

High-Resolution Direction Finding on Uniform Linear Arrays: A Comparative Evaluation of MUSIC Algorithm Derivatives

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Graphical/Tabular Abstract (Grafik Özet)

This study compares variants of the MUSIC algorithm for Direction of Arrival (DoA) estimation using a Uniform Linear Array (ULA) with different antenna numbers and spacings. Simulations with both coherent and non-coherent sources assess Traditional, Root, FBSS, and Improved MUSIC in terms of accuracy and efficiency. / Bu çalışma, farklı anten sayısı ve aralıklarına sahip uniform doğrusal dizilerde (ULA) DoA kestirimi için çeşitli MUSIC algoritması türevlerini karşılaştırmaktadır. Eşfazlı ve eşfazlı olmayan kaynaklarla yapılan benzetimler, Geleneksel, Root, FBSS ve İyileştirilmiş MUSIC yöntemlerini doğruluk ve verimlilik açısından değerlendirmektedir.

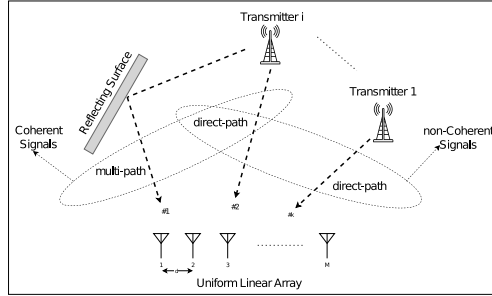


Figure A: DoA Scenario with ULA /Şekil A: Düzgün Doğrusal Dizilim ile Yön Bulma Senaryosu

Highlights (Önemli noktalar)

- Comprehensive comparison of four MUSIC algorithm variants under unified conditions / Dört farklı MUSIC algoritması türevinin ortak koşullar altında kapsamlı karşılaştırması
- Evaluation includes both accuracy (RMSE) and computational efficiency / Değerlendirme, hem doğruluk (RMSE) hem de hesaplama verimliliğini içermektedir
- Offers practical guidance for algorithm selection based on system needs / Sistem ihtiyaçlarına göre algoritma seçimi için pratik bir rehber sunmaktadır

Aim (Amaç): The purpose of this study is to conduct a detailed comparative analysis of various MUSIC algorithm variants for DoA estimation in array signal processing. / Bu çalışmanın amacı, dizi işaret işleme alanında DoA (Geliş Yönü) kestirimi için çeşitli MUSIC algoritması türevlerinin ayrıntılı bir karşılaştırmalı analizini yapmaktır.

Originality (Özgünlük): This study provides a unified comparison of four prominent MUSIC algorithm variants within a consistent simulation framework. Using extensive MATLAB-based Monte Carlo simulations under coherent and non-coherent conditions, it offers a deeper evaluation than most prior works. / Bu çalışma, dört önde gelen MUSIC algoritması türevinin tutarlı bir benzetim çerçevesinde karşılaştırmasını sunmaktadır. Eşfazlı ve eşfazlı olmayan koşullar altında kapsamlı MATLAB tabanlı Monte Carlo benzetimleri kullanılarak önceki çalışmalara göre daha derinlemesine bir değerlendirme sağlamaktadır.

Results (Bulgular): The simulation results revealed that each algorithm has particular strengths: Traditional MUSIC provides a solid reference point, Root MUSIC excels in computational performance, FBSS MUSIC effectively handles coherent sources, and Improved MUSIC offers enhanced accuracy in complex conditions. / Simülasyon sonuçları, her bir algoritmanın kendine özgü güçlü yönleri olduğunu ortaya koymuştur: Geleneksel MUSIC sağlam bir referans noktası sunarken, Root MUSIC hesaplama performansında öne çıkmaktadır. FBSS MUSIC, eşfazlı kaynakları etkili bir şekilde işlerken, Geliştirilmiş MUSIC karmaşık koşullarda artırılmış doğruluk sağlamaktadır.

Conclusion (Sonuç): In conclusion, this study has systematically explored and compared several MUSIC-based algorithms for DoA estimation, emphasizing their unique features, performance metrics, and practical suitability. / Sonuç olarak, bu çalışma DoA kestirimi için çeşitli MUSIC tabanlı algoritmaları sistematik bir şekilde incelemiş ve karşılaştırmış, onların kendine özgü özelliklerini, performans ölçütlerini ve pratik uygunluklarını vurgulamıştır.



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Abstract

This paper investigates various Multiple Signal Classification (MUSIC) algorithms for Direction of Arrival (DoA) estimation, including 'Traditional MUSIC', 'Root MUSIC', 'Forward/Backward Spatial Smoothing (FBSS) MUSIC', and 'Improved/Modified MUSIC'. The study elaborates on the principles underlying each algorithm and explores the factors influencing the accuracy of DoA estimation, such as the number of array elements, antenna spacing, number of snapshots, Signal-to-Noise Ratio (SNR), and scanning angle resolution. Through extensive MATLAB simulations employing the Monte Carlo method, the performance of these algorithms is evaluated and compared in scenarios involving coherent, non-coherent, and single/dual-source incident signals. The Root Mean Square Error (RMSE) is calculated as a function of the aforementioned parameters to quantify estimation accuracy. Additionally, the computational efficiency of each algorithm is assessed by comparing their execution durations. The results provide valuable insights into the strengths and limitations of each MUSIC variant, offering guidance for their application in practical DoA estimation tasks.

Düzgün Doğrusal Dizilerde Yüksek Çözünürlüklü Yön Bulma: MUSIC Algoritması Türevlerinin Karşılaştırmalı Değerlendirmesi

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Öz

Bu çalışma, Geleneksel MUSIC, Kök MUSIC (Root MUSIC), İleri/Geri Uzaysal Yumuşatma (FBSS) MUSIC ve Geliştirilmiş/Değiştirilmiş MUSIC dahil olmak üzere, Geliş Açısı (DoA) kestirimi için çoklu sinyal sınıflandırma (MUSIC) algoritmalarını incelemektedir. Çalışma, her bir algoritmanın temel prensiplerini detaylandırmakta ve dizi eleman sayısı, anten aralığı, anlık görüntü sayısı, Sinyal-Gürültü Oranı (SNR) ve tarama açısı çözünürlüğü gibi DoA kestirimi doğruluğunu etkileyen faktörleri araştırmaktadır. Monte Carlo yöntemi kullanılarak gerçekleştirilen kapsamlı MATLAB simülasyonları aracılığıyla, bu algoritmaların performansı, eşfazlı (koherent), eşfazlı olmayan (non-koherent) ve tek/çift kaynaklı gelen sinyal senaryolarında değerlendirilmiş ve karşılaştırılmıştır. Kestirim doğruluğunu nicelendirmek için, bahsedilen parametrelerin bir fonksiyonu olarak Ortalama Karekök Hata (RMSE) hesaplanmıştır. Ek olarak, her bir algoritmanın hesaplama verimliliği, yürütme süreleri karşılaştırılarak değerlendirilmiştir. Sonuçlar, her bir MUSIC varyantının güçlü ve zayıf yönlerine dair değerli bilgiler sunarak, pratik DoA kestirimi uygulamalarında kullanımlarına yönelik rehberlik sağlamaktadır.

1. INTRODUCTION (GİRİŞ)

Direction of Arrival (DoA) estimation is a fundamental problem in array signal processing, with wide-ranging applications in radar, sonar, wireless communications, and acoustic sensing. Accurate DoA estimation enables the localization of signal sources [1], which is critical for tasks such as target tracking [2], beamforming, and spatial

filtering. From past to present, numerous parametric algorithms [3]-[6] and deep learning-based methods [7]-[10] have been developed for direction-of-arrival estimation. Among the various techniques developed for DoA estimation, the Multiple Signal Classification (MUSIC) algorithm has emerged as one of the most prominent and widely used methods due to its high resolution and robustness in resolving closely spaced sources.

The MUSIC algorithm [11], introduced by Schmidt in 1986, leverages the eigenstructure of the covariance matrix of received signals to estimate the DoA of multiple sources. Its ability to achieve super-resolution capabilities has made it a cornerstone in array signal processing. However, the performance of MUSIC is influenced by several factors, including the array configuration, signal coherence, and environmental noise. Over the years, numerous variants of the MUSIC algorithm have been proposed to address these challenges and enhance its applicability in diverse scenarios. These include 'Root MUSIC' [12], which improves estimation accuracy by transforming the spectral search into a polynomial rooting problem; 'Forward/Backward Spatial Smoothing (FBSS) MUSIC' [13]-[14], which mitigates the degradation caused by coherent signals; and 'Improved/Modified MUSIC' [15], which incorporates advanced techniques to further refine estimation performance.

