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A Comparison of Five Tests for the Equality of Inverse Gaussian Means under Heteroscedasticity of Scale Parameters

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Abstract

Analysis of Reciprocals F-test developed by Miura [1] is used to test the equality of Inverse Gaussian (IG) means based on the assumption of homogeneity of scale parameters. However, this method is not valid when this assumption is not satisfied. There are some methods developed for comparing the equality of the IG means under heteroscedasticity of scale parameters. In this study, the goal of this study is to compare these methods under different combinations of parameters and various sample sizes. We compare the performances of the five commonly used tests in the literature via Monte Carlo simulation study. The tests considered are analysis of reciprocals (ANORE) F-test, parametric bootstrap approach (PBA), generalized p-value approach proposed by Tian (GPA), generalized p-value approach proposed by Shi and Lv (GPSL) and computational approach test (CAT). According to simulation results, GPSL and CAT have satisfactory type I error rates for all parameter combinations. Except for the number of groups is k=7, when the sample sizes are different and scale parameters are both different and inversely proportional, the power of GPA is higher than the other tests.

1. INTRODUCTION

In applied statistics, a very common problem is to compare the means of k independent experiments. Analysis of variance (ANOVA) F-test is used for testing the equality of means. It is known that ANOVA procedures in the framework of experimental design are traditionally based on the several basic assumptions, such as normality, homogeneity of variances and independence of observations [2-8]. However, it is encountered positive valued random variables in many real life problems. One of the most important distribution that is used to model positive valued random variables is IG distribution. It can represent a highly skewed to an almost normal distribution. IG distribution introduced by Schrödinger [9] and von Smoluchowski [10] as the probability distribution of the first passage time in Brownian motion, is especially useful for analyzing the right skewed data sets.

It attracts a great deal of attention various fields of scientific studies, such as demography, linguistics, air born communication receiver performance, histomorphometry, electrical networks, management sciences, mental health, linguistics, product/device reliability studies, cardiology and so forth. Also, it is one of the most widely used distributions to model lifetimes or reaction-times and describe exposure concentration, failure of a system. [11-14]

A number of authors have discussed and investigated certain properties of the IG distribution. Interest readers may refer to [15-23] for further details.

The IG distribution has the following probability distribution function (pdf)

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$$f(x|\mu,\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2} exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \quad x>0, \ \ \mu,\lambda>0$$

where μ and λ represent mean and scale parameters, respectively.

In this article, the testing of the equality of IG means has been considered. Just like ANOVA F-test is based upon homogeneity of variances assumption for testing equality of normal means, ANORE F-test developed by Miura [1] for testing equality of IG means is based on the assumption of homogeneity of scale parameters. However, in many real-life problems, scale parameters are usually not homogeneous. Many tests have been proposed in the statistical literature in order to testing the equality of IG means under the assumption of homogeneity and heterogeneity of scale parameters. Chhikara [24] considered the two-sample case to compare two IG means. Miura [1] considered a test for one-way layout, two-way layout and two-sided test for the mean of the IG distribution when scale parameter is unknown. According to power studies by computer simulation, the test proposed by Miura [1] for (ANOVA) is overall preferred to Fisher's F-test, with or without variance-stabilization of data. In Davis [25] the likelihood ratio test was determined for four tests of hypotheses involving the IG distribution. Samanta [26] considered the problem of testing the equality of two IG means on the basis of independent random samples from these distributions. They derived the likelihood ratio test and propose another test. Tian [27] developed an approach using generalized test variables and generalized p-values. Ma and Tian [28] proposed a parametric bootstrap approach. Ye et al. [29] proposed two approaches that were readily applicable for inferences on the common mean in several IG populations. The proposed approaches are hybrids between the generalized inference method and the large-sample theory. Lin and Wu [30] proposed an interval estimation method for the common mean of several heterogeneous IG populations. The proposed method is based upon a higher order likelihood-based procedure. Shi and Lv [31] defined a new generalized pivot quantity and gave the generalized p-value based on this generalized pivot quantity. Gökpınar et al. [32] proposed a testing procedure based on Computational Approach Test (CAT) for the equality of several IG means under heterogeneity. CAT is a special case of parametric bootstrap test. Recently, CAT has received great attention by most researchers; see, for example, Pal et al., [33], Chang and Pal [34], Chang et al. [35], Gökpınar et al. [32], Gökpınar and Gökpınar [36], Jafari and Abdollahnezhad [37], Gökpınar and Gökpınar [8], and Jafari and Kazami [38].

The objective of this paper is to compare the most commonly used tests for testing equality of IG means.

The rest of this paper is arranged as follows: In Section 2, details of considered tests are presented. Section 3 gives the simulation results for comparisons of the tests in terms of type I error and power. Section 4 summarizes concluding remarks.

