Evolution equations in Fréchet spaces

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Article Info

Abstract

This paper deals with the existence of mild solutions for a class of evolution equations. The technique used is a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness.

1. Introduction

There has been a significant development in functional evolution equations in recent years; see the monographs [2, 3, 14, 17, 18] and the references therein. By means of a nonlinear alternative of Leray–Schauder type for contraction operators on Fréchet spaces [13], Baghli and Benchohra [4, 5] provided sufficient conditions for the existence of mild solutions of some classes of evolution equations, while in [6, 7, 8] the authors presented some global existence and stability results for functional evolution equations and inclusions in the space of continuous and bounded functions. In [1], an iterative method is used for the existence of mild solutions of evolution equations and inclusions. Using the Tichonov's fixed point theorem, Olszowy and Wędrychowicz [16] considered a class of evolution equations on unbounded intervals. However in the previous papers some restrictions like, the compactness of the semigroup, the Lipschitz condition on the nonlinear term or the boundedness of the obtained mild solutions, are supposed.

In the present paper, we discuss the existence of mild solutions for the evolution equation

$$u'(t) = A(t)u(t) + f(t, u(t)); \ \ \text{if } t \in \mathbb{R}_+ := [0, \infty), \tag{1.1}$$

with the initial condition

$$u(0) = u_0 \in E, \tag{1.2}$$

where $f : \mathbb{R}_+ \times E \to E$ is a given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $\{A(t)\}_{t>0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of bounded linear operators $\{U(t,s)\}_{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+}$; for $(t,s) \in \Lambda := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \le s \le t < +\infty\}$.

This paper initiates the existence of solutions for evolution equations with an application of a generalization of the classical Darbo fixed point theorem, and the concept of measure of noncompactness in Fréchet spaces, The paper is organized as follows. In Section 2 some preliminary results are introduced. The main results is presented in Section 3, while the last section is devoted to an illustrative example.

2. Preliminaries

Let $I := [0,T]$; $T > 0$. A measurable function $u : I \to E$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. For properties of the Bochner integral, see for instance, Yosida [19].

By $B(E)$ we denote the Banach space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup_{\|u\|=1} \|N(u)\|.$$

As usual, $L^1(I,E)$ denotes the Banach space of measurable functions $u : I \to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^T \|u(t)\|dt.$$

As usual, by $\mathscr{C} := C(I)$ we denote the Banach space of all continuous functions from I into E with the norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{t \in I} \|u(t)\|.$$

In what follows, for the family $\{A(t),\ t \geq 0\}$ of closed densely defined linear unbounded operators on the Banach space E we assume that it satisfies the following assumptions (see [3], p. 158).

(P_1) The domain $D(A(t))$ is independent of t and is dense in E,

(P_2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant K independent of λ and t such that
$$\|R(t,A(t))\| \leq K(1+|\lambda|)^{-1}, \text{ for } Re\lambda \leq 0,$$

(P_3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that
$$\|(A(t) - A(\theta))A^{-1}(\tau)\| \leq L|t-\tau|^\alpha, \text{ for } t,\theta,\tau \in I.$$

Lemma 2.1. *([3], p. 159) Under assumptions $(P_1) - (P_3)$, the Cauchy problem*

$$u'(t) - A(t)u(t) = 0,\ t \in I,\ u(0) = y_0,$$

has a unique evolution system $U(t,s)$, $(t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leq s \leq t \leq T\}$ satisfying the following properties:

1. *$U(t,t) = I$ where I is the identity operator in E,*
2. *$U(t,s)\,U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t \leq T$,*
3. *$U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(t,s) \in \Delta$ and for each $u \in E$, the mapping $(t,s) \to U(t,s)u$ is continuous.*

More details on evolution systems and their properties can be found in the books of Ahmed [3] and Pazy [17].

Let $C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t \in [0,n]} \|v(t)\|;\ n \in \mathbb{N},$$

and the distance

$$d(u,v) = \sum_{n=1}^\infty 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v \in C(\mathbb{R}_+).$$

We recall the following definition of the notion of a sequence of measures of noncompactness [10, 11].

Definition 2.2. *Let $\mathscr{M}_X$ be the family of all nonempty and bounded subsets of a Fréchet space X. A family of functions $\{\mu_n\}_{n \in \mathbb{N}}$ where $\mu_n : \mathscr{M}_X \to [0,\infty)$ is said to be a family of measures of noncompactness in the real Fréchet space X if it satisfies the following conditions for all $B,B_1,B_2 \in \mathscr{M}_X$:*

(a) *$\{\mu_n\}_{n \in \mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,*

(b) *$\mu_n(B_1) \leq \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,*

(c) *$\mu_n(ConvB) = \mu_n(B)$ for $n \in \mathbb{N}$,*

(d) *If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathscr{M}_X$ such that $B_{i+1} \subset B_i$; $i = 1,\cdots$ and if $\lim_{i \to \infty} \mu_n(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty := \cap_{i=1}^\infty B_i$ is nonempty.*

Some Properties:

(e) We call the family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}}$ to be homogeneous if $\mu_n(\lambda B) = |\lambda|\mu_n(B)$; for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.