In the literature, numerous studies are available related to the traditional MUSIC algorithm [16]-[19], the Root MUSIC algorithm [20]-[23], the FBSS MUSIC algorithm [24]-[27], and the Improved/Modified MUSIC algorithm [28]-[31]. Despite the advancements in MUSIC-based algorithms, a comprehensive comparison of their performance under varying conditions remains an area of active research. Factors such as the number of array elements, antenna spacing, number of snapshots, Signal-to-Noise Ratio (SNR), and scanning angle resolution significantly impact the accuracy and efficiency of DoA estimation. Understanding the interplay between these factors is essential for optimizing the design and deployment of array systems in practical applications.

This paper aims to provide a detailed analysis of four MUSIC algorithms—Traditional MUSIC, Root MUSIC, FBSS MUSIC, and Improved/Modified MUSIC—in the context of DoA estimation. Through extensive MATLAB simulations employing the Monte Carlo method, the performance of these algorithms is evaluated and compared under scenarios involving coherent, non-coherent, and single/dual-source incident signals. The Root Mean Square Error (RMSE) is used as a metric to quantify estimation accuracy, while computational efficiency is assessed by comparing algorithm durations. By examining the strengths and limitations of each algorithm, this study seeks to offer valuable insights for researchers and practitioners in selecting and implementing the most suitable MUSIC variant for their specific application needs.

The main contributions of this paper are summarized as follows:

1. **Comprehensive Evaluation of MUSIC Variants:** It is performed a detailed comparative analysis of four key MUSIC algorithm derivatives—Traditional MUSIC, Root MUSIC, FBSS MUSIC, and Improved/Modified MUSIC—highlighting their operational principles and performance trade-offs.
2. **Performance Assessment under Diverse Signal Conditions:** It is systematically evaluated each algorithm's accuracy in estimating Direction of Arrival (DoA) for coherent, non-coherent, and single/dual-source signal scenarios, addressing practical challenges in real-world signal environments.
3. **Impact Analysis of System Parameters:** It is investigated how various system-level factors—including number of array elements, inter-element spacing, snapshot count, SNR, and angle scanning resolution—affect the DoA estimation performance, using Root Mean Square Error (RMSE) as a performance metric.
4. **Computational Efficiency Comparison:** The study includes an execution time analysis to assess the computational demands of each algorithm, providing insights into their suitability for real-time applications.
5. **Simulation-Driven Insights via Monte Carlo Analysis:** Extensive MATLAB-based Monte Carlo simulations support our findings, ensuring statistical robustness and offering practical guidance for algorithm selection based on performance and complexity.

The remainder of this paper is organized as follows: Section 2 provides a brief overview of the theoretical foundation underlying Direction of Arrival (DoA) estimation using Uniform Linear Arrays (ULAs), with a particular focus on the subspace-based MUSIC family of algorithms. Section 3 details the specific formulations and processing steps of each MUSIC variant studied in this work. In Section 4, it is presented and discussed the comparative performance results based on RMSE and computational time metrics. Finally, Section 5 concludes the paper with a summary of key findings and recommendations for future research directions in high-resolution DoA estimation.

2. MATHEMATICAL MODEL (MATEMATİKSEL MODEL)

A uniform linear array with M elements is considered for capturing L narrowband coherent and/or non-coherent signals. Here, each source impinges on the array from a distinct direction, denoted as θ_k for $k = 1, 2, \dots, L$; where the index k distinguishes each individual source. Moreover, the spacing between adjacent elements is bounded by $d \leq \lambda/2$, where λ represents the wavelength of the incoming signals [32]-[33].

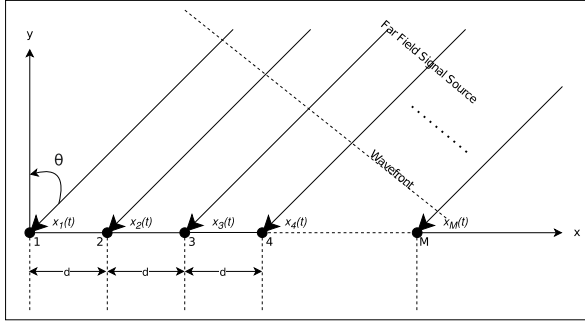


Figure 1. Structure of a uniform linear antenna array (Düzgün doğrusal anten dizisinin yapısı)

Figure 1. illustrates the configuration of the antenna array, where both the array and the incoming signals are assumed to lie within the same plane.

Here, the matrix representation of the L incident signal sources on the array elements can be formulated in terms of the signal vector as:

$$S(t) = [S^1(t), S^2(t), \dots, S_L(t)]^T \quad (1)$$

By selecting the first element of the antenna array as the reference, the received signals across the M array elements can be arranged into the following matrix form:

$$R(t) = AS(t) + N(t) \quad (2)$$

$$R(t) = [R_1(t) \cdot R_2(t) \cdots R_M(t)]^T \quad (3)$$

In this formulation, $N(t)$ is defined as the noise vector representing the total noise across all array elements, while A denotes the steering vector matrix, which can be expressed as:

$$A = [\alpha(\theta_1) \cdot \alpha(\theta_2) \cdots \alpha(\theta_L)] \quad (4)$$

and

$$\alpha(\theta_k) = \left[1 e^{-j \frac{2\pi d_1}{\lambda} \sin(\theta_k)} \dots e^{-j(M-1) \frac{2\pi d_{M-1}}{\lambda} \sin(\theta_k)} \right]^T \quad (5)$$

3. THE MUSIC ALGORITHMS (MUSIC ALGORİTMALARI)

3.1. Traditional MUSIC Algorithm (Geleneksel MUSIC Algoritması)

MUSIC algorithm is an advanced spectral estimation technique that employs eigenvalue decomposition to determine the direction of arrival (DOA) of incoming signals. Using the orthogonality principle, the method decomposes the received signals correlation matrix into signal and noise subspaces, where the DOA is estimated by identifying steering vectors that are orthogonal to the noise subspace. This approach ensures the accuracy of the DOA estimation [11].

The covariance matrix V_C is defined as:

$$V_C = E[RR^H] \quad (6)$$

Since the signal and noise are independent, the data covariance matrix can be separated into distinct signal and noise components:

$$V_C = AV_S A^H + V_N \quad (7)$$

Here, V_S represents the signal correlation matrix, while $AV_S A^H$ corresponds to the signal component. A corresponds to the steering vector matrix, which characterizes the directional response of the array to incoming signals from different angles and V_N is the noise correlation matrix and can be expressed as:

$$V_C = P_S \Sigma P_S^H + P_N \sum P_N^H \quad (9)$$

The correlation matrix can be decomposed into components associated with the signal and noise subspaces, where P_S and P_N serve as the respective bases, as illustrated below:

$$V_C = P_S \Sigma P_S^H + P_N \sum P_N^H \quad (9)$$

The DOA can be estimated through a minimal optimization search, as:

$$\alpha^H(\theta) P_N = 0 \quad (10)$$

$$\theta_{MUSIC} = \operatorname{argmin} \alpha^H(\theta) P_N P_N^H \alpha(\theta) \quad (11)$$

By reformulating the above equation in inverse form, a spectral function is obtained, with peaks detected using spectral peak search to estimate the DOA signals, as demonstrated below:

$$P_{MUSIC} = \frac{1}{\alpha^H(\theta) P_N P_N^H \alpha(\theta)} \quad (12)$$

3.2. Improved/Modified MUSIC Algorithm

(Geliştirilmiş/Değiştirilmiş MUSIC Algoritması)

Traditional MUSIC struggles with coherent signals. To enhance performance, a transformation matrix F and the complex conjugate of R are applied, yielding the received signal matrix W and its covariance V_W [15]:

$$V_W = E[WW^H] = FV_C^*F \quad (13)$$

The matrices V_C and V_W are integrated to construct V :

$$V = AV_S A^H + F[AV_S A^H]^*F + 2\sigma^2 I \quad (14)$$

The noise subspace derived from the eigen-decomposition of V is used to construct the spatial spectrum and detect its peaks.

3.3. Root MUSIC Algorithm (Kök MUSIC Algoritması)

The Root MUSIC algorithm recasts conventional spectral peak search into a polynomial root-finding approach via a defined polynomial [12], [20] -[22].