2. TESTS FOR THE QUALITY OF IG MEANS

In this section we give briefly tests used in the study. Suppose $X_{i1}, X_{i2}, \cdots, X_{in_i}$ is a random sample from (μ_i, λ_i), i=1, 2, …, k. We are interested in testing

$$\begin{aligned} &H_0 : \mu_1 = \mu_2 = ... = \mu_k = \mu_0 \\ &against \\ &H_1 : \mu_i \neq \mu_j, \ \exists i \neq j \ \ (i, j = 1, 2, ..., k) \end{aligned} \tag{1}$$

2.1. ANORE F Test

Under the null hypothesis, that is, $\lambda_1 = \lambda_2 = ... = \lambda_k$ likelihood ratio test can be given as follows:

$$F_h = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\bar{X}_i^{-1} - \bar{X}_{..}^{-1}\right)\Big/(k-1)}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\bar{X}_{ij}^{-1} - \bar{X}_i^{-1}\right)\Big/(n-k)} .$$

where $\sum_{i=1}^{l} n_i = n$, $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij} \Big/ n_i$ and $\bar{X}_{..} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij} \Big/ n$. Reject H_0 at the α-level of significance if

$F_h > F_{k-1,n-k;1-\alpha}$

and accept otherwise.

2.1. Parametric Bootstrap Approach (PBA)

In this section, the PBA proposed by Ma and Tian [28] is presented that is used for testing equality of several IG means.

Let $X_{i1}, \mathrm{X}_{i2}, \cdots, \mathrm{X}_{in_i}$ be a random sample from an $IG(\mu_i, \lambda_i)$ $i = 1, \cdots, k$. It is well known that

$$\bar{X}_i \sim IG(\mu_i, n_i\lambda_i) \quad \text{and} \quad \lambda_i V_i \sim \chi^2_{n_i-1},$$

where $V_i = \sum_{j=1}^{n_i}\left(\frac{1}{X_{ij}} - \frac{1}{\bar{X}_i}\right)$. Under null hypothesis, Q statistic is given as

$$Q(\lambda_1, \lambda_2, \cdots, \lambda_k) = \sum_{i=1}^{k} n_i\lambda_i\left(\frac{1}{\bar{X}_i} - \frac{1}{\hat{\mu}}\right) \sim \chi^2_{k-1},$$

where $\hat{\mu} = \sum_{i=1}^{k} n_i\lambda_i\bar{X}_i \Big/ \sum_{i=1}^{k} n_i\lambda_i$.

The PB pivot variable based on the statistic Q is given by

$$Q_B\left(\bar{X}_{B1}, \cdots, \bar{X}_{Bk}, \lambda_{B1}, \cdots, \lambda_{Bk} \,\middle|\, \bar{X}_1, \cdots, \bar{X}_k, \mathrm{V}_1, \cdots, \mathrm{V}_k\right) = \sum_{i=1}^{k} n_i\lambda_{Bi}\left(\frac{1}{\bar{X}_{Bi}} - \frac{1}{\hat{\mu}_B}\right),$$

where

$$\bar{X}_{Bi} \sim IG\left(\frac{\sum_{i=1}^{k} n_i\hat{\boldsymbol{\lambda}}_i\bar{X}_i}{\sum_{i=1}^{k} n_i\hat{\boldsymbol{\lambda}}_i}, n_i\hat{\boldsymbol{\lambda}}_i\right), \qquad \hat{\lambda}_i = n_i / \mathrm{V}_i$$

$$\hat{\mu}_B = \frac{\sum_{i=1}^{k} n_i\lambda_{Bi}\bar{X}_i}{\sum_{i=1}^{k} n_i\boldsymbol{\lambda}_{Bi}} \qquad \text{and} \qquad \boldsymbol{\lambda}_{Bi} \sim \frac{\chi^2_{n_i-1}}{V_i}, i = 1, \cdots, k .$$

For a given data set with $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_k)$ and $(v_1,v_2,\cdots,v_k)$, m bootstrap samples, $Q_B^{(i)}, i=1,\cdots,m$, are drawn. The PBA rejects H_0 at significance level α, as shown in

$$p=\frac{\#\left(Q_B^{(i)}\geq Q_{B0}\right)}{m}\leq\alpha$$

where Q_{B0} is an observed value of Q_B

$$Q_{B0}=\sum_{i=1}^{k}n_i\hat{\lambda}_{i(RML)}\left(\frac{1}{\bar{x}_i}-\frac{1}{\hat{\mu}_{RML}}\right),$$

where $\hat{\mu}_{RML}$ and $\hat{\lambda}_{i(RML)}$, $i=1,\cdots,k$ are the restricted maximum likelihood estimates (RMLEs) from the observed samples and the RMLEs of the μ and λ parameters have no closed forms. Therefore, the RMLEs of these parameters can be obtained through the iterations as below: updating the estimates from *l*-step estimates $\left(\lambda_1^{(l)},\cdots,\lambda_k^{(l)},\mu^{(l)}\right)$ by (2). Here, initial value $\mu^{(0)}$ could set as grand mean $\frac{1}{k}\sum_{i=1}^{k}\bar{x}_i$. $\left(\lambda_1^{(l)},\cdots,\lambda_k^{(l)},\mu^{(l)}\right)$ converge to the MLEs under H_0 denoted as $\hat{\mu}_{RML}$, $\hat{\lambda}_{i(RML)}$.

