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On Finite Extension and Conditions on Infinite Subsets of Finitely Generated FC and FN_k-groups

Mourad Chelgham[1] <chelghamm@yahoo.fr>
Mohamed Kerada[2,*] <mkerada@yahoo.fr>
Lemnouar Noui[3] <nouilem@yahoo.fr>

[1]*Freres Mentouri Constantine1 University, Mathematics Department, 25000 Constantine, Algeria.*
[2]*LMAM Laboratory and Department of Informatics, Mohamed Seddik Ben Yahia University, Jijel, Algeria.*
[3]*Colonel El-Hadj Lakhdar University, Mathematics Department, 05000 Batna, Algeria.*

Abstract – Let k>0 an integer. F, τ, N, N_k, $N_k^{(2)}$ and A denote, respectively, the classes of finite, torsion, nilpotent, nilpotent of class at most k, group in which every two generator subgroup is in N_k and abelian groups. The main results of this paper is, firstly, to prove that in the class of finitely generated FN-group, the property FC is closed under finite extension. Secondly, we prove that a finitely generated τN-group in the class $((\tau N_k)\tau,\infty)$ (respectively $((\tau N_k)\tau,\infty)^*$) is a $\tau N_k^{(2)}$-group (respectively τN_c for certain integer c=c(k)) and deduce that a finitely generated FN-group in the class $((FN_k)F,\infty)$ (respectively $((FN_k)F,\infty)^*$) is $FN_k^{(2)}$-group (respectively FN_c for certain integer c=c(k)). Thirdly we prove that a finitely generated NF-group in the class $((FN_k)F,\infty)$ (respectively $((FN_k)F,\infty)^*$) is $N_k^{(2)}F$-group (respectively N_cF for certain integer c=c(k)). Finally and particularly, we deduce that a finitely generated FN-group in the class $((FA)F,\infty)$ (respectively $((FC)F,\infty)^*$, $((FN_2)F,\infty)^*$) is in the class FA (respectively FN_2, $FN_3^{(2)}$).

Keywords *– FC-group, (FC)F-group, $(\tau N_k)\tau$-group, $(FN_k)F$-group, $((FN_k)F,\infty)$-group, $((FN_k)F,\infty)^*$-group, finitely generated group.*

1 Introduction

Definition 1.1. A group G is said to be with finite contumacy classes (or shortly FC-group) if and only if every element of G has a finite contumacy class in G.

It is known that FIZ⊆FA⊆FC, where FIZ denotes the class of center-by-finite groups, and that for finitely generated equalities FIZ=FA=FC hold. These results and other have been studied and developed by Baer, Neumann, Erdos and Tomkinson and others in [5, 8, 13, 15, 22]. FC-groups have many similar properties with abelian groups and finite groups.

Corresponding Author.

On the one hand, several authors have studied the class of (χ,∞)-groups, where χ is a given property of groups, with some conditions on these groups. The question that interests mathematicians is the following: If G is a group in the class (χ,∞), where χ is a given property, and then does G have a property in relation to the property χ? For example is that G has the property $\chi\gamma$ or $\gamma\chi$, where γ is another group property, or in particular is it in the same class χ. For example, in 1976, B.H Neumann in [14], has shown that a group is in the class (A, ∞), if and only if, it is FIZ-group, where A is the class of abelian groups. In 1981, Lennox and Wiegold in [12] proved that a finitely generated solvable group is in the class (N, ∞) (resp. (P, ∞), (Co, ∞)) if and only if, it is FN, (resp. P, Co), where P, N, Co and F designates respectively polycyclic, nilpotent, coherent and finite class of groups. Other results of this type of this class can be found in section 2.

On the other hand, some authors give another extension of the problem of Paul Erdos and noted it $(\chi, \infty)^*$. For example in 2005, Trabelsi in [21] proved that a finitely generated soluble group in the class $(CN, \infty)^*$, where C is the class of cernikov group. Other results these types are given in section 2.

2 Preliminary

Before giving proof to the results in next section, we need some definitions and basic known facts from the theory of isolators in nilpotent groups, which has been developed in [11] (see also [6]).