Here, the signal angles are determined by selecting steering vectors that, due to their orthogonality with the noise subspace, yield a null response, thereby facilitating the formulation of a corresponding polynomial [12]:

$$P(y) = A^H P_N P_N^H A \quad (15)$$

where $K \times K$ correlation matrix is assumed and the steering vector A is represented as:

$$A = \begin{bmatrix} e^{-j\frac{2\pi d}{\lambda} \sin(\theta) \cdot 0} \\ e^{-j\frac{2\pi d}{\lambda} \sin(\theta) \cdot 1} \\ e^{-j\frac{2\pi d}{\lambda} \sin(\theta) \cdot 2} \\ \vdots \\ e^{-j\frac{2\pi d}{\lambda} \sin(\theta) \cdot (K-1)} \end{bmatrix} \quad (16)$$

Due to noise, the L roots of $P(y)$ closest to the unit circle are selected and the corresponding signal angles are calculated using:

$$y = e^{j\frac{2\pi d}{\lambda} \sin(\theta)} \quad (17)$$

$$\theta_k = \arcsin \left[\frac{\lambda}{2\pi d} \arg(y_k) \right] \quad (18)$$

3.4. FBSS MUSIC Algorithm (FBSS MUSIC Algoritması)

Considering the ULA defined in the mathematical model section, spatial smoothing is applied to

obtain the average covariance matrices for the forward and backward subarrays, denoted as V_f and V_b , respectively, as in [13]-[14]:

$$V_f = \frac{1}{U} \sum_{i=1}^U E \left[R_x^f(t) R_x^{fH}(t) \right] \quad (19)$$

$$V_b = \frac{1}{U} \sum_{i=1}^U E \left[R_x^b(t) R_x^{bH}(t) \right] \quad (20)$$

Here, U denotes the total number of subarrays, with the x^{th} forward and backward subarrays represented by $R_x^f(t)$ and $R_x^b(t)$, respectively. The corresponding covariance matrices for spatial smoothing are formulated in:

$$V_{fb} = \frac{V_f + V_b}{2} \quad (21)$$

After constructing the covariance matrix and obtaining the noise subspace P_N via eigenvalue decomposition, the DOAs are estimated from the peaks of the following pseudo-spectrum:

$$P_{FBSS}(\theta) = \frac{1}{\alpha^H(\theta) P_N P_N^H \alpha(\theta)} \quad (22)$$

4. SIMULATION RESULTS AND ANALYSIS (BENZETİM SONUÇLARI VE ANALİZLER)

Simulations were conducted in different scenarios to compare the performances of MUSIC algorithms. Direction-finding estimations of the algorithms were obtained, and errors were calculated by generating coherent and noncoherent signals for single source and dual sources cases.

4.1. Single Source Incident Signal (Tek Kaynaktan Gelen Sinyal)

The results of single source direction of arrival estimation include 'RMSE-Number of Array Elements', 'RMSE-Spacing between Array Elements', 'RMSE-Number of Snapshot' and 'RMSE-SNR' graphs obtained via the Monte Carlo method. Lastly, the effect of the scanning angle resolution on the error has been observed and the duration of the traditional MUSIC algorithm for different scanning angle resolution has been calculated. For each iteration of each scenario, the direction of arrival is randomly generated between -80° and 80° . The angle of arrival of the generated signal is not an integer, and calculations are

performed up to the fourth decimal place. Besides, the noise is assumed to be Additive White Gaussian noise for all these experiments. The simulation parameters are provided in Table 1.

Table 1. Simulation parameters for single source experiments (Tek kaynaklı deneyler için simülasyon parametreleri)

Parameters	Values
Number of sources (non-coherent)	1
Number of sources (coherent)	0
Direction of arrival (azimuth)	Randomly between -80° and 80° . (e.g., $-73,6139^\circ$)
Direction of arrival (elevation)	Fixed at $90,0000^\circ$
Noise type	Gaussian

4.1.1. Number of Array Elements (Dizi Elemanlarının Sayısı)

Simulations were performed for thirteen different numbers of array elements (M) from 8 to 32 using the parameters given in Table 2 to examine the effect of SNR on direction-finding performance.

Table 2. Simulation parameters for different number of array elements (M) (Farklı sayıda dizi elemanı (M) için simülasyon parametreleri)

Parameters	Fig 2a	Fig 2b	Fig 2c
Number of array elements (M)	8:2:32	8:2:32	8:2:32
Spacing between array elements (d)	$\lambda/10$, $\lambda/2$	$\lambda/10$	$\lambda/10$
Number of snapshots (N)	100	10, 1000	100
Signal to noise ratio (SNR)	-15 dB	-15 dB	-15 dB, 25 dB
Number of iterations per M	1000	1000	1000

Based on these simulations, it can be observed that an increase in number of antennas M reduces the RMSE for all algorithms. In all graphs of Figure 2; Traditional MUSIC, FBSS MUSIC, and Improved MUSIC algorithms provide nearly identical results as expected since there is no second coherent/non-coherent source. The ROOT MUSIC algorithm yields the best results among the algorithms. As can be seen in Figure 2(a), as expected, lower RMSE values are obtained with a half-wavelength ($\lambda/2$) antenna spacing (d), compared to a one-tenth wavelength ($\lambda/10$) antenna spacing. Besides, using

more snapshots (N), and higher SNR levels results in lower RMSE values as can be seen in Figure 2(b) and Figure 2(c), as expected.

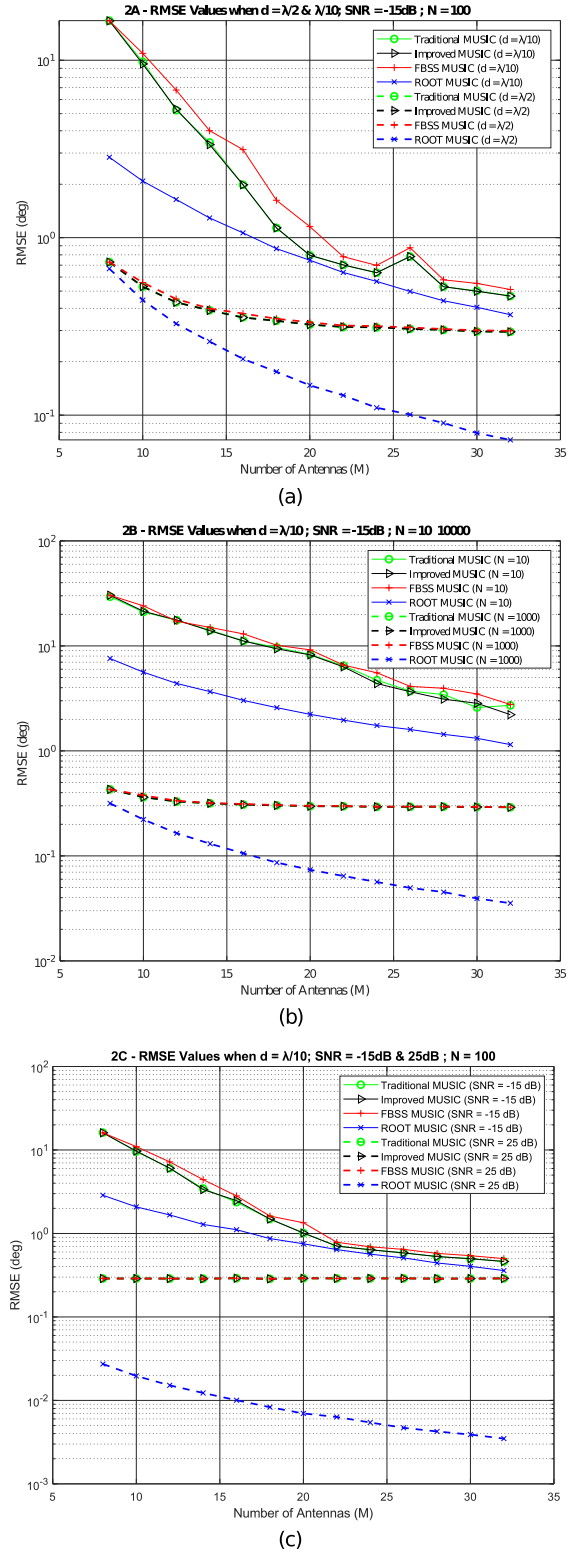


Figure 2. RMSE vs. number of array elements (M) graphs for a) 2 different spacing between array elements (d), b) 2 different number of snapshot (N), c) 2 different SNR levels (a) Dizi elemanları arasında 2 farklı uzunluk (d), b) 2 farklı örnek sayısı (N), c) 2 farklı SNR seviyesi için RMSE ve dizi elemanları sayısı (M) grafikleri)

The number of array elements plays a critical role in determining the resolution and accuracy of MUSIC-based DoA estimation algorithms. Increasing the number of elements in a Uniform Linear Array enhances the array aperture, thereby improving angular resolution and enabling the detection of closely spaced sources. For all MUSIC variants studied, performance generally improves with more array elements, as the signal and noise subspaces become more distinguishable, leading to sharper spatial spectra and reduced estimation errors. Notably, algorithms like Root MUSIC and FBSS MUSIC particularly benefit from a higher number of elements, with FBSS MUSIC leveraging the increased spatial diversity to mitigate the effects of signal coherence. However, this performance gain comes at the cost of increased computational complexity, particularly for methods involving matrix eigendecomposition and root solving.

