Y^X$ and $\mathfrak{A} = F(X)$. If (f_n) is statistically convergent to f belonging to the space (Y^X, τ_{sp}), then (f_n) is pointwise statistically convergent to f. That is, if $f_n \overset{st-\tau_{sp}}{\rightarrow} f$, then $f_n \overset{st}{\rightarrow} f$.

Proposition 5. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in Y^X, $f \in Y^X$ and $\mathfrak{A} = \{X\}$. *If* (f_n) is statistically convergent to f belonging to the space (Y^X, τ_{su}), then (f_n) is uniformly statistically convergent to f. That is, *if* $f_n \overset{st-\tau_{su}}{\rightarrow} f$, then $f_n \overset{st-u}{\rightarrow} f$.

Theorem 6. Let $(Y, \mathfrak{U})$ be a Hausdorff semi-uniform space. If a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is statistically convergent, then its statistical limit is unique.

Proof. Let $(Y, \mathfrak{U})$ be a Hausdorff semi-uniform space. Suppose that $(f_n)_{n\in\mathbb{N}}$ is statistically convergent to both f and g. That is,

$$\delta(\{n \in \mathbb{N}: f_n \notin V_A[f]\}) = 0, \quad \forall A \in \mathfrak{A}, V \in \mathfrak{U}.$$

$$\delta(\{n \in \mathbb{N}: f_n \notin V_A[g]\}) = 0, \quad \forall A \in \mathfrak{A}, V \in \mathfrak{U}.$$

Since $(Y,\mathfrak{U})$ is Hausdorff, the intersection of all entourages is the diagonal

$$\bigcap_{V\in\mathfrak{U}} V = \{(y,y) \mid y \in Y\}.$$

If for some $x \in X$, $f(x)$ differs from $g(x)$, then we can find an entourage $V \in \mathfrak{U}$ such that $(f(x), g(x))$ does not belong to V . Thus, there is a set A containing x and an entourage V such that

$$V_A[f] \cap V_A[g] = \emptyset.$$

However, since f_n statistically converges to both f and g, we have

$$\delta(\{n \in \mathbb{N}: f_n \in V_A[f]\}) = 1,$$

$$\delta(\{n \in \mathbb{N}: f_n \in V_A[g]\}) = 1.$$

Since $V_A[f]$ and $V_A[g]$ are disjoint, this would imply that there exist infinitely many n such that f_n belongs to both sets, which is a contradiction. However, both have natural density 1, and two disjoint subsets of $\mathbb{N}$ cannot each have density 1. Hence, we conclude that $f = g$, proving the uniqueness of the statistical limit.

Theorem 7. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* Y^X *and* $f \in Y^X$. *If* $f_n \overset{st-\tau_{su}}{\rightarrow} f$, *then* $f_n \overset{st-\tau_{sp}}{\rightarrow} f$.

Proof. Assume that $f_n \overset{st-\tau_{su}}{\rightarrow} f$. That is, for each entourage $V \in \mathfrak{U}$,

$$\delta(\{n \in \mathbb{N}: f_n \notin V_X[f]\}) = 0.$$

By definition of $V_X[f]$, we obtain

$$V_X[f] = \{g \in Y^X: (g(x), f(x)) \in V, \forall x \in X\}.$$

This means that whenever $f_n \in V_X[f]$, it satisfies

$$(f_n(x), f(x)) \in V, \quad \forall x \in X.$$

Thus, for each fixed x, the event

$$\{n \in \mathbb{N}: (f_n(x), f(x)) \notin V\}$$

is a subset of

$$\{n \in \mathbb{N}: f_n \notin V_X[f]\}.$$

Since the latter has density zero, the former must also have density zero. That is, $f_n \overset{st-\tau_{sp}}{\rightarrow} f$.

The converse of the previous theorem need not be true in general.

Example 8. Let $X = [0,1]$ and let Y be a semi-uniform space that is not necessarily a uniform space. Define the function sequence (f_n) in Y^X as follows

$$f_n(x) = \begin{cases} 1, & x \in [0, \frac{1}{n}], \\ 0, & \text{otherwise.} \end{cases}$$

Define $f(x) = 0$ for all $x \in [0,1]$. We show that (f_n) is pointwise statistically convergent to f, but it fails to be statistically uniformly convergent.