$$\lambda_i^{(l+1)}=n_i\Big/\sum_{i=1}^{n_i}\left(\frac{1}{x_{ij}}-\frac{1}{\mu^{(l)}}\right)^2$$

$$i=1,\cdots,k \qquad (2)$$

$$\mu^{(l+1)}=\frac{\sum_{i=1}^{k}n_i\lambda_i^{(l+1)}\bar{x}_i}{\sum_{i=1}^{k}n_i\lambda_i^{(l+1)}}$$

The algorithm for calculating p-value using the PBA see Ma and Tian [28].

2.3. Generalized p-Value Approach proposed by Tian (GPA)

In this section, the GPA proposed by Tian [27] is presented. The generalized pivot for λ_i could be written as

$$R_{\lambda_i}=\frac{\lambda_i V_i}{v_i}\sim\frac{\chi^2_{n_i-1}}{v_i}$$

Here, v_i is the observed value of V_i. R_{λ_i} coincides with the traditional pivot for λ_i. For the equality of IG means the GPA is defined as

$$T=\frac{Q(\lambda_1,\lambda_2,\cdots,\lambda_k)}{q\left(R_{\lambda_1},R_{\lambda_2},\cdots,R_{\lambda_k}\right)},$$

where

$$q\left(R_{\lambda_1}, \mathrm{R}_{\lambda_2}, \cdots, \mathrm{R}_{\lambda_k}\right) = \sum_{i=1}^{k} n_i R_{\lambda_i}\left(\frac{1}{\bar{x}_i} - \frac{1}{R_{\hat{\mu}}}\right) \quad \text{and} \quad R_{\hat{\mu}} = \frac{\sum_{i=1}^{k} n_i R_{\lambda_i} \bar{x}_i}{\sum_{i=1}^{k} n_i R_{\lambda_i}}.$$

The proposed test statistic *T* satisfies the three conditions:

- As $\bar{X}_i = \bar{x}_i$, $V_i = v_i$, $R_{\lambda_i} = \lambda_i\left(i = 1, 2, \cdots, k\right)$ and $Q\left(\lambda_1, \lambda_2, \cdots, \lambda_k\right) = q\left(R_{\lambda_1}, \mathrm{R}_{\lambda_2}, \cdots, \mathrm{R}_{\lambda_k}\right)$.

Hence, the observed value of *T* is $\mathrm{t}_{\mathrm{obs}} = 1$.

- Under null hypothesis, $Q\left(\lambda_1, \lambda_2, \cdots, \lambda_k\right) \sim \chi^2_{k-1}$ and $q\left(R_{\lambda_1}, \mathrm{R}_{\lambda_2}, \cdots, \mathrm{R}_{\lambda_k}\right)$ are functions

of a set of independent random variables $\chi^2_{\mathrm{n_i}-1}$ and observed values $\bar{\mathrm{x}}_{\mathrm{i}}$ and v_{i} for $i = 1, 2, \cdots, k$. Thus, *T* does not depend on any unknown parameters.

The generalized p-value can be obtained as

$$\begin{aligned} p-value &= \mathrm{P}\left(T \geq t_{obs} = 1 \middle| H_0\right) \\ &= \mathrm{P}\left(Q\left(\lambda_1, \lambda_2, \cdots, \lambda_k\right) \geq q\left(R_{\lambda_1}, \mathrm{R}_{\lambda_2}, \cdots, \mathrm{R}_{\lambda_k}\right) \middle| H_0\right). \end{aligned}$$

2.4. Generalized p-Value Proposed by Shi and Lv (GPSL)

In this section generalized p-value approach proposed by Shi and Lv [31] is presented. It is easily proved that ($\bar{X}_i, V_i$) forms a set of complete sufficient statistics for (μ_i, λ_i). It is desired to develop a generalized pivot quantity for the testing problem based on the above complete sufficient statistics.
The testing problem (1) is equivalent to testing

$$H_0 : \frac{1}{\mu_1} = \frac{1}{\mu_2} = \ldots = \frac{1}{\mu_k}. \tag{3}$$

Define

$$\theta_i = \frac{1}{\mu_i}, \quad \theta = (\theta_1, \theta_2, \ldots, \theta_k)' \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 & . & . & . & 0 & -1 \\ 0 & 1 & . & . & . & 0 & -1 \\ . & . & & . & & . & . \\ . & . & & . & & . & . \\ . & . & & . & & . & . \\ 0 & 0 & . & . & . & 1 & -1 \end{pmatrix}_{k-1,k}.$$