Definition 2.1. A group G is said to be with finite contumacy classes (or shortly FC-group) if and only if every element of G has a finite contumacy class in G.

Nishigoryin [15] showed that every extension of a finite group by an FC-group is likewise an FC-group; in other words F(FC)=FC. As we mentioned in introduction the property FC is not closed under finite extension that means (FC)F is not always FC.

Therefore, we add some conditions on these groups so that it is. We prove in Theorem 1. that, in the class of finitely generated finite-by-nilpotent-group, the property FC is closed under taking finite extension.

Definition 2.2. If H is a subgroup of a group G. The isolator of H in G noted $I_G(H)$ is the set of elements $x \in G$ such that, for some integer $r > 0$, we have $x^r \in H$.

To prove the Theorem1 below in the next section, we begin by giving the next Lemma.

Lemma 2.1. Let G be a group and H a subgroup of G.

(i) If G is aτN-group, then the set of elements of finite order is a characteristic subgroup $\tau(G)$ of G and the group quotient $G/\tau(G)$ is torsion-free.

(ii) If G is a finitely generated FN-group, then $\tau(G)$ is finite.

(iii) If G is locally nilpotent torsion-free group, then the isolator $I_G(H) = \{x \in G \,/\exists\, n\in \mathbb{N}: x^n \in H\}$ is a subgroup of G containing H . If H is nilpotent of class k then $I_G(H)$ is nilpotent of class k as well. In particular, if H is abelian then $I_G(H)$ is abelian as well.

Definition 2.3. Let χ is a given property of groups. A group G is said to be in the class (χ,∞) (respectively $(\chi, \infty)^*$) if and only if every infinite subset X of G contains two distinct elements x, y such that the subgroup $<x, y>$ (respectively $<x, x^y>$) is a χ-group.

Note that if χ is a subgroup closed class, then $\chi \subset (\chi, \infty) \subset (\chi, \infty)^*$.

In addition to the first results mentioned in the introduction concerning category (χ, ∞), we recall other results. In 2000, 2002 and 2005, Abdollahi and Trabelsi, proved in [1, 19, 21] that a finitely generated solvable group is in the class (FN_k, ∞) (resp. (FN, ∞), (NF, ∞), $(\tau N, \infty)$) if and only if it is $FN_k^{(2)}$, (resp. FN, NF, τN). Other results of this type have been obtained, for example in [3, 4, 7, 9, 10, 20].

In this note we prove that a finitely generated τN-group G is in the class $((\tau N_k)\tau, \infty)$ is in the class $\tau N_k^{(2)}$ and deduce that a finitely generated FN-group (respectively NF-group) G in the class of $((FN_k)F, \infty)$-groups, is in the class of $FN_k^{(2)}$-groups (respectively in the class of $N_k^{(2)}F$-groups) and In particular a finitely generated FN-group G is in the class $((FC)F, \infty)$, if and only if, it is FA-group.

About other results on the class $(\chi, \infty)^*$. In 2007, Rouabehi and Trabelsi proved in [18] that a finitely generated soluble group in the class $(CN, \infty)^*$ where C is the class of cernikov group (respectively in the class $(\tau N, \infty)^*$) is FN-group (respectively τN-group).In 2007 too, Guerbi and Rouabhi proved in [9] that a finitely generated Hyper (abelian-by-finite) group in the class $(\Omega, \infty)^*$ is FN-group, where Ω the class of groups of finite depth, i.e. $G \in \Omega$, if and only if, there exists $k \in \mathbb{N}: \gamma_{k+1}(G)= \gamma_k(G)$ where $(\gamma_i(G))$ is the lower central series of G.
In this paper we prove that a finitely generated τN-group in the class $((\tau N_k)\tau, \infty)^*$ is in the class $(\tau N_c)\tau$ for certain integer c=c(k) and deduce that a finitely generated FN-group (respectively NF-group) G in the class $((FN_k)F, \infty)^*)$ is in the class FN_c (respectively N_cF). Finally, if G is a finitely generated FN-group in the class $((FC)F, \infty)^*$ (respectively $((FN_2)F, \infty)^*$) then G is in the class of FN_2-groups (respectively in the class of $FN_3^{(2)}$-groups) .