(f) If the family $\{\mu_n\}_{n \in \mathbb{N}}$ satisfied the condition $\mu_n(B_1 \cup B_2) \leq \mu_n(B_1) + \mu_n(B_2)$, for $n \in \mathbb{N}$, it is called subadditive.

(g) It is sublinear if both conditions (e) and (f) hold.

(h) We say that the family of measures $\{\mu_n\}_{n \in \mathbb{N}}$ has the maximum property if

$$\mu_n(B_1 \cup B_2) = \max\{\mu_n(B_1), \mu_n(B_2)\},$$

(i) The family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if and only if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

Example 2.3. *For $B \in \mathscr{M}_X$, $x \in B$, $n \in \mathbb{N}$ and $\varepsilon > 0$, let us denote by $\omega^n(x,\varepsilon)$ for $n \in \mathbb{N}$; the modulus of continuity of the function x on the interval $[0,n]$; that is,*

$$\omega^n(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0,n], |t - s| \le \varepsilon\}.$$

Further, let us put

$$\omega^n(B,\varepsilon) = \sup\{\omega^n(x,\varepsilon) : x \in B\},$$

$$\omega_0^n(B) = \lim_{\varepsilon \to 0^+} \omega^n(B,\varepsilon),$$

$$\bar{\alpha}^n(B) = \sup_{t \in [0,n]} \alpha(B(t)) := \sup_{t \in [0,n]} \alpha(\{x(t) : x \in B\}),$$

and

$$\beta_n(B) = \omega_0^n(B) + \bar{\alpha}^n(B).$$

The family of mappings $\{\beta_n\}_{n\in\mathbb{N}}$ where $\beta_n : \mathscr{M}_X \to [0,\infty)$, satisfies the conditions (a)-(d) from Definition 2.2.

Definition 2.4. *A nonempty subset $B \subset X$ is said to be bounded if*

$$\sup_{v \in X} \|v\|_n < \infty;$$

for $n \in \mathbb{N}$.

Lemma 2.5. *[9] If Y is a bounded subset of Fréchet space X, then for each $\varepsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\mu_n(Y) \le 2\mu_n(\{y_k\}_{k=1}^{\infty}) + \varepsilon; \; for \; n \in \mathbb{N}.$$

Lemma 2.6. *[15] If $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ is uniformly integrable, then $\mu_n(\{u_k\}_{k=1}^{\infty})$ is measurable for $n \in \mathbb{N}$, and*

$$\mu_n\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \le 2\int_0^t \mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0,n]$.

Definition 2.7. *Let Ω be a nonempty subset of a Fréchet space X, and let $A : \Omega \to X$ be a continuous operator which transforms bounded subsets of onto bounded ones. One says that A satisfies the Darbo condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$, if*

$$\mu_n(A(B)) \le k_n\mu_n(B)$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$. If $k_n < 1$; $n \in \mathbb{N}$ then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

In the sequel we will make use of the following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 2.8. *[10, 11] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space F and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

3. Existence of mild solutions

In this section, we present the main results for the global existence of solutions for our problem. Let us introduce the definition of the mild solution of the problem (1.1)-(1.2).

Definition 3.1. *A continuous function $u(\cdot) : I \to E$ is said a mild solution of the problem (1.1)-(1.2), if u satisfies the following integral equation*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s) \, f(s,u(s))ds, \quad for \; each \; t \in \mathbb{R}_+.$$

Let us introduce the following hypotheses.

(H_1) There exists a constant $M \ge 1$ such that

$$\|U(t,s)\|_{B(E)} \le M; \; for \; every \; (t,s) \in \Lambda.$$

(H_2) The function $t \mapsto f(t,u)$ is measurable on $\mathbb{R}_+$ for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous on E for a.e. $t \in \mathbb{R}_+$.

(H_3) There exists a continuous function $p : \mathbb{R}_+ \to [0,\infty)$ such that

$$\|f(t,u)\| \le p(t)(1 + \|u\|); \; for \; a.e. \; t \in \mathbb{R}_+, \; and \; each \; u \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t,B)) \le p(t)\mu(B),$$

where μ is a measure of noncompactness on the Banach space E.

For $n \in \mathbb{N}$, let

$$p_n^* = \sup_{t \in [0,n]} p(t),$$

and define on $C(\mathbb{R}_+)$ the family of measure of noncompactness by

$$\mu_n(D) = \sup_{t \in [0,n]} e^{-4Mp_n^*\tau t} \mu(D(t)),$$

where $\tau > 1$, and $D(t) = \{v(t) \in E : v \in D\}$; $t \in [0,n]$.

Theorem 3.2. *Assume that the hypotheses* $(H_1) - (H_4)$ *are satisfied, and* $nMp_n^* < 1$ *for each* $n \in \mathbb{N}$*. Then the problem (1.1)-(1.2) has at least one mild solution.*

Proof. Consider the operator $N : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(Nu)(t) = U(t,0)u_0 + \int_0^t U(t,s) \ f(s,u(s))ds. \tag{3.1}$$

Clearly, the fixed points of the operator N are solution of the problem (1.1)-(1.2).

