European Journal of Technique

journal homepage: https://dergipark.org.tr/en/pub/ejt

Vol.15, No.2, 2025

Research Article

Basin Structure and Unpredictability in a Two-Dimensional Memristor-Based Cubic Map

Serpil Yilmaz Kutluay

İzmir Katip Çelebi University, Faculty of Engineering and Architecture, Department of Computer Engineering, İzmir, Turkey.
(e-mail: serpil.yilmaz@ikcu.edu.tr).

ARTICLE INFO

Received: May, 09. 2025
Revised: Aug., 19. 2025
Accepted: Sep., 04. 2025

Keywords:
Basins of attraction
Basin Entropy
Memristive systems
Multistability

Corresponding author: *Serpil Yılmaz Kutluay*

ISSN: *2536-5010* | e-ISSN: *2536-5134*

DOI: https://doi.org/10.36222/ejt.1696324

ABSTRACT

This study investigates the dynamics of a two-dimensional memristor-based cubic map, with emphasis on the structure of its basins of attraction and the unpredictability of its long-term behavior. A recurrence-based automated method is employed to identify attractors without prior knowledge of their locations, which enables a comprehensive analysis of multistability. The resulting basin structures exhibit fractal boundaries and regions of divergence, reflecting a high sensitivity to initial conditions. To quantify the complexity of the attractor basins, basin entropy values are evaluated across various parameter sets as a measure of the system's unpredictability. The results show regions of high basin entropy, which highlight the emergence of intricate fractal-like basin boundaries and robust chaotic behavior. These findings suggest that memristive elements can enhance complexity and unpredictability in discrete dynamical systems, with potential applications in the design of secure and resilient digital systems that exploit chaotic dynamics.

1. INTRODUCTION

In recent years, there has been growing interest in incorporating memristors into the study and design of chaotic systems. The memristor, short for "memory resistor", was postulated in [1] as the fourth fundamental circuit element. It is characterized by a nonlinear resistance that depends on the integral of current or voltage over time, leading to memory-dependent behavior. This property allows the memristor to retain information about past electrical states. Its intrinsic memory effect distinguishes it from traditional passive elements and introduces internal state dynamics and nonlinearity, which make it highly effective for modeling and constructing complex dynamical systems. Due to these features, memristors have attracted significant attention in areas such as secure communications, neural computing, and signal processing. Memristive systems are particularly valuable for tasks requiring complexity and unpredictability, and they can operate near the edge of chaos. This is a regime where systems exhibit high computational power and strong sensitivity to initial conditions [2].

Extensive research has focused on integrating memristors into discrete chaotic systems. A memristive chaotic map with amplitude control and bifurcation characteristics was proposed in [3]. A three-dimensional discrete memristive chaotic system with an infinitely wide parameter range was developed in [4], demonstrating complex dynamics and strong sensitivity to initial states. A class of discrete memristor-based maps was introduced by perturbing the sine map internally in [5], thereby expanding the chaos range and enhancing ergodicity. In [6], cosine-based memristors were incorporated into the Logistic and Hénon maps, revealing richer dynamics through Lyapunov exponents, bifurcation diagrams, and complexity metrics.

Further developments in memristive systems include the integration of multistable discrete memristors into the FitzHugh–Nagumo neuron model, which enabled the observation of coexisting hidden attractors with potential implications for neural networks [7]. An improved memristive Hénon map was proposed using state-variable differences and validated through analog circuit implementations [8]. In another study, a discrete second-order memristor coupled with a sine map was designed to construct a three-dimensional chaotic system with enhanced dynamical properties [9]. More recently, a novel method to improve attractor stability in discrete systems was proposed in [10] by employing a memristive function.

The two-dimensional memristive cubic map (2D-MCM), presented in [11], has been shown to exhibit rich dynamical behaviors with improved performance in video encryption.

Similarly, the memristive Gaussian Map (MGM), characterized by hyperchaos and coexisting attractors, was proposed as a chaotic system for image encryption in [12]. A range of coupling strategies for memristor-based maps has been investigated, revealing diverse complex behaviors suitable for cryptographic applications [13]. A generalized two-dimensional memristive model with enhanced complexity and hyperchaotic dynamics was introduced in [14]. Memristor-based chaotic and hyperchaotic systems have also been extensively investigated for secure communication in [15–18], while their use in achieving low-power consumption for logic circuits has been demonstrated in [19]. Furthermore, extensions into the fractional-order domain have been implemented on FPGAs for image encryption in [20]. More recently, a dual-memristor-based discrete chaotic system, termed the three-dimensional memristive cubic map (3D-MCM), was introduced in [21], representing a significant advancement in the design of memristive chaotic systems for cryptographic applications.