3 Main results

3.1. Stability by finite extension

As we know, the property FC is not closed under the formation of extension. The following example shows that even, a finite extension of a FC-group is not always a FC-group.

Example 3.1. Let $G = D_\infty =< a; b/ a^2 = 1$ and $aba = b^{-1}>$ the infinite dihedral group, which is a finitely generated soluble group, generated by the involutions a, b. We have $K=C_\infty =<b>$ which is a infinite cyclic group isomorphic to Z therefore it is a FC-group and the quotient group G/K is isomorphic to $C_2 =< a >$ which is finite of order 2, thus G is a finite extension of a FC-group, but as the center of the infinite dihedral group is trivial then it is not a FC-group.

This example shows also that, in the class of finitely generated soluble groups, the property FC is not closed under the formation of finite extension. So we consider the class of finitely generated finite-by-nilpotent groups. We prove that, in this class, the property FC is closed under taking finite extension. Precisely we prove the following Theorem.

Theorem 3.1. Let G a finitely generated finite-by-nilpotent group. G is FC-by-finite group, if and only if, G is FC-group.

Proof. If G is FC-group, it is clear that, G is FC-by-finite. Conversely, since G is finitely generated finite-by-nilpotent group, there exists a finite normal subgroup F of G such that the quotient group G/F is nilpotent group. As the property FC-by-finite is closed under quotient, it is enough to show that G/F is a FC-group. For this it is sufficient to show that every FC-by-finite group G in the class of finitely generated nilpotent groups is a FC-group too. Assume that G is (FC)F, so there exists a normal FC-subgroup N with of finite index in the group G. Since G is finitely generated and nilpotent, it checks the maximal condition on subgroups. So N is finitely generated FC-subgroup. According to ([5], Theorem 6.2) N is center-by-finite which means that Z(N) is of finite index in N. Or N is of finite index in G. It follows that Z(N) is of finite index in G. Let T = τ(G), the torsion subgroup of G, by Lemma 2.1, (ii) above T is finite. Note that, since F(FC) = FC as pointed out above in [15], it is enough to prove the statement for G=T, that is we may assume T = 1, that is G is nilpotent torsion-free group. Since Z(N) is of finite index in G then $I_G(H) = G$. So by using Lemma 2.1, (iii) with H = Z(N) we deduce that G is abelian group. This completes the proof.

Remark 3.1. The example below shows that Theorem 1. is falls when the condition "finitely generated" is omitted.

Example 3.2 Let A = $F_2[X]$ algebra of polynoms on the field F_2 and the isomorphism $\varphi: A \times A \rightarrow A \times A$, $(P,Q) \rightarrow (P + Q,Q)$. We put $H = A \times A$ and K =<φ> such that $\varphi^2 = Id_{A\times A}$ the identity application on $A\times A$. Since H is an abelian group, it is a FC-group. K is a finite group of order 2 and so it is FC too. We consider G = H ©K, the semi-direct product of H by K. G is a non finitely generated nilpotent group, which is a finite extension of the FC-group H. But G is not a FC-group.

3.2 τN_k and FN_k-groups and conditions on infinite subsets

Our first elementary propositions below follows from lemmas below.

Lemma 3.1. ([1], Corollary 1.8. (i)) If G a finitely generated soluble group in the class (FN_k,∞), then G is in the class of $FN_k^{(2)}$ -groups and there exists an integer t, depending only on k, such $G=Z_t(G)$ is finite.

Lemma 3.2. ([4], Theorem) Let G be a finitely generated soluble group. Then G has the property (N_k,∞) if and only if G is a $FN_k^{(2)}$-group.

Proposition 3.1. If G is a finitely generated finite-by-soluble group in the class (FN_k,∞); then G is in the class of $FN_k^{(2)}$-groups.