For each fixed $x \neq 0$, there exists N_x such that for all $n \geq N_x$ we have $f_n(x) = 0$. Thus, for each x, $\delta(\{n \in \mathbb{N}: f_n(x) \neq f(x)\}) = 0$. This implies that $(f_n(x))$ statistically converges to $f(x)$ pointwise.

However, uniform statistical convergence fails. Consider the set

$$S_n = \{x \in [0,1] \mid f_n(x) \neq f(x)\} = [0, 1/n].$$

Although S_n shrinks as $n \to \infty$, the semi-uniform structure on Y does not necessarily induce a fully uniform topology. Since the entourages in a semi-uniform structure are not necessarily transitive, there exists an entourage V such that

$$\delta(\{n \in \mathbb{N}: f_n \notin V_X[f]\}) > 0.$$

This implies that (f_n) does not statistically converge to f uniformly. Thus, we have an example where statistical pointwise convergence holds but statistical uniform convergence fails.

Definition 9. A filter $\mathfrak{U}$ of subsets on $Y \times Y$ is called *locally-uniformity* if it is a semi-uniformity and for all $V \in \mathfrak{U}$ and $y \in Y$, *there is a* $W \in \mathfrak{U}$ such that

$$(W \circ W)[y] \subseteq V[y].$$

Then the pair $(Y,\mathfrak{U})$ is called *locally-uniform space* [16].

Lemma 10. Let $(Y, \mathfrak{U})$ be a locally-uniform space and $V \in \mathfrak{U}$. Then, there exists a symmetric entourage $W \in \mathfrak{U}$ such that

$$W \circ W \circ W \subseteq V.$$

Proof. Let $V \in \mathfrak{U}$ be given. By the definition of a locally-uniform space, for each $y \in Y$ there exists an entourage $U_y \in \mathfrak{U}$ such that

$$(U_y \circ U_y)[y] \subseteq V[y].$$

For every finite subset $\{y_1, \dots, y_k\} \subset Y$, set

$$U_{y_1,\dots,y_k} = U_{y_1} \cap \dots \cap U_{y_k}.$$

Since $\mathfrak{U}$ is a filter, each finite intersection $U_{y_1,\dots,y_k}$ belongs to $\mathfrak{U}$. Choose one such U and note that for each y_i in the chosen finite family,

$$(U \circ U)[y_i] \subseteq (U_{y_i} \circ U_{y_i})[y_i] \subseteq V[y_i].$$

Hence $U \circ U \subseteq V$ on the corresponding part of Y. Because $\mathfrak{U}$ is a semi-uniformity, there exists $W' \in \mathfrak{U}$ such that $W' \circ W' \subseteq U$. Now define

$$W = W' \cap (W')^{-1}.$$

Then $W \in \mathfrak{U}$ and W is symmetric. Finally,

$$W \circ W \circ W \subseteq W' \circ W' \circ W' \subseteq U \circ W' \subseteq U \circ U \subseteq V.$$

Thus, there exists a symmetric entourage $W \in \mathfrak{U}$ such that $W \circ W \circ W \subseteq V$.

Theorem 11. Let $(Y, \mathfrak{U})$ be a locally-uniform space. If a sequence $(f_n)_{n\in\mathbb{N}}$ in $C(X, Y)$ and $f_n \overset{st-\tau_{su}}{\rightarrow} f$, then $f \in C(X, Y)$.

Proof. If (f_n) is statistically convergent to f belonging to the space $(C(X, Y), \tau_{su})$ and let $x_0 \in X$. We prove that f is continuous at x_0. Let $V \in \mathfrak{U}$ be given. Since $(Y, \mathfrak{U})$ is locally-uniform, there exists a symmetric $W \in \mathfrak{U}$ such that

$$W \circ W \circ W \subseteq V.$$

Since $f_n \to f$ uniformly statistically (appears from Proposition 5), the set

$$A = \{n \in \mathbb{N}: (f_n(x), f(x)) \notin W \text{ for all } x \in X\}$$

has asymptotic density 0. Thus, we can choose $n_0 \in \mathbb{N}\backslash A$ such that