4.1.2. Spacing Between Array Elements (Dizi Elemanları Arasındaki Boşluk)

Simulations were performed for ten different spacing values between array elements (d), from $(\lambda/10)$ to $(\lambda/2)$ using the parameters given in Table 3 to examine the effect of SNR on DoA performance.

Table 3. Simulation parameters for different space values between array elements (d) (Farklı sayıda antenler arası uzaklık (d) için simülasyon parametreleri)

Parameters	Fig 3a	Fig 3b	Fig 3c
Number of array elements (M)	8, 24	8	8
Spacing between array elements (d)	λ/k ; $k = 2:1:10$	λ/k ; $k = 2:1:10$	λ/k ; $k = 2:1:10$
Number of snapshots (N)	100	10, 1000	100
Signal to noise ratio (SNR)	-15 dB	-15 dB	-15 dB, 25 dB
Number of iterations per d	1000	1000	1000

Based on these simulations, it can be observed that an increase in spacing between array elements (d) reduces the RMSE for all algorithms. In all graphs of Figure 3; Traditional MUSIC, FBSS MUSIC, and Improved MUSIC algorithms provide nearly identical results. The ROOT MUSIC algorithm yields the best results among the algorithms. Besides, as can be seen in all graphs of Figure 3, using more antennas (M) or using more snapshots (N) or higher SNR levels result in lower RMSE values as expected.

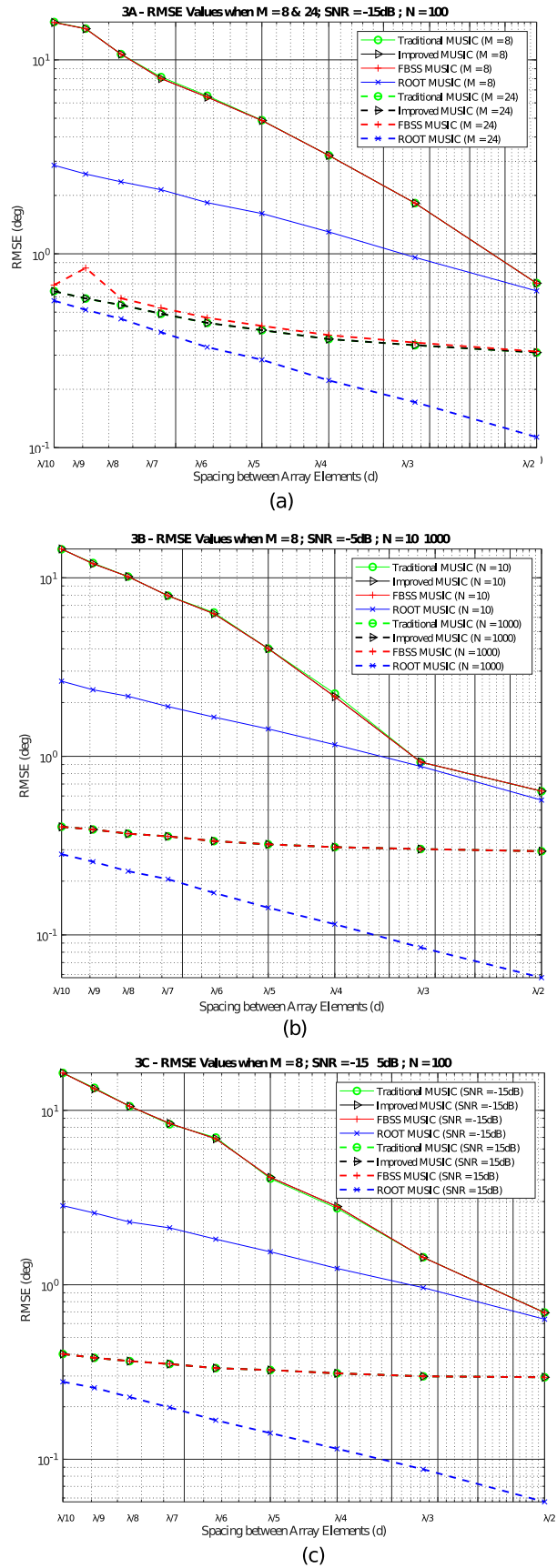


Figure 3. RMSE vs. spacing (d) graphs for a) 2 different number of array elements (M), b) 2 different number of snapshot (N), c) 2 different SNR levels (a) 2 farklı sayıda dizi elemanı (M), b) 2 farklı sayıda örnek sayısı (N), c) 2 farklı SNR seviyesi için RMSE ve antenler arası uzaklık (d) grafikleri)

The spacing between array elements significantly influences the performance of MUSIC and its variants in DoA estimation. Ideally, the inter-element spacing should be set to half the signal wavelength ($\lambda/2$) to prevent spatial aliasing and ensure unambiguous angle detection. If the spacing exceeds $\lambda/2$, grating lobes may appear in the spatial spectrum, leading to incorrect DoA estimates, especially in Traditional and Root MUSIC. Conversely, spacing less than $\lambda/2$ results in reduced array aperture, which degrades angular resolution and limits the ability to distinguish closely spaced sources. Across all MUSIC variants, optimal spacing helps maintain a well-conditioned array manifold and improves subspace separation, thereby enhancing estimation accuracy. Variants like FBSS MUSIC are particularly sensitive to element spacing, as their smoothing operations assume a uniform and alias-free array geometry.

4.1.3. Number of Snapshots (Anlık Örnek Sayısı)

Simulations were performed for ten different numbers of snapshots from (N) 10 to 10,000 using the parameters given in Table 4 to examine the effect of SNR on direction-finding performance.

Table 4. Simulation parameters for different number of snapshots (N) (Farklı sayıda Örnek Sayısı (N) için simülasyon parametreleri)

Parameters	Fig 4a	Fig 4b	Fig 4c
Number of array elements (M)	8, 24	8	8
Spacing between array elements (d)	$\lambda/10$	$\lambda/10, \lambda/2$	$\lambda/10$
Number of snapshots (N)	10:10000	10:10000	10:10000
Signal to noise ratio (SNR)	-15 dB	-15 dB	-15 dB, 25 dB
Number of iterations per d	1000	1000	1000

Based on these simulations, it can be observed that an increase in number of snapshots (N) reduces the RMSE for all algorithms. In all graphs of Figure 4; Traditional MUSIC, FBSS MUSIC, and Improved MUSIC algorithms provide nearly identical results. The ROOT MUSIC algorithm yields the best results among the algorithms. As can be seen in Figure 4(b), as expected, lower RMSE values are obtained with a half-wavelength ($\lambda/2$) antenna spacing (d) compared to a one-tenth wavelength ($\lambda/10$) antenna spacing. Besides, using more array

elements (M) or higher SNR levels results in lower RMSE values as can be seen in Figure 4(a) and Figure 4(c), as expected.