Proof. Suppose that G is finite-by-soluble, there exists finite normal subgroup N such that G/N is soluble. As the class of (FN_k,∞)-group, is closed under taking quotient, then the quotient group G/N is a finitely generated soluble group in the class of (FN_k,∞)-group. By

Lemma 3.2 above, the quotient group G/N is in the class of $FN_k^{(2)}$-groups. Therefore G is finite-by- $FN_k^{(2)}$-groupand this gives that G is $FN_k^{(2)}$-group. This completes the proof.

Proposition 3.2. If G is a finitely generated torsion-by-soluble group in the class $(\tau N_k,\infty)$; then G is in the class of $\tau N_k^{(2)}$-groups.

Proof. Suppose that G is torsion-by-soluble, there exists a torsion and normal subgroup N such that G/N is soluble. As the class of $(\tau N_k,\infty)$-group, is closed under taking quotient, then the quotient group G/N is a finitely generated soluble group in the class of $(\tau N_k,\infty)$which is included in $(\tau N, \infty)$.By a result in [21], G/N is in the class of τN-groups. Using Lemma 2.1, (i), G/N admits a torsion group $\tau(G/N) = T/N$ such that the quotient G/T is torsion-free in the class $(\tau N_k,\infty)$. So G/T is a finitely generated soluble group in the class $(\tau N_k,\infty)$. It results by Lemma 3.2 above that G/T is in the class $FN_k^{(2)}$, therefore G is torsion-by-$FN_k^{(2)}$, and this gives that G is $\tau N_k^{(2)}$-group. This completes the proof.

Theorem 3.2 Let G a finitely generated τN-group. If G is in the class $((\tau N_k)\tau,\infty)$, then, G is $\tau N_k^{(2)}$-group.

Proof. Assume that G is finitely generated τN- group in the class $((\tau N_k)\tau,\infty)$. There exists a normal and torsion subgroup H of G such that G/H is nilpotent quotient group. Since G/H is finitely generated nilpotent group, it has a torsion subgroup T/H of finite order and as H is torsion group then T is torsion group too. So G/T is torsion-free nilpotent group in the class $((\tau N_k)\tau,\infty)$ which gives that G/T is in the class $(N_k\tau,\infty)$. We deduce by ([16], Lemma 6.33) that G/T is in the class $(N_k\tau,\infty)$ and so G/T is a finitely generated soluble group in the class (N_k,∞). It follows by ([4] Theorem) that G/T belongs in the class of $FN_k^{(2)}$-groups and as T is torsion, it gives that G is in the class of $\tau N_k^{(2)}$-groups. This completes the proof.

If we replace the property τN by the property FN, we obtain the result in the lemma below.

Lemma 3.3. Let G a finitely generated FN-group.
(i) If G is in the class $((FN_k)F,\infty)$, then G is in the class of $FN_k^{(2)}$-groups.
(ii) G is in the class $((FC)F,\infty)$, if and only if, G is FC-group.

Proof. (i) Assume that G is finitely generated FN-group in the class $((FN_k)F,\infty)$which is in the class$((\tau N_k)\tau,\infty)$. As G is FN-group, there exists a normal and finite subgroup H of G such that G/H is nilpotent. As in the above theorem, we found that the torsion subgroup T/Hof G/H is finite and so T is finite too. As the property $((\tau N_k)\tau,\infty)$ is closed under quotient then the quotient group G/T a torsion-free nilpotent group which verifies the conditions of the above theorem. It follows that G/T belongs in the class of $FN_k^{(2)}$-groups,which gives that G/Tis in the class$N_k^{(2)}$ and hence G is in the class $FN_k^{(2)}$.

(ii) As finitely generated FN-group verifies maximal condition on subgroups, then, FC = FA = FN_1 = $FN_1^{(2)}$ and $((FC)F,\infty) = ((FN_1)F,\infty)$. This completes the proof.

The Example 1 above shows that nilpotency is necessary for the results of the above theorem to remain true.

Remark 3.2. (i) As $(FN_k)F$ is a subgroup closed class, then$(FN_k)F\subset((FN_k)F,\infty)$, we deduce that a finitely generated FN-group in the class $(FN_k)F$, is in the class $FN_k^{(2)}$.