Then, the testing problem (3) can be expressed as

$$H_0 : H\theta = 0$$

According to Ye et al. [29], the generalized pivot quantity for λ_i based on the i*th* sample is given by

$$R_{\lambda_i} = \frac{n_i \lambda_i V_i}{n_i v_i} \sim \frac{\chi^2_{n_i - 1}}{n_i v_i}$$

and the generalized pivot quantity for μ_i based on the i*th* sample is given by

$$T_{\mu_i} = \frac{\overline{x}_i}{\left| 1 + \frac{\sqrt{n_i \lambda_i}\left(\overline{X}_i - \mu\right)}{\overline{X}_i} \sqrt{\frac{\overline{x}_i}{n_i R_i}} \right|} \overset{d}{\sim} \frac{\overline{x}_i}{\left| 1 + Z_i \sqrt{\frac{\overline{x}_i}{n_i R_i}} \right|} .$$

Define

$$T_{\theta_i} = 1 / T_{\mu_i} \overset{d}{\sim} \frac{\left| 1 + Z_i \sqrt{\frac{\overline{x}_i}{n_i R_i}} \right|}{\overline{x}_i} \tag{4}$$

T_{θ_i} is generalized pivot quantity for θ_i. However, actually T_{θ_i} is an approximate generalized pivot quantity by the last expression in (4). Then it is easy to obtain the generalized pivot quantity for $H\theta$:

$$T_{H\theta} = H(T_{\theta_1}, T_{\theta_2}, ..., T_{\theta_k})' \tag{5}$$

Let $\overline{X} = (\overline{X}_1, \overline{X}_2, ..., \overline{X}_k)$ and $V = (V_1, V_2, ..., V_k)$. From (5), the conditional expectation of $T_{H\theta}$ given $(\overline{\mathrm{X}}, V) = (\overline{x}, v)$ is

$$\hat{\mu}_T = E(T_{H\theta} | (\overline{x}, v)) = HE(T_\theta | (\overline{x}, v)) = H(E(T_{\theta_1} | (\overline{x}, v)), ..., E(T_{\theta_k} | (\overline{x}, v)))' \tag{6}$$

and the conditional covariance matrix of $T_{H\theta}$ given $(\overline{\mathrm{X}}, V) = (\overline{x}, v)$ is

$$\hat{\Sigma}_T = Cov(T_{H\theta} | (\overline{x}, v)) = HCov(T_\theta | (\overline{x}, v))H' = Hdiag(\mathrm{Cov}(T_{\theta_1} | (\overline{x}, v), ..., T_{\theta_k} | (\overline{x}, v)))H', \tag{7}$$

Where

$$E(T_{\theta_i} | (\overline{x}, v)) = \frac{1}{x_i} \quad \text{and} \quad \mathrm{Cov}(T_{\theta_i} | (\overline{x}, v)) = \frac{v_i}{(n_i - 3)\overline{x}_i}, \quad n_i > 3, \text{ i=1,2,…,k.}$$

Let $\overline{T}$ denote the standardized expression for $T_{H\theta}$ with $\overline{T} = \hat{\Sigma}_T^{-1/2}(T_{H\theta} - \hat{\mu}_T)$ where $\hat{\mu}_T$ and $\hat{\Sigma}_T$ are given by (6) and (7), respectively. For a given $(\overline{x}_i, v_i)$, $\overline{t} = \hat{\Sigma}_T^{-1/2}(H\theta - \hat{\mu}_T)$ and the distribution of $\overline{T}$ is free of any unknown parameter. The generalized p value can be given by

$$p = P\left(\|\overline{T}\|^2 \geq \|\overline{t}\|^2 \middle| H_0 \right) = P\left(\left(T_{H\theta} - \hat{\mu}_T\right)' \hat{\Sigma}_T^{-1} \left(T_{H\theta} - \hat{\mu}_T\right) \geq \hat{\mu}_T' \hat{\Sigma}_T^{-1} \hat{\mu}_T \right).$$

2.5. Computational Approach Test (CAT)

In this section, a test procedure is given based on CAT proposed by Gökpınar et al. [32] for testing equality of several IG means under the unequal scale parameters. The null hypothesis H_0 is expressed in terms of suitable scalar η. The testing problem (1) is equivalent to testing

$$H_0 : \sum_{i=1}^{k} n_i \left(\frac{1}{\mu_i} - \frac{1}{\bar{\mu}} \right) = 0 .$$

where $\bar{\mu} = \sum_{i=1}^{k} n_i \mu_i / n$. Gökpınar et al. [32] obtained MLE of η and used it as a test statistic. The CAT could be given as shown below:

Step 1. The MLE of the parameters are obtained as $\hat{\boldsymbol{\mu}}_{i(ML)} = \bar{\boldsymbol{X}}_i$ and $\hat{\boldsymbol{\lambda}}_{i(ML)}^{-1} = \frac{1}{n_i} \sum_{j=1}^{n_i} \left(\boldsymbol{X}_{ij}^{-1} - \bar{\boldsymbol{X}}_i^{-1} \right)$.