These developments consistently demonstrate that embedding memristors into discrete chaotic systems enhances the chaotic range, increases system complexity, and highlights the potential of memristor-based chaotic maps for applications ranging from secure communications and image encryption to high-performance pseudorandom number generation.

One of the central challenges in the study of nonlinear dynamical systems is the analysis of unpredictability, particularly in multistable regimes where multiple attractors coexist. In such systems, small variations in initial conditions can lead to significantly different long-term behaviors. A key concept in understanding these dynamics is the basin of attraction, which refers to the region in phase space associated with a particular attractor. A basin of attraction contains the set of initial conditions that converge to a specific long-term behavior, such as a fixed point, periodic orbit, or chaotic trajectory. These regions provide valuable insights into system dynamics, especially in multistable regimes.

Recent efforts have focused on automating basin estimation. A fully automated method based on finite state machines was proposed in [22], enabling attractor and basin identification without prior assumptions or approximations. Additionally, basin entropy, introduced in [23], provides a global and quantitative measure of unpredictability. By partitioning the phase space into grid cells and computing the Shannon entropy of final-state distributions, basin entropy characterizes the uncertainty of long-term behavior, with higher values corresponding to more complex and unpredictable basin geometries.

This study investigates the two-dimensional memristor-based cubic map introduced in [11], with particular emphasis on the unpredictability of its basins of attraction. The automated method from [22] is employed for attractor and basin identification, and basin entropy is calculated to quantify unpredictability and assess basin complexity. The objective is to analyze how memristor-induced dynamics reshape the phase space, focusing on the structure and intricacy of basin geometries. Such analysis provides insights into the influence of memory elements on basin structures and supports the design of secure and reliable nonlinear systems in digital hardware.

2. THE MEMRISTIVE CUBIC MAP MODEL

According to the definition proposed by Chua [1], a general charge-controlled memristor is characterized by the following equations:

$$\begin{cases} v(t) = M(q)i(t) \\ \dfrac{dq(t)}{dt} = f(q,i) \end{cases} \tag{1}$$

where $v(t)$ and $i(t)$ denote the voltage and current across the memristor, respectively, and q represents the internal charge state variable. The memristance $M(q)$ and the function $f(q,i)$ are specified as:

$$M(q) = \delta_1 q(t)^2 + \delta_2, \;\; f(q,i) = k_1 i(t) + k_2 q(t) \tag{2}$$

where δ_1, δ_2, k_1, and k_2 are adjustable parameters. By discretizing this continuous-time model, the general discrete memristor model can be expressed as:

$$\begin{cases} v(n) = (\delta_1 q(n)^2 + \delta_2)i(n) \\ q(n+1) = Tk_1 i(n) + (Tk_2 + 1)q(n) \end{cases} \tag{3}$$

where T denotes the discretization step size, and n is the discrete time index.

In [11], a two-dimensional discrete chaotic system, termed the 2D memristive cubic map, was proposed by coupling a memristor with the classical cubic map. The original one-dimensional cubic map is given by:

$$x_{n+1} = \mu x_n^3 + (1-\mu)x_n \tag{4}$$

where $\mu \in (0,4]$ is a control parameter that regulates the nonlinear behavior of the system.

To incorporate memory effects, the discrete memristor model introduced in (3) is integrated into the cubic map, leading to the formulation of the two-dimensional memristive cubic map:

$$\begin{cases} x_{n+1} = \alpha(\mu x_n^3 + (1-\mu)x_n) + \beta sin\big((\delta_1 y_n^2 + \delta_2)x_n\big) \\ y_{n+1} = Tk_1 x_n + (Tk_2 + 1)y_n \end{cases} \tag{5}$$

By selecting $\delta_1 = 1$, $\delta_2 = -2$, Tk_1=0.9, and $Tk_2 = 0$, as adopted in [11], the system simplifies to:

$$\begin{cases} x_{n+1} = \alpha(\mu x_n^3 + (1-\mu)x_n) + \beta sin\big((y_n^2 - 2)x_n\big) \\ y_{n+1} = 0.9x_n + y_n \end{cases} \tag{6}$$

where α, β, and μ are control parameters that shape the dynamics and complexity of the map, with $\alpha \in (0,1.5)$, $\beta \in (0,1.5)$, and $\mu \in (0,4)$.

3. NUMERICAL RESULTS

In this section, numerical simulation results are presented, including phase portraits, bifurcation diagrams, Lyapunov exponents, and basins of attraction of the two-dimensional memristor-based cubic map for various control parameter sets as defined by (6). In addition, the unpredictability of the system's final states is quantified using basin entropy.

 ISSN 2536-5010 | **e-ISSN** 2536-5134

EUROPEAN JOURNAL OF TECHNIQUE, Vol.15, No.2, 2025

The system was simulated over the time interval $t \in (0, 40000)$, with the first 10000 units treated as transient and discarded from the analysis. For the simulations presented in Figures 1–3, the initial conditions were set to $x_0 = 0.1$ and $y_0 = 0.1$.