(ii) Theorem 1 can be proved by using (ii) in the lemma above and by seeing that (FC)F is a subgroup closed class so $(FC)F\subset((FC)F, \infty)$.

(iii) In (i) of the above lemma, as G is in the class $FN_k^{(2)}$ and as nilpotent groups of class at most k are k-Engel then G is finite-by-(k-Engel, torsion-free and soluble of derived length an integer d). So by a result of Gruenberg [16, Theorem 7.36 (i)] G is in the class of$FN_{k^{d-1}}$and by P. Hall [10] there exists an integer c=c(k, l) depending on k, d such that $G/Z_c(G)$.

Recall that FN-groups are NF-groups (see[9]).

Theorem 3.3. Let G a finitely generated NF-group.
(i) If G is in the class $((FN_k)F,\infty)$, then G is in the class of $N_k^{(2)}F$-groups.
(ii) In particular, if G is in the class $((FC)F,\infty)$, then G is in the class of AF-group.

Proof. (i) Assume that G is finitely generated NF-group in the class $((FN_k)F,\infty)$. As the group G is NF- group, and then it contains a normal nilpotent subgroup N such that G/N is finite. As the subgroup N is finitely generated and nilpotent of finite index then N is polycyclic so by ([14], Theorem 5.4.15) there exists a subgroup M normal in N and poly-infinite cyclic hence torsion-free and of finite index in N. Let $K=M_G$ the core of the subgroup M, so K is nilpotent torsion-free of finite index in G. Since the class $((FN_k)F,\infty)$ is closed under taking subgroups, then K is nilpotent subgroup in the class $((FN_k)F,\infty)$ and according to (i) in the above lemma we deduce that K is torsion-free subgroup in the class of $FN_k^{(2)}$-groups which gives that K is $N_k^{(2)}$-group and so G is $N_k^{(2)}F$-group. In particular, for k=1 $(FC)F=(FA)F=(FN_1)F$ and $N_1^{(2)}F=AF$.This completes the proof.

If we replace the property $((\tau N_k)\tau, \infty)$ by the property $((\tau N_k)\tau, \infty)^*$ in the above Theorem, we obtain the next result.

Theorem 3.4. Let G a finitely generated τN-group. G is in the class $((\tau N_k)\tau, \infty)^*$, then there exists an integer c=c(k) such that G is in the class of τN_c-group.

Proof. Assume that G is finitely generated τN- group in the class $((\tau N_k)\tau, \infty)^*$. Let $T=\tau(G)$ the torsion group of G. So by Lemma 2.1. (i) G/T is torsion-free nilpotent group and as $((\tau N_k)\tau, \infty)^*$is quotient closed class then G/T belongs in $((\tau N_k)\tau, \infty)^*$ and hence G/T is in the class $(N_k\tau, \infty)^*$. We deduce by ([16], Lemma 6.33) that G/T is in the class $(N_k, \infty)^*$. Note that the class $(N_k, \infty)^*$is included in the class $\varepsilon_{k+1}(\infty)$, where $\varepsilon_{k+1}(\infty)$ is the class of groups whose every infinite subset X contain two distinct elements x, y such that $[x,_{k+1}y]=1$. We deduce that G/T belongs in $\varepsilon_{k+1}(\infty)$. Since G/T is nilpotent so soluble then by ([2], Theorem 3) there exists an integer c=c(k) depending only on k such that $(G/T)/Z_c(G/T)$ is finite. By a result in ([10], Theorem 1) $\gamma_{c+1}(G/T)= \gamma_{c+1}(G)T/T$ is finite and so is torsion, and since T is torsion group , we deduce that $\gamma_{c+1}(G)$ is torsion group too. Therefore G is in the class of τN_c-group. This completes the proof.

Lemma 3.4. Let G a finitely generated FN-group.
(i) If G is in the class $((FN_k)F,\infty)^*$, then there exists an integer c=c(k) depending only on k such that G is in the class ofFN_c-group.
(ii) G is in the class $((FC)F,\infty)^*$, then, $G/Z_2(G)$ is finite and G is in the class of FN_2-groups.
(iii) If G is in the class $((FN_2)F, \infty)^*$, then, G is in the class of $FN_3^{(2)}$-groups.