Therefore, the test statistic is rewritten as $\hat{\eta}_{ML} = \sum_{i=1}^{k} n_i \left(\frac{1}{\bar{X}_i} - \frac{1}{\bar{X}} \right)$. The observed value of $\hat{\eta}_{ML}$ is $\hat{\eta}_{ML0}$.

Step 2. Generate artificial sample $X_{i1}, X_{i2}, \cdots, X_{in_i}$ i.i.d i=1, 2, …, k from $\mathrm{IG}\left(\hat{\mu}_{\mathrm{RML}}, \hat{\lambda}_{\mathrm{i(RML)}}\right)$, a large of number of times (say, m times). Here the RMLEs of the μ and λ_i parameters are obtained from equation (4). After that for each of these replicated samples, recalculate the values of $\hat{\eta}_{ML}^{(j)} \quad (j = 1, \ldots, m)$.

Step 3. Calculate the p-value as $p = \frac{\#\left(\hat{\eta}_{ML}^{(j)} > \hat{\eta}_{ML0}\right)}{m}$. In the case of $p < \alpha$, H_0 is rejected.

3. SIMULATION STUDY

In this section, for testing equality of IG means, the ANORE F-test, PBA, GPA, GPSL and CAT are compared with respect to their estimated type I error rates and powers via Monte Carlo simulation. To estimate the type I error rates and powers of all the tests under specified nominal level of α=0,05, we consider some cases from smaller to larger sample sizes with different number of groups. The number of groups for the simulation considered are k=3, 4, 5, 7. For each parameter setting, 5000 random samples from IG distribution are generated. For the specified nominal level of α=0,05, the estimated type I error rates of five tests are presented in from Table 1 to Table 4.

Table 1. *The estimated type I error rates for k=3*

n	λ	ANORE F	PBA	GPA	GPSL	CAT
5 5 5	10 10 10	0,0484	0,0648	0,0392	0,0394	0,0212
5 5 5	5 10 15	0,0626	0,0574	0,0440	0,0414	0,0294
5 5 5	15 10 5	0,0634	0,0634	0,0450	0,0324	0,0268
10 10 10	10 10 10	0,0586	0,0692	0,0538	0,0406	0,0484
10 10 10	5 10 15	0,0600	0,0636	0,0494	0,0418	0,0454
10 10 10	15 10 5	0,0620	0,0638	0,0500	0,0374	0,0478
15 15 15	10 10 10	0,0554	0,0630	0,0502	0,0400	0,0510
15 15 15	5 10 15	0,0588	0,0588	0,0476	0,0362	0,0464
15 15 15	15 10 5	0,0662	0,0640	0,0528	0,0426	0,0528
30 30 30	10 10 10	0,0458	0,0508	0,0456	0,0408	0,0448
30 30 30	5 10 15	0,0506	0,0494	0,0464	0,0392	0,0440
30 30 30	15 10 5	0,0554	0,0546	0,0504	0,0436	0,0474

5 10 15	10 10 10	0,0502	0,0564	0,0492	0,0414	0,0338
5 10 15	5 10 15	0,1158	0,0546	0,0512	0,0432	0,0330
5 10 15	15 10 5	0,0274	0,0650	0,0466	0,0414	0,0442
10 20 30	10 10 10	0,0508	0,0548	0,0474	0,0356	0,0450
10 20 30	5 10 15	0,1138	0,0574	0,0564	0,0404	0,0450
10 20 30	15 10 5	0,0286	0,0604	0,0496	0,0360	0,0502

Table 2. *The estimated type I error rates for k=4*

n	λ	ANORE F	PBA	GPA	GPSL	CAT
5 5 5 5	10 10 10 10	0,0488	0,0474	0,0534	0,0390	0,0176
5 5 5 5	5 10 10 15	0,0572	0,0498	0,0564	0,0448	0,0196
5 5 5 5	15 10 10 5	0,0602	0,0516	0,0570	0,0404	0,0212
10 10 10 10	10 10 10 10	0,0510	0,0576	0,0544	0,0478	0,0392
10 10 10 10	5 10 10 15	0,0514	0,0480	0,0482	0,0440	0,0372
10 10 10 10	15 10 10 5	0,0560	0,0524	0,0524	0,0492	0,0412
15 15 15 15	10 10 10 10	0,0564	0,0528	0,0520	0,0438	0,0434
15 15 15 15	5 10 10 15	0,0588	0,0532	0,0532	0,0500	0,0450
15 15 15 15	15 10 10 5	0,0580	0,0526	0,0518	0,0462	0,0452
30 30 30 30	10 10 10 10	0,0502	0,0544	0,0514	0,0495	0,0478
30 30 30 30	5 10 10 15	0,0594	0,0542	0,0542	0,0494	0,0498
30 30 3030	15 10 10 5	0,0554	0,0502	0,0492	0,0442	0,0458
5 10 10 15	10 10 10 10	0,0486	0,0506	0,0496	0,0420	0,0324
5 10 10 15	5 10 10 15	0,0976	0,0474	0,0550	0,0476	0,0264
5 10 10 15	15 10 10 5	0,0336	0,0576	0,0584	0,0514	0,0398
10 20 20 30	10 10 10 10	0,0528	0,0524	0,0550	0,0508	0,0474
10 20 20 30	5 10 10 15	0,1020	0,0524	0,0544	0,0518	0,0404
10 20 20 30	15 10 10 5	0,0302	0,0546	0,0518	0,0460	0,0460