$$(f_{n_0}(x), f(x)) \in W \quad \text{for all } x \in X.$$

In particular, we conclude

$$(f_{n_0}(x_0), f(x_0)) \in W.$$

Since f_{n_0} is continuous at x_0, there exists an open neighborhood O_{x_0} of x_0 such that

$$(f_{n_0}(x_0), f_{n_0}(x)) \in W \quad \text{for all } x \in O_{x_0}.$$

Let $x \in O_{x_0}$. Using (1), (2), and (3), we obtain

$$(f(x_0), f(x)) \in W \circ W \circ W \subseteq V.$$

Hence, for all $x \in O_{x_0}$, we arrive at $(f(x_0), f(x)) \in V$, and so f is continuous at x_0. Since x_0 was arbitrary, we conclude that $f \in C(X, Y)$.

4. Conclusion

In this paper, we have extended the concept of statistical convergence to function spaces equipped with semi-uniform convergence topologies. We have examined the relationships among pointwise, uniform, and semi-uniform statistical convergence and identified conditions under which one type implies another. The preservation of continuity under statistical convergence in locally uniform spaces was also established. These results generalize several known statistical convergence properties from uniform to semi-uniform settings, broadening the theoretical foundation of convergence structures in function spaces. Possible directions for further research include the study of statistical convergence in more generalized frameworks, such as grill-based or filter-based convergence structures, and exploring semi-uniform statistical Cauchy sequences. Another promising line of investigation is the development of analogous results for ideal or net convergence in semi-uniform spaces.

References

[1] Kelley, J. L. **(1975)**. *General Topology*, Springer, New York–Berlin.

[2] Bourbaki, N. **(1989)**. *General Topology*, Springer, Berlin.

[3] Husek, M. **(1964)**. *Generalized proximity and uniform spaces I*, Comment. Math. Univ. Carolinae, 5, 247–266.

[4] Husek, M. **(1965)**. *Generalized proximity and uniform spaces II*, Comment. Math. Univ. Carolinae, 6, 119–139.

[5] Čech, E. **(1966)**. *Topological Spaces* (revised by Z. Frolík and M. Katětov), Academia Publishing House of the Czechoslovak Academy of Sciences, Prague.

[6] Page, W. **(1989)**. *Topological Uniform Structures*, Dover Publications Inc., New York.

[7] Fast, H. **(1951)**. *Sur la convergence statistique*, Colloq. Math. 2, 241–244.

[8] Steinhaus, H. **(1951)**. *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. 2, 73–74.

[9] Zygmund, A. **(1979**). *Trigonometric Series* (2nd ed.), Cambridge University Press, Cambridge.

[10] Balcerzak, M., Dems, K., Komisarski, A. **(2007)**. *Statistical convergence and ideal convergence for sequences of functions*, *J. Math. Anal. Appl.* 328(1), 715–729. DOI: 10.1016/j.jmaa.2006.05.040

[11] Caserta, A., Maio, G. Di., Kočinac, L. D. R. **(2011)**. *Statistical convergence in function spaces*, Abstr. Appl. Anal., Art. ID 420419, 11 pp. DOI: 10.1155/2011/420419

[12] Megaritis, A. C. **(2017)**. *Ideal convergence of nets of functions with values in uniform spaces*, *Filomat* 31(20), 6281–6292. DOI: 10.2298/FIL1720281M

[13] Osmanoğlu, İ. **(2025)**. *Statistical and Uniformly Statistical Convergence of Sequences of Functions*, *Filomat* 39(30), Article in Press.

[14] Georgiou, D. N., Megaritis, A. C., Özçağ, S. **(2018)**. *Statistical convergence of sequences of functions with values in semi-uniform spaces*, Comment. Math. Univ. Carolin. 59(1), 103–117. DOI: 10.14712/1213-7243.2015.231

[15] Maio, G. Di, Kočinac, L. D. R. **(2008)**. *Statistical convergence in topology*, Topology Appl. 156, 28–45. DOI: 10.1016/j.topol.2008.01.015

[16] Williams, J. **(1972)**. *Locally uniform spaces*, Trans. Amer. Math. Soc. 168, 435–469. DOI: 10.1090/S0002-9947-1972-0296891-5