Proof. (i) Assume that G is finitely generated FN- group in the class $((FN_k)F,\infty)^*$. Let $T=\tau(G)$ the torsion subgroup of G. So by Lemma 2.1. (ii) T is a characteristic (so normal) and finite subgroup in G and as the same way in the above theorem, we deduce by ([16], Lemma 6.33) that G/T is in the class $(N_k, \infty)^*$which is included in the class $\varepsilon_{k+1}(\infty)$and according to ([2], Theorem 3) we found that there exists an integer c=c(k) depending only on k such that $(G/T)/Z_c(G/T)$ is finite. By a result in ([10], Theorem 1) $\gamma_{c+1}(G/T)= \gamma_{c+1}(G)T/T$ is finite and since T is finite, $\gamma_{c+1}(G)$ is finite too. Therefore G is in the class of FN_c-groups.

(ii) As the same way in (i) and the above Theorem we found that G/T is in the class $(A,\infty)^*$which is included in the class $\varepsilon_2(\infty)$, where $\varepsilon_2(\infty)$ is the class of groups whose every infinite subset X contain two distinct elements x, y such that $[x,_2y]=1$. we deduce by ([7], Theorem) that $(G/T)/Z_2(G/T)$ is finite and as T is finite then $G/Z_2(G)$ is finite equivalently $\gamma_3(G)$ is finite. It follows that G is in the class of FN_2-groups.

(iii) For k=2, as the same way in the above theorem we found that G/T is in the class $(N_2,\infty)^*$which is included in the class $\varepsilon_3(\infty)$, where $\varepsilon_3(\infty)$ is the class of groups whose every infinite subset X contain two distinct elements x, y such that $[x,_3y]=1$.we deduce by ([2], Theorem 1) that G/T is in the class $FN_3^{(2)}$ and as the torsion subgroup T is finite, then G is $F(FN_3^{(2)})$-group. It follows that G is $FN_3^{(2)}$-group. This completes the proof.

Theorem 3.5. Let G a finitely generated NF-group.
(i) If G is in the class $((FN_k)F,\infty)^*$, then there exists an integer c=c(k) depending only on such that G is in the class of N_cF-groups.
(ii) If G is in the class of $((FC)F, \infty)^*$-groups, then, G is in the class of N_2F-group.
(iii) If G is in the class $((FN_2)F, \infty)^*$, then, G is in the class of $N_3^{(2)}F$-groups.

Proof. As the group G is NF- group, and then it contains a normal nilpotent subgroup N such that G/N is finite. As the subgroup N is finitely generated and nilpotent of finite index then N is polycyclic so by ([14], Theorem 5.4.15) there exists a normal subgroup M in N and poly-infinite cyclic hence torsion-free and of finite index in N. Let $K=M_G$ the core of the subgroup M, so K is nilpotent torsion-free of finite index in G. Since the class $((FN_k)F,\infty)^*$is closed under taking subgroups, then K is in this class too, so by (i) in the above lemma, we obtains that there exists an integer c=c(k) depending only on k such that K is FN_c-group and as K is torsion-free, it is N_c-group and so G is N_cF-group

(ii) Particulary for k=1, we have $((FC)F, \infty)^*=((FN_1)F, \infty)^*$, in this case the subgroup K is a finitely generated torsion-free nilpotent group in the class $((FN_1)F, \infty)^*$ and according to

(ii) in the above lemma, we deduce that K is in the class FN_2-groups and as K is torsion-free, it is N_2-group of finite index in G, this gives that G is N_2F-group.

(iii) In particular for k=2, we have the subgroup K in (i) is a finitely generated torsion-free nilpotent group in the class $((FN_2)F, \infty)^*$ and according to (iii) in the above lemma, we deduce that K is in the class $FN_3^{(2)}$-groups and as K is torsion-free it is the class $N_3^{(2)}$-group and as G/K if finite this gives that G is in the class of $N_3^{(2)}F$-groups.

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