Table 3. *The estimated type I error rates for k=5*

n	λ	ANORE F	PBA	GPA	GPSL	CAT
5 5 5 5 5	10 10 10 10 10	0,0466	0,0434	0,0618	0,0388	0,0160
5 5 5 5 5	5 5 10 15 15	0,0672	0,0398	0,0696	0,0446	0,0208
5 5 5 5 5	15 15 10 5 5	0,0706	0,0468	0,0746	0,0492	0,0212
10 10 10 10 10	10 10 10 10 10	0,0496	0,0518	0,0570	0,0454	0,0360
10 10 10 10 10	5 5 10 15 15	0,0664	0,0542	0,0676	0,0564	0,0386
10 10 10 10 10	15 15 10 5 5	0,0580	0,0490	0,0610	0,0520	0,0376
15 15 15 15 15	10 10 10 10 10	0,0486	0,0494	0,0536	0,0442	0,0392
15 15 15 15 15	5 5 10 15 15	0,0614	0,0480	0,0548	0,0474	0,0410
15 15 15 15 15	15 15 10 5 5	0,0592	0,0504	0,0540	0,0472	0,0406
30 30 30 30 30	10 10 10 10 10	0,0512	0,0526	0,0542	0,0482	0,0486
30 30 30 30 30	5 5 10 15 15	0,0572	0,0470	0,0490	0,0464	0,0440
30 30 30 30 30	15 15 10 5 5	0,0610	0,0490	0,0516	0,0468	0,0472
5 5 10 15 15	10 10 10 10 10	0,0438	0,0434	0,0566	0,0410	0,0276
5 5 10 15 15	5 5 10 15 15	0,1362	0,0460	0,0724	0,0554	0,0230
5 5 10 15 15	15 15 10 5 5	0,0214	0,0514	0,0574	0,0444	0,0344
10 10 20 30 30	10 10 10 10 10	0,0476	0,0490	0,0578	0,0500	0,0396
10 10 20 30 30	5 5 10 15 15	0,1398	0,0450	0,0602	0,0560	0,0380
10 10 20 30 30	15 15 10 5 5	0,0274	0,0570	0,0596	0,0486	0,0494

***Table 4.** The estimated type I error rates for k=7*

n	λ	ANORE F	PBA	GPA	GPS	CAT
5 5 5 5 5 5 5	10 10 10 10 10 10 10	0,0480	0,0382	0,0946	0,0432	0,0154
5 5 5 5 5 5 5	5 5 10 10 10 15 15	0,0606	0,0424	0,0946	0,0480	0,0202
5 5 5 5 5 5 5	15 15 10 10 10 5 5	0,0578	0,0382	0,0924	0,0422	0,0146
10 10 10 10 10 10 10	10 10 10 10 10 10 10	0,0500	0,0462	0,0748	0,0542	0,0338
10 10 10 10 10 10 10	5 5 10 10 10 15 15	0,0612	0,0454	0,0688	0,0522	0,0366
10 10 10 10 10 10 10	15 15 10 10 10 5 5	0,0638	0,0464	0,0732	0,0554	0,0352
15 15 15 15 15 15 15	10 10 10 10 10 10 10	0,0514	0,0490	0,0636	0,0488	0,0408
15 15 15 15 15 15 15	5 5 10 10 10 15 15	0,0608	0,0522	0,0698	0,0540	0,0426
15 15 15 15 15 15 15	15 15 10 10 10 5 5	0,0600	0,0494	0,0656	0,0522	0,0418
30 30 30 30 30 30 30	10 10 10 10 10 10 10	0,0458	0,0488	0,0552	0,0450	0,0408
30 30 30 30 30 30 30	5 5 10 10 10 15 15	0,0592	0,0462	0,0528	0,0452	0,0462
30 30 30 30 30 30 30	15 15 10 10 10 5 5	0,0572	0,0476	0,0546	0,0462	0,0462
5 5 10 10 10 15 15	10 10 10 10 10 10 10	0,0474	0,0452	0,0784	0,0498	0,0278
5 5 10 10 10 15 15	5 5 10 10 10 15 15	0,1236	0,0422	0,0826	0,0556	0,0250
5 5 10 10 10 15 15	15 15 10 10 10 5 5	0,0330	0,0504	0,0714	0,0460	0,0404
10 10 20 20 20 30 30	10 10 10 10 10 10 10	0,0506	0,0490	0,0634	0,0482	0,0390
10 10 20 20 20 30 30	5 5 10 10 10 15 15	0,1150	0,0392	0,0572	0,0478	0,0340
10 10 20 20 20 30 30	15 15 10 10 10 5 5	0,0274	0,0504	0,0612	0,0496	0,0480