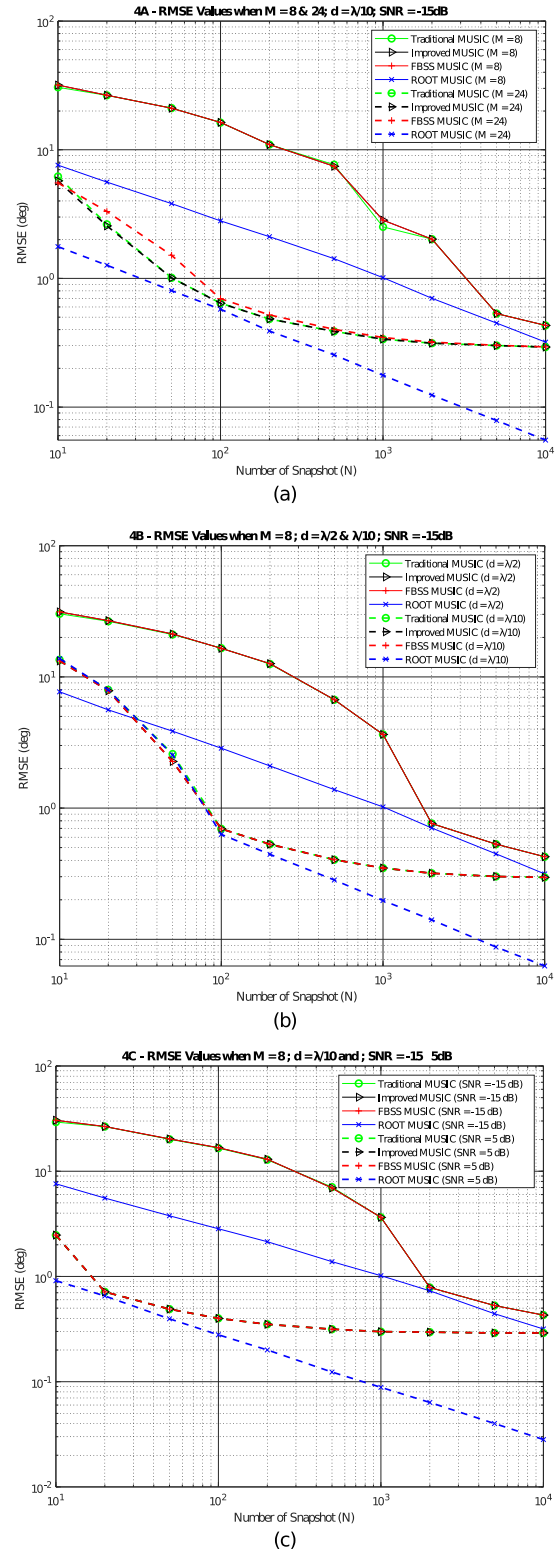


Figure 4. RMSE vs. number of snapshot (N) graphs for a) 2 different number of array elements (M), b) 2 different spacing between array elements (d), c) 2 different SNR levels (a) 2 farklı dizi elemanı sayısı (M), b) dizi elemanları arasında 2 farklı uzunluk (d), c) 2 farklı SNR seviyesi için RMSE ve anlık örnek sayısı (N) grafikleri)

The number of snapshots (N), representing the temporal samples collected during signal observation, directly affects the estimation accuracy of MUSIC-based algorithms. A higher number of snapshots leads to a more accurate estimation of the sample covariance matrix, resulting in improved separation between the signal and noise subspaces. This, in turn, enhances the resolution and robustness of all MUSIC variants, particularly in low SNR environments. Traditional MUSIC and Root MUSIC show noticeable improvements in peak sharpness and stability with increased snapshots, while FBSS MUSIC benefits from more reliable smoothing operations. However, in practical scenarios where real-time processing or limited observation time is required, the number of snapshots may be constrained, which can degrade performance and increase sensitivity to noise and source correlation.

4.1.4. Signal to Noise Ratio (Sinyal Gürültü Oranı)

Simulations were performed for eleven different levels of SNR from -20 dB to 30 dB using the parameters given in Table 5 to examine the effect of SNR on direction-finding performance.

Table 5. Simulation parameters for different SNR Levels (Farklı Sinyal Gürültü Oranı için simülasyon parametreleri)

Parameters	Fig 5a	Fig 5b	Fig 5c
Number of array elements (M)	8, 24	8	8
Spacing between array elements (d)	$\lambda/10$	$\lambda/10, \lambda/2$	$\lambda/10$
Number of snapshots (N)	100	100	10, 1000
Signal to noise ratio (SNR)	-20:5:30 dB	-20:5:30 dB	-20:5:30 dB
Number of iterations per d	1000	1000	1000

Based on these simulations, it can be observed that an increase in SNR level reduces the RMSE for all algorithms. In all graphs of Figure 5; Traditional MUSIC, FBSS MUSIC, and Improved MUSIC algorithms provide nearly identical results. The ROOT MUSIC algorithm yields the best results among the algorithms. As can be seen in Figure 5(b), as expected, lower RMSE values are obtained with a half-wavelength ($\lambda/2$) antenna spacing (d) compared to a one-tenth wavelength ($\lambda/10$) antenna spacing. Besides, using more array

elements (M) or using more snapshots (N) results in lower RMSE values as can be seen in Figure 5(a) and Figure 5(c), as expected.

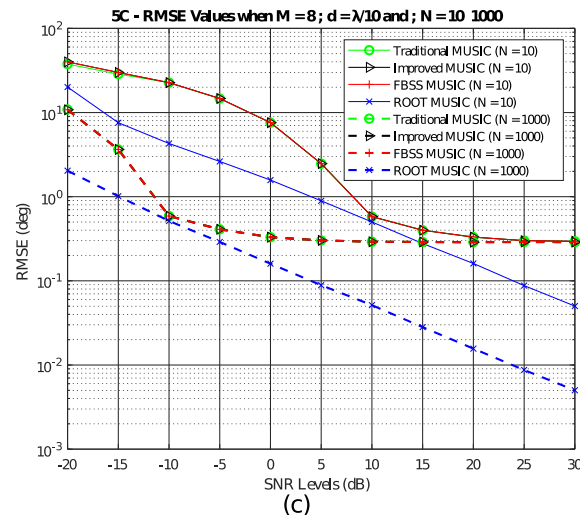
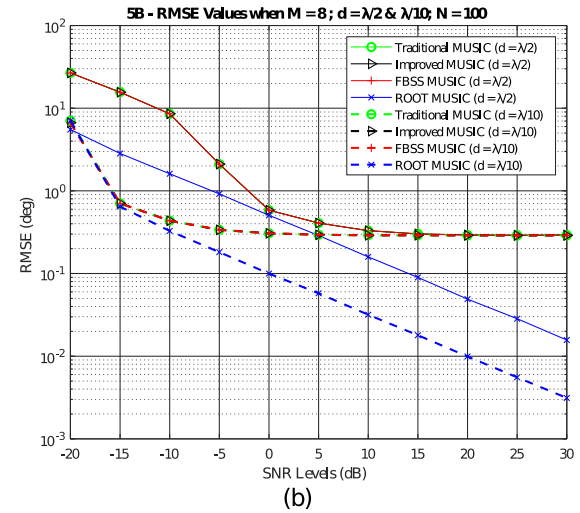
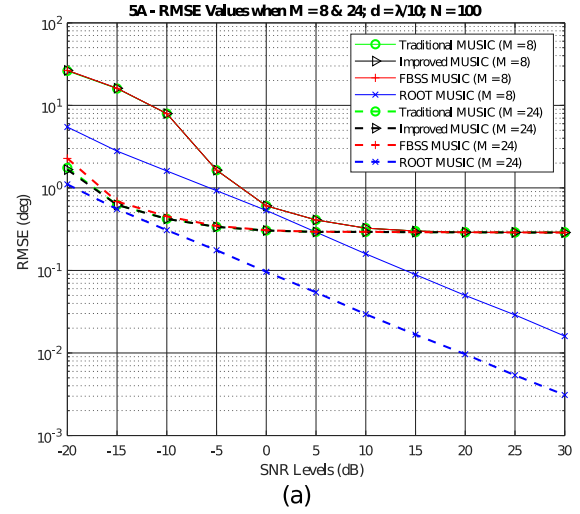


Figure 5. RMSE vs. SNR graphs for a) 2 different number of array elements (M), b) 2 different spacing between array elements (d), c) 2 different number of snapshot (N) (a) 2 farklı sayıda dizi elemanı (M), b) dizi elemanları arasında 2 farklı uzunluk (d), c) 2 farklı sayıda anlık örnek (N) için RMSE ve SNR grafikleri)

Signal-to-Noise Ratio (SNR) is a critical factor influencing the accuracy and reliability of DoA estimation in MUSIC-based algorithms. At higher SNR levels, the distinction between the signal and noise subspaces becomes clearer, resulting in well-defined spectral peaks and more accurate angle estimation across all MUSIC variants. Root MUSIC exhibits strong performance in high-SNR conditions. However, all variants' performance deteriorates significantly as SNR decreases, leading to blurred or misplaced peaks. Extremely low SNR conditions remain challenging for all variants, underscoring the need for careful algorithm selection based on the expected noise environment.

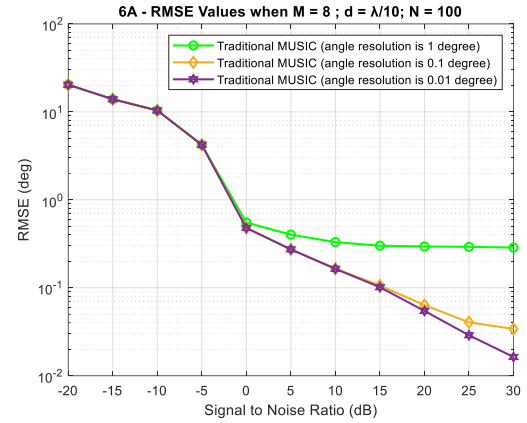
4.1.5. Scanning Angle / Angle Resolution (Tarama Açısı / Açık Çözünürlüğü)

Simulations were conducted for three different resolutions from 1° to 0.01° using the parameters given in the Table 6 to examine the effect of scan angle resolution on direction-finding performance.