For any $n \in \mathbb{N}$, let R_n be a positive real number with

$$R_n \geq \frac{M\|u_0\| + np_n^*M}{1 - nMp_n^*},$$

and we consider the ball

$$B_{R_n} := B(0, R_n) = \{w \in C(\mathbb{R}_+) : \|w\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}$, and each $u \in B_{R_n}$ and $t \in [0,n]$ we have

$$
\begin{aligned}
\|(Nu)(t)\| & \leq \|U(t,0)\|_{B(E)}\|u_0\| + \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u(s))\|ds \\
& \leq M\|u_0\| + M\left(\int_0^t p(s)(1 + \|u(s)\|)ds\right) \\
& \leq M\|u_0\| + M(1 + \|u\|_n)\int_0^t p(s)ds \\
& \leq M\|u_0\| + nMp_n^*(1 + R_n) \\
& \leq R_n.
\end{aligned}
$$

Thus

$$\|N(u)\|_n \leq R_n. \tag{3.2}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 2.8. The proof will be given in several steps.

Step 1. $N : B_{R_n} \to B_{R_n}$ *is continuous.*
Let $\{u_k\}_{k \in \mathbb{N}}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in [0,n]$, we have

$$
\begin{aligned}
\|(Nu_k)(t) - (Nu)(t)\| & \leq \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u_k(s)) - f(s,u(s))\|ds \\
& \leq M\int_0^t \|f(s,u_k(s)) - f(s,u(s))\|ds.
\end{aligned}
$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(u_k) - N(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

Step 2. $N(B_{R_n})$ *is bounded.*
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.

Step 3. *For each bounded subset D of B_{R_n}, $\mu_n(N(D)) \leq \ell_n \mu_n(D)$.*

From Lemmas 2.5 and 2.6, for any $D \subset B_{R_n}$ and any $\varepsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$
\begin{aligned}
\mu((ND)(t)) &= \mu\left(\left\{U(t,0)u_0 + \int_0^t U(t,s)\,f(s,u(s))ds;\ u \in D\right\}\right) \\
&\leq 2\mu\left(\left\{\int_0^t U(t,s)f(s,u_k(s))ds\right\}_{k=1}^{\infty}\right) + \varepsilon \\
&\leq 4\int_0^t \mu\left(\left\|U(t,s)\right\|_{B(E)}\{f(s,u_k(s))\}_{k=1}^{\infty}\right)ds + \varepsilon \\
&\leq 4M\int_0^t \mu\left(\{f(s,u_k(s)\}_{k=1}^{\infty}\right)ds + \varepsilon \\
&\leq 4M\int_0^t p(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right)ds + \varepsilon \\
&\leq 4Mp_n^*\int_0^t e^{4Mp_n^*\tau s}e^{-4Mp_n^*\tau s}\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right)ds + \varepsilon \\
&\leq \frac{\left(e^{4Mp_n^*\tau t}-1\right)}{\tau}\mu_n(D) + \varepsilon \\
&\leq \frac{e^{4Mp_n^*\tau t}}{\tau}\mu_n(D) + \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, then

$$
\mu((ND)(t)) \leq \frac{e^{4Mp_n^*\tau t}}{\tau}\mu_n(D).
$$

Thus

$$
\mu_n(N(D)) \leq \frac{1}{\tau}\mu_n(D).
$$

As a consequence of steps 1 to 3 together with Theorem 2.8, we can conclude that N has at least one fixed point in B_{R_n} which is a mild solution of problem (1.1)-(1.2).

$\square$

4. An example

As an application of our results, we consider the following functional evolution problem of the form

$$
\begin{cases}
\dfrac{\partial z}{\partial t}(t,x) = a(t,x)\dfrac{\partial^2 z}{\partial x^2}(t,x) + Q(t,z(t,x)); & t \in \mathbb{R}_+,\ x \in [0,\pi], \\[2mm]
z(t,0) = z(t,\pi) = 0; & t \in \mathbb{R}_+. \\[2mm]
z(0,x) = \Phi(x); & x \in [0,\pi],
\end{cases}
\tag{4.1}
$$

where $a(t,x) : \mathbb{R}_+ \times [0,\pi] \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t, $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,x)w''$ with domain

$$
D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}.
$$

Then $A(t)$ generates an evolution system $U(t,s)$ (see [12]).

For $x \in [0,\pi]$, we have

$$
y(t)(x) = z(t,x); \quad t \in \mathbb{R}_+,
$$

$$
f(t,y)(x) = Q(t,z(t,x)); \quad t \in \mathbb{R}_+,
$$

and

$$
u_0(x) = \Phi(x); \quad x \in [0,\pi].
$$

Thus, under the above definitions of f, u_0 and $A(\cdot)$, the system (4.1) can be represented by the functional evolution problem (1.1)-(1.2). Furthermore, more appropriate conditions on Q ensure the hypotheses $(H_1) - (H_4)$. Consequently, Theorem 3.2 implies that the evolution problem (4.1) has at least one mild solution.

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