From the numerical results, when the number of groups is k=3 and the sample sizes are small, moderate and equal, especially when the scale parameters are different, the type error rates of ANORE F-test larger than nominal α=0,05. In addition, when the scale parameters are equal and the above situation is valid, type error rates of PBA exceed nominal level of α=0,05. When the sample sizes are different, and the scale parameters are different and directly proportional, type I error rate of ANORE F-test can be as high as 0,11. On the contrary, when the scale parameters are inversely proportional, type I error rate of PBA exceeds nominal level of α=0,05. When k=4, the sample sizes are equal, small and scale parameters are inversely proportional type I error rate of ANORE F-test is larger than nominal level of α=0,05. Just like k=3, when the number of groups is k=4 and when the sample sizes and the scale parameters are different and directly proportional, type I error rate of ANORE F-test can be as high as 0,10. As the number of groups go up, k=7, GPA tends to be liberal for small and moderate samples.

For specified nominal level of $\alpha = 0,05$ Table 5-Table 8 present the estimated powers of five tests.

Table 5. *The estimated powers of tests for k=3*

μ=[3 4 5]

	n=[5 5 5]					**n=[10 10 10]**					**n=[15 15 15]**				
λ	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**
10 10 10	0,1540	0,1758	0,1218	0,1030	0,0750	0,3096	0,3296	0,2750	0,2538	0,2840	0,4652	0,4808	0,4350	0,4216	0,4510
5 10 15	0,1586	0,1392	0,0994	0,0840	0,0728	0,2868	0,2552	0,2212	0,2010	0,2368	0,4014	0,3720	0,3386	0,3226	0,3616
15 10 5	0,1180	0,1444	0,0994	0,0870	0,0478	0,2536	0,3230	0,2744	0,2626	0,1980	0,3652	0,4494	0,4084	0,3954	0,3240

	n=[30 30 30]					**n=[5 10 15]**					**n=[10 20 30]**				
	ANORE F	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**
10 10 10	0,8024	0,8028	0,7896	0,7842	0,7990	0,2410	0,2080	0,2090	0,1770	0,1504	0,4762	0,4414	0,4360	0,4156	0,4230
5 10 15	0,7100	0,6736	0,6586	0,6512	0,6850	0,3244	0,1688	0,1690	0,1468	0,1150	0,5114	0,3370	0,3400	0,3138	0,3028
15 10 5	0,6850	0,7730	0,7562	0,7516	0,6630	0,1070	0,1922	0,1836	0,1596	0,1242	0,2716	0,4516	0,4536	0,4226	0,3548

Table 6. *The estimated powers of tests for k=4*

μ=[3 4 4 5]

	n=[5 5 5 5]					**n=[10 10 10 10]**					**n=[15 15 15 15]**				
λ	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**
10 10 10 10	0,1204	0,1126	0,1168	0,0792	0,0498	0,2732	0,2612	0,2542	0,2094	0,2332	0,3922	0,3786	0,3734	0,3458	0,3700
5 10 10 15	0,1340	0,0984	0,1034	0,0698	0,0524	0,2492	0,2096	0,1996	0,1642	0,1954	0,3580	0,3058	0,2990	0,2670	0,3110
15 10 10 5	0,1084	0,1046	0,1178	0,0812	0,0356	0,2180	0,2502	0,2526	0,2214	0,1652	0,3428	0,3980	0,3934	0,3654	0,2946

	n=[30 30 30 30]					**n=[5 10 10 15]**					**n=[10 20 20 30]**				
	ANORE F	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**
10 10 10 10	0,7468	0,7326	0,7290	0,7188	0,7392	0,2000	0,1726	0,1880	0,1454	0,1318	0,4222	0,3794	0,3996	0,3676	0,3742
5 10 10 15	0,6620	0,6148	0,6096	0,5962	0,6278	0,2878	0,1774	0,1878	0,1482	0,1080	0,4730	0,3352	0,3434	0,3070	0,2800
15 10 10 5	0,6468	0,7342	0,7272	0,7162	0,6168	0,1086	0,1644	0,1830	0,1422	0,1200	0,2654	0,3760	0,3978	0,3748	0,3220