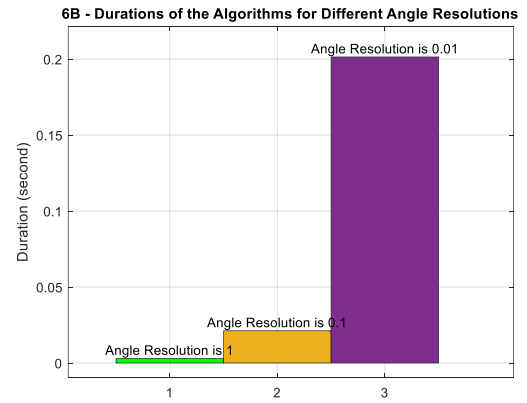
Table 6. Simulation parameters for different angle resolutions (Farklı açı çözünürlükleri için simülasyon parametreleri)

Parameters	Fig 6a
Number of array elements (M)	8
Spacing between array elements (d)	$\lambda/10$
Number of snapshots (N)	100
Signal to noise ratio (SNR)	-20:5:30 dB
Scanning angle resolution	$-90^\circ:1^\circ:90^\circ$ $-90^\circ:0.1^\circ:90^\circ$ $-90^\circ:0.01^\circ:90^\circ$
Number of iterations per N	1000

As can be understood from the simulation results in Figure 6(a), the error decreases as the SNR level increases but never reaches zero. This is because the required angle resolution to determine the exact arrival angle of the signal is extremely small. As seen in the graph, reducing the scan angle resolution leads to lower RMSE values at high SNR levels, but it produces similar results at low SNR levels. Additionally, reducing the angle resolution by a factor of 10 increases the number of scanned angles by the same factor, which proportionally extends the algorithm's runtime as seen in Figure 6(b). Therefore, a trade-off exists in this scenario. In a real system, this trade-off should be optimized appropriately.



(a)



(b)

Figure 6. a) RMSE vs. SNR graph and b) Durations of the algorithms for 3 different scanning angle resolutions (3 farklı tarama açısı çözünürlüğü için a) RMSE ve SNR grafiği ve b) algoritmaların süreleri)

The scanning angle resolution, which defines the angular step size used when evaluating the MUSIC spatial spectrum, significantly impacts the performance and accuracy of the Traditional MUSIC algorithm. Finer resolution allows for more precise localization of the spectral peaks corresponding to the Direction of Arrival (DoA), leading to improved estimation accuracy—especially when sources are closely spaced. However, this comes at the cost of increased computational burden, as more angle points must be evaluated during the search. Conversely, a coarser resolution reduces computational load but may result in peak broadening or missed detections, particularly under challenging conditions such as low SNR or limited snapshots. Therefore, selecting an appropriate scanning resolution involves a trade-off between accuracy and processing efficiency, and should be guided by the application's performance requirements and available computational resources.

4.2. Dual Source Non-Coherent Incident Signal (Çift Kaynaktan Gelen Eşfazlı Olmayan Sinyaller)

The results of dual non-coherent source direction of arrival estimation include 'RMSE-Number of Snapshot' and 'RMSE-SNR' graphs obtained via the Monte Carlo method. For each iteration of each scenario, the direction of arrival of first source is randomly generated between -80° and 80° . The angle of the second source is plus/minus 20° of first source. The angles of the generated signals are not integers, and calculations are performed up to the fourth decimal place. Besides, the noise is assumed to be Additive White Gaussian noise for all these experiments. The simulation parameters are provided in Table 7.

Table 7. Simulation parameters for dual non-coherent sources experiments (Çift kaynak eşfazlı olmayan sinyallerle yapılan deneyler için simülasyon parametreleri)

Parameters	Values
Number of sources (non-coherent)	2
Number of sources (coherent)	0
Direction of arrival (azimuth)	Randomly between -80° and 80° . (e.g., $-73,6139^\circ, -53,6139^\circ$)
Direction of arrival (elevation)	Fixed at $90,0000^\circ$
Noise type	Gaussian

Based on these simulations, it can be observed that an increase in number of snapshots (N) and an increase in SNR reduce the RMSE for all algorithms. In all graphs of Figure 7 and Figure 8; Traditional MUSIC, FBSS MUSIC, and Improved MUSIC algorithms provide nearly identical results. The ROOT MUSIC algorithm yields the best results among the algorithms. As can be seen in Figure 7(b), as expected, lower RMSE values are obtained with a half-wavelength ($\lambda/2$) antenna spacing (d) compared to a one-tenth wavelength ($\lambda/10$) antenna spacing. Besides, using more array elements (M) or higher SNR levels results in lower RMSE values as can be seen in Figure 7(a) and Figure 7(c), as expected.

As can be seen in Figure 8(b), as expected, lower RMSE values are obtained with a half-wavelength ($\lambda/2$) antenna spacing (d) compared to a one-tenth wavelength ($\lambda/10$) antenna spacing. Besides, using more array elements (M) or using more snapshots (N) results in lower RMSE values as can be seen in Figure 8(a) and Figure 8(c), as expected.

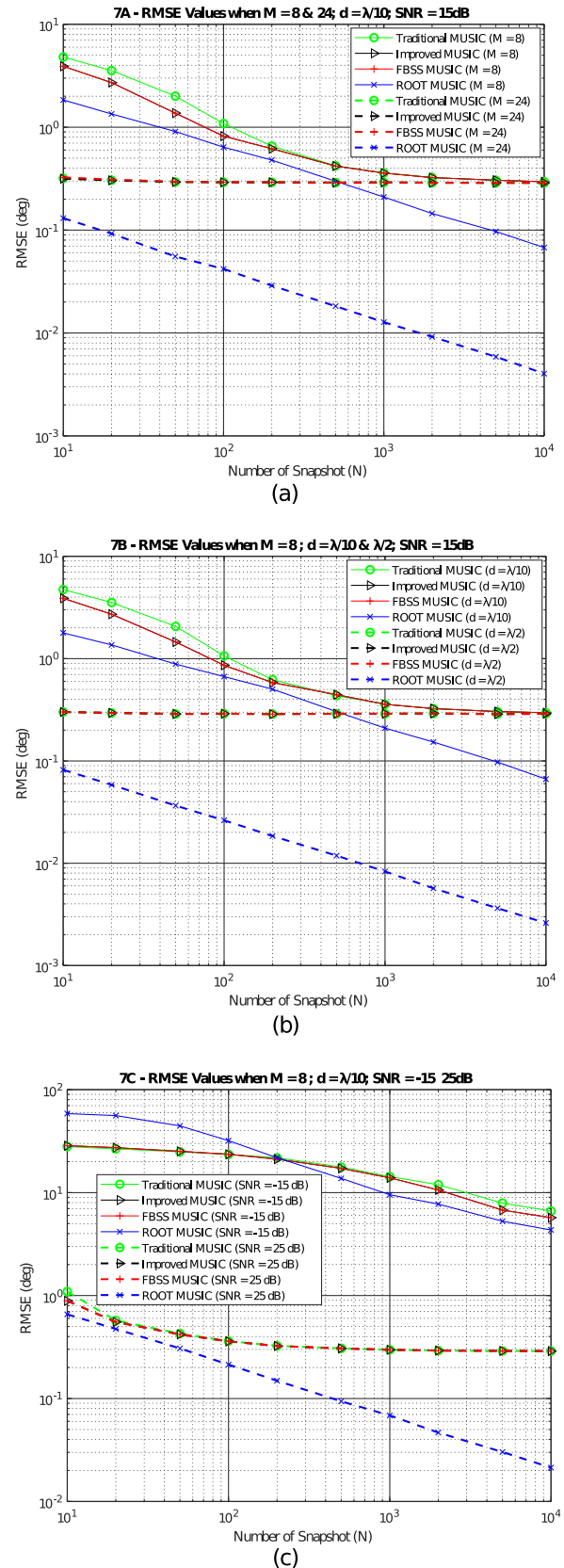


Figure 7. RMSE vs. number of snapshot (N) graphs for a) 2 different number of array elements (M), b) 2 different spacing between array elements (d), c) 2 different SNR levels (a) 2 farklı dizi elemanı sayısı (M), b) dizi elemanları arasında 2 farklı uzunluk (d), c) 2 farklı SNR seviyesi için RMSE ve anlık örnek sayısı (N) grafikleri)