Table 7. *The estimated powers of tests for k=5*

μ=[3 3 4 5 5]

	n=[5 5 5 5 5]					**n=[10 10 10 10 10]**					**n=[15 15 15 15 15]**				
λ	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**
10 10 10 10 10	0,2126	0,1724	0,2224	0,1280	0,0878	0,4662	0,4526	0,4660	0,3866	0,4102	0,6712	0,6688	0,6700	0,6182	0,6412
5 5 10 15 15	0,1876	0,1328	0,1714	0,0994	0,0660	0,3940	0,3238	0,3432	0,2696	0,2926	0,5646	0,4942	0,5060	0,4524	0,4796
15 15 10 5 5	0,1584	0,1566	0,1994	0,1218	0,0458	0,3424	0,3996	0,4156	0,3654	0,2304	0,5450	0,6142	0,6212	0,5754	0,4590

	n=[30 30 30 30 30]					**n=[5 5 10 15 15]**					**n=[10 10 20 30 30]**				
	ANORE F	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**
10 10 10 10 10	0,9600	0,9582	0,9586	0,9530	0,9570	0,3270	0,2594	0,3330	0,2282	0,1958	0,6858	0,6108	0,6568	0,5888	0,6114
5 5 10 15 15	0,8970	0,8702	0,8738	0,8596	0,8662	0,4456	0,1656	0,2450	0,1708	0,1066	0,7244	0,4228	0,4802	0,4156	0,4124
15 15 10 5 5	0,8924	0,9254	0,9238	0,9146	0,8640	0,1286	0,2864	0,3420	0,2402	0,1496	0,3600	0,6298	0,6492	0,5956	0,4644

Table 8. *The estimated powers of tests for k=7*

μ=[3 3 4 4 4 5 5]

	n=[5 5 5 5 5 5 5]					**n=[10 10 10 10 10 10 10]**					**n=[15 15 15 15 15 15 15]**				
λ	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**
10 10 10 10 10 10 10	0,1812	0,1520	0,2568	0,1198	0,0756	0,4048	0,3526	0,4266	0,3202	0,3376	0,6104	0,5808	0,6212	0,5524	0,5738
5 5 10 10 10 15 15	0,1704	0,1142	0,2086	0,0928	0,0628	0,3484	0,2762	0,3364	0,2362	0,2518	0,5142	0,4336	0,4760	0,4030	0,4230
15 15 10 10 10 5 5	0,1540	0,1354	0,2356	0,1080	0,0466	0,3246	0,3492	0,4112	0,3178	0,2228	0,4922	0,5624	0,5980	0,5296	0,4076

	n=[30 30 30 30 30 30 30]					**n=[5 5 10 10 10 15 15]**					**n=[10 10 20 20 20 30 30]**				
	ANORE F	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**	**ANORE F**	**PBA**	**GPA**	**GPSL**	**CAT**
10 10 10 10 10 10 10	0,9402	0,9340	0,9404	0,9276	0,9366	0,3016	0,2428	0,3444	0,2178	0,1900	0,6290	0,5640	0,6330	0,5558	0,5632
5 5 10 10 10 15 15	0,8622	0,8236	0,8384	0,8134	0,8256	0,4090	0,2032	0,2846	0,1856	0,1140	0,6716	0,4664	0,5228	0,4466	0,4018
15 15 10 10 10 5 5	0,8692	0,9166	0,9252	0,9128	0,8322	0,1502	0,2324	0,3292	0,2044	0,1612	0,3720	0,5394	0,6058	0,5280	0,4400

As seen from Table 5, when k=3 and scale parameters are equal, for small, moderate and equal sample sizes, ANORE F-test is more powerful than the other tests. When the scale parameters are inversely proportional the power of GPA is slightly better than the other tests. For large and equal sample sizes, the powers of ANORE F-test and PBA are close to each other. When the sample sizes and scale parameters are different, the power of GPA is better than the other tests.

The similar interpretations are made for Table 6, that is when k=4. As seen from Table 7 and Table 8, when the scale parameters are equal, usually ANORE F-test is more powerful than the other tests. However, for small and moderate sample sizes, when the scale parameters are different, the power of PBA is higher than the other tests. Also, when sample sizes are different and scale parameters are inversely proportional, GPA is better than the others. While k=7, for moderate and equal sample sizes, when the scale parameters are different, PBA is more powerful than the others. For large and equal sample sizes, when scale parameters are inversely proportional GPA is more powerful than the other tests. Also for k=7, while sample sizes are different, especially as the difference between sample sizes go up and when the scale parameters are directly proportional, GPA is highly powerful.

4. CONCLUSION

In this study, the powers of five tests are compared. Simulation study indicates that, regardless of the number of groups, GPSL and CAT have satisfactory type I error rates for all parameter combinations. While the number of groups is small, k=3, in most of the situations, type I error rates of the ANORE F-test and PBA is higher than nominal level of α=0,05. As the number of groups are getting higher, especially when k=7, ANORE F-test and GPA exceed nominal level of α=0,05.
When the scale parameters are equal, as expected, ANORE F- test is more powerful than the others. Except for k=7, when the sample sizes are different and scale parameters are both different and inversely proportional, the power of GPA is higher than the other tests.

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