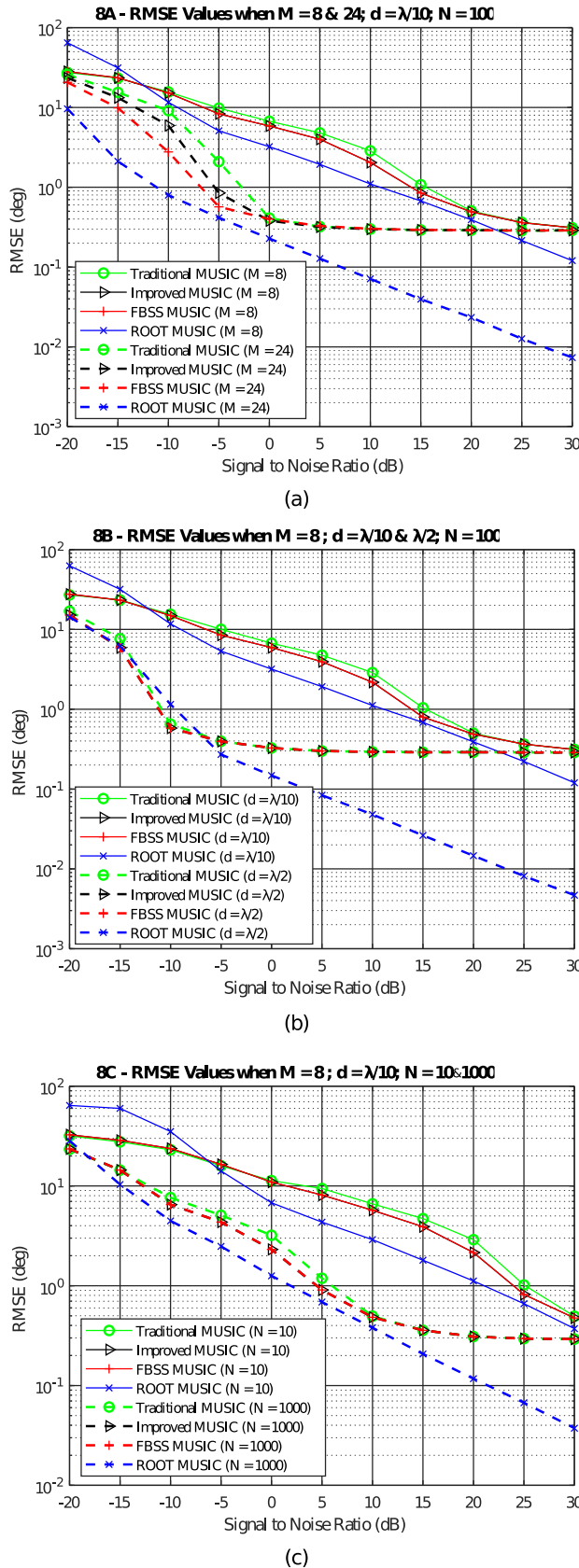


Figure 8. RMSE vs. SNR graphs for a) 2 different number of array elements (M), b) 2 different spacing between array elements (d), c) 2 different number of snapshot (N) (a) 2 farklı sayıda dizi elemanı (M), b) dizi elemanları arasında 2 farklı uzunluk (d), c) 2 farklı sayıda anlık örnek (N) için RMSE ve SNR grafikleri)

4.3. Dual Source Coherent Incident Signal

(Çift Kaynaktan Gelen Eşfazlı Sinyaller)

The results of dual coherent source direction of arrival estimation include 'RMSE-Number of Snapshot' and 'RMSE-SNR' graphs obtained via the Monte Carlo method. For each iteration of each scenario, the direction of arrival of first source is randomly generated between -80° and 80° . The angle of the second (coherent) source is plus/minus 20° of first source. The angles of the generated signals are not integers, and calculations are performed up to the fourth decimal place. Besides, the noise is assumed to be Additive White Gaussian noise for all these experiments. The simulation parameters are provided in Table 8.

Table 8. Simulation parameters for dual coherent sources experiments (Çift kaynak eşfazlı sinyallerle yapılan deneyler için simülasyon parametreleri)

Parameters	Values
Number of sources (non-coherent)	1
Number of sources (coherent)	1
Direction of arrival (azimuth)	Randomly between -80° and 80° . (e.g., $-73,6139^\circ, -53,6139^\circ$)
Direction of arrival (elevation)	Fixed at $90,0000^\circ$
Noise type	Gaussian

It can be observed that Traditional MUSIC and ROOT MUSIC algorithms cannot work for all SNR and Number of Snapshot values as expected because of the coherent signal. Besides, it is seen that an increase in number of snapshots (N) and an increase in SNR reduce the RMSE for FBSS MUSIC and Improved/Modified MUSIC algorithms. In all graphs of Figure 9 and Figure 10 FBSS MUSIC, and Improved MUSIC algorithms provide nearly identical results. As can be seen in Figure 9(b), as expected, lower RMSE values are obtained with a half-wavelength ($\lambda/2$) antenna spacing (d) compared to a one-tenth wavelength ($\lambda/10$) antenna spacing. Besides, using more array elements (M) or higher SNR levels results in lower RMSE values for FBSS MUSIC and Improved/Modified MUSIC algorithms as can be seen in Figure 9(a) and Figure 9(c), as expected. In Figure 10(b), as expected for FBSS MUSIC and Improved/Modified MUSIC algorithms, lower RMSE values are obtained with a half-wavelength ($\lambda/2$) antenna spacing (d) compared to a one-tenth wavelength ($\lambda/10$) antenna spacing. Besides, using more array elements (M) or using more snapshots (N) results in lower RMSE values as can be seen in Figure 10(a) and Figure 10(c), as expected.

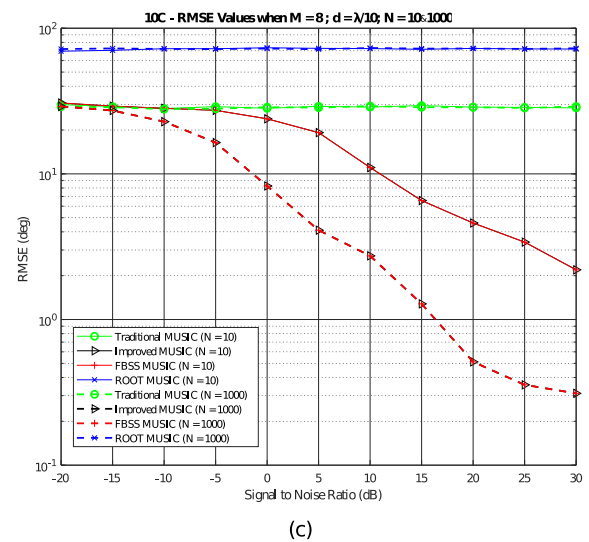
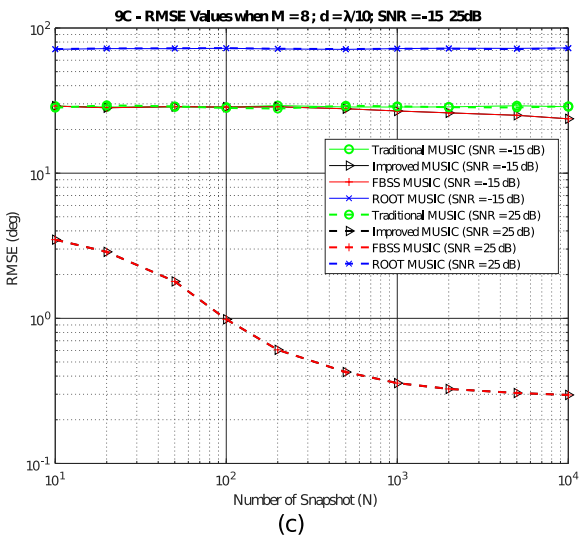
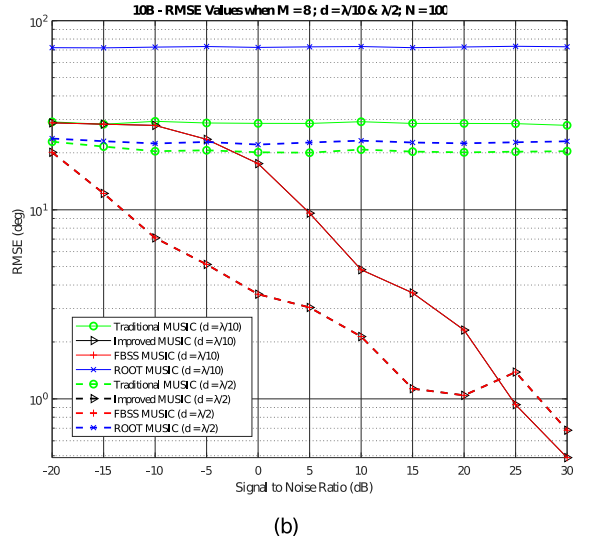
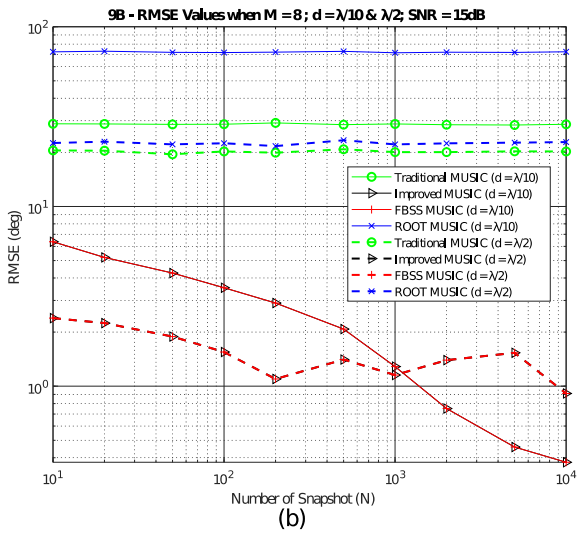
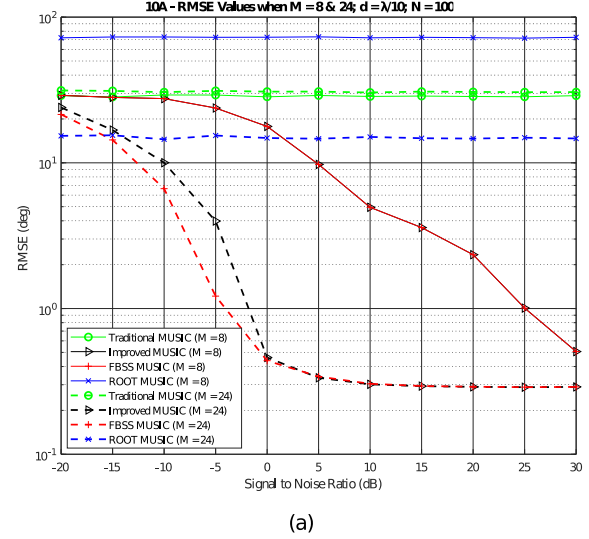
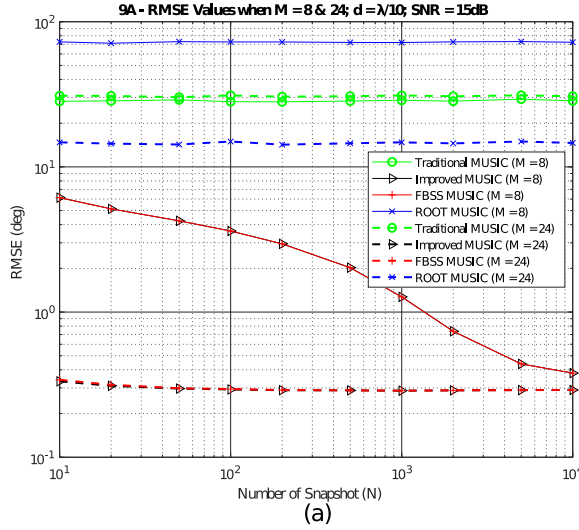


Figure 9. RMSE vs. number of snapshot (N) graphs for a) 2 different number of array elements (M), b) 2 different spacing between array elements (d), c) 2 different SNR levels (a) 2 farklı dizi elemanı sayısı (M), b) dizi elemanları arasında 2 farklı uzunluk (d), c) 2 farklı SNR seviyesi için RMSE ve anlık örnek sayısı (N) grafikleri)

Figure 10. RMSE vs. SNR graphs for a) 2 different number of array elements (M), b) 2 different spacing between array elements (d), c) 2 different number of snapshot (N) (a) 2 farklı sayıda dizi elemanı (M), b) dizi elemanları arasında 2 farklı uzunluk (d), c) 2 farklı sayıda anlık örnek (N) için RMSE ve SNR grafikleri)

4.4. Duration of the Algorithms (Algoritma Süreleri)

As can be seen from the Figure 11, the ROOT MUSIC algorithm is the fastest-performing algorithm. The second fastest algorithm is FBSS MUSIC. Improved MUSIC and Traditional MUSIC algorithms are the slowest-performing algorithms, and they have nearly identical execution times.

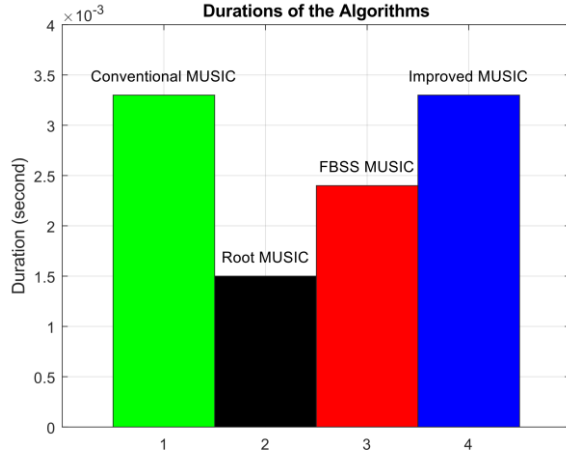


Figure 11. Durations of all MUSIC algorithms (Tüm MUSIC algoritmalarının süreleri)

5. CONCLUSIONS AND FUTURE WORK (SONUÇLAR VE GELECEK ÇALIŞMALAR)

This paper has provided a comprehensive investigation of various Multiple Signal Classification (MUSIC) algorithms for Direction of Arrival (DoA) estimation, including Traditional MUSIC, Root MUSIC, Forward/Backward Spatial Smoothing (FBSS) MUSIC, and Improved/Modified MUSIC. By elucidating the underlying principles of each algorithm and examining the factors that influence their performance—such as the number of array elements, antenna spacing, number of snapshots, Signal-to-Noise Ratio (SNR), and scanning angle resolution—this study has highlighted the critical parameters that impact the accuracy and efficiency of DoA estimation. Through extensive MATLAB simulations utilizing the Monte Carlo method, the performance of these algorithms was evaluated and compared under scenarios involving coherent, non-coherent, and single/dual-source(s) incident signals. The Root Mean Square Error (RMSE) was employed as a metric to quantify estimation accuracy, while computational efficiency was assessed through comparisons of execution times and variances.

The results demonstrate that each MUSIC variant has distinct strengths and limitations, making them suitable for different practical applications. Traditional MUSIC serves as a robust baseline,

while Root MUSIC offers improved computational efficiency and accuracy for uniform linear arrays. FBSS MUSIC effectively addresses the challenges posed by coherent signals, and Improved/Modified MUSIC provides enhanced performance in complex environments through advanced preprocessing and noise reduction techniques. These findings offer valuable guidance for selecting the most appropriate MUSIC algorithm based on specific application requirements, such as array configuration, signal environment, and computational constraints.

Future research could explore the multiple sources, including multiple coherent and non-coherent signals with different amplitudes from different angles. Besides, the integration of these algorithms with emerging technologies, such as deep learning or compressed sensing, to further enhance their performance in challenging scenarios can be a new area of exploration. Additionally, the development of hybrid approaches that combine the strengths of multiple MUSIC variants could provide a pathway for achieving even greater accuracy and efficiency in DoA estimation tasks. Overall, this study contributes to the ongoing advancement of array signal processing by providing a detailed analysis of MUSIC algorithms and their practical implications for real-world applications.

DECLARATION OF ETHICAL STANDARDS (ETİK STANDARTLARIN BEYANI)

The author of this article declares that the materials and methods they use in their work do not require ethical committee approval and/or legal-specific permission.

Bu makalenin yazarı çalışmalarında kullandıkları materyal ve yöntemlerin etik kurul izni ve/veya yasal-özel bir izin gerektirmediğini beyan ederler.

AUTHORS' CONTRIBUTIONS (YAZARLARIN KATKILARI)

Emrah ONAT: He conducted the experiments, analyzed the results and performed the writing process.

Deneyleri yapmış, sonuçlarını analiz etmiş ve maklenin yazım işlemini gerçekleştirmiştir.

CONFLICT OF INTEREST (ÇIKAR ÇATIŞMASI)

There is no conflict of interest in this study.

Bu çalışmada herhangi bir çıkar çatışması yoktur.

CODE (KOD)

www.github.com/emrahonat/MUSICs

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