Figure 1 illustrates the long-term behavior of the system through the state trajectories $(x,\ y)$ and reveals the structure of the attractor under the specified parameter settings.

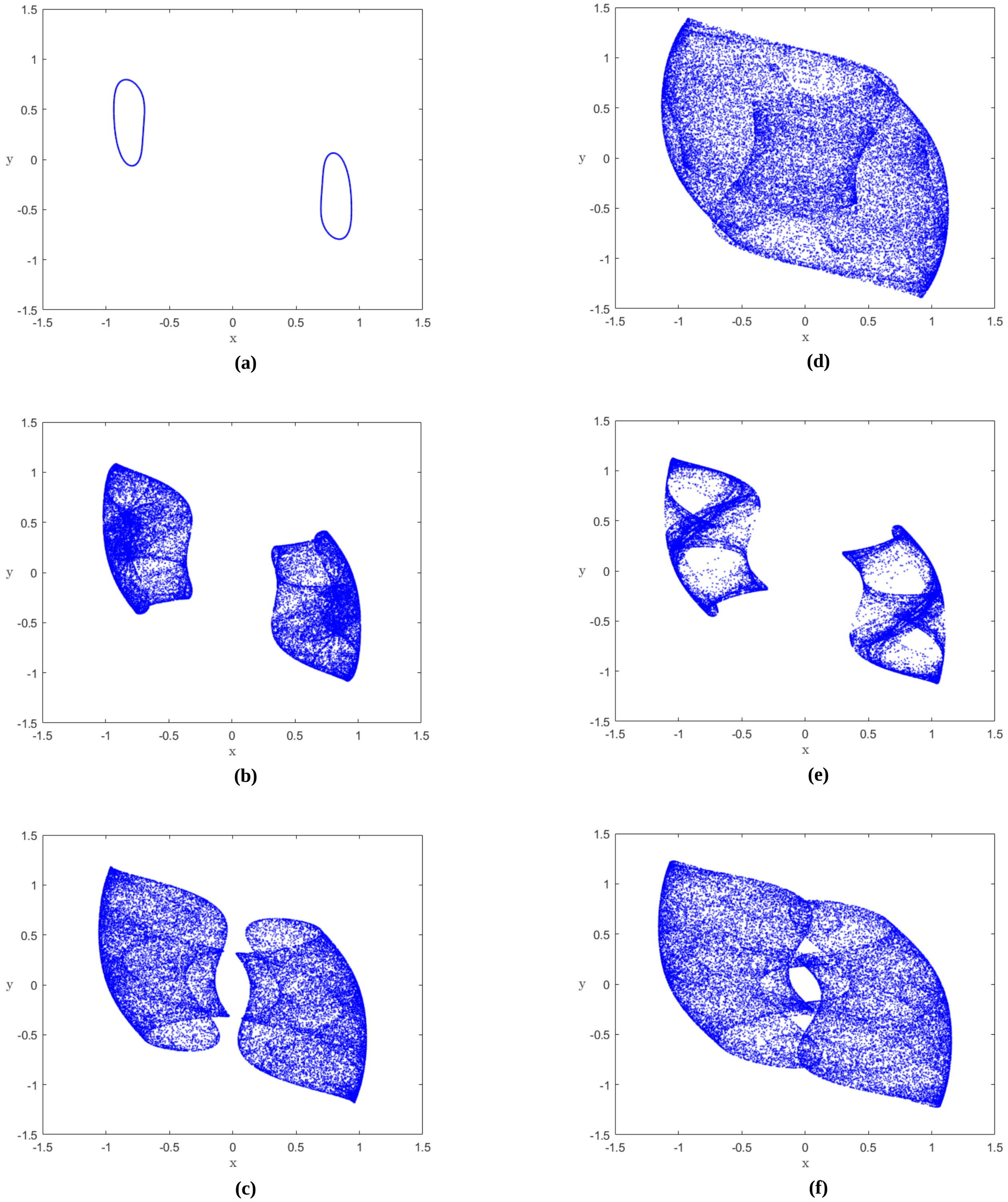

Figure 1. Phase plots of the system (6) for various combinations of parameters α, β, and μ, with fixed initial conditions $x_0 = 0.1, y_0 = 0.1$. Each subplot corresponds to a different parameter set: **(a)** α = 0.3, β = 0.9, μ = 2.5; **(b)** α = 0.5, β = 0.9, μ = 2.5; **(c)** α = 0.6, β = 0.9, μ = 2.5; **(d)** α = 0.8, β = 0.9, μ = 2.5; **(e)** α = 0.3, β = 1.0, μ = 3.0; **(f)** α = 0.4, β = 1.0, μ = 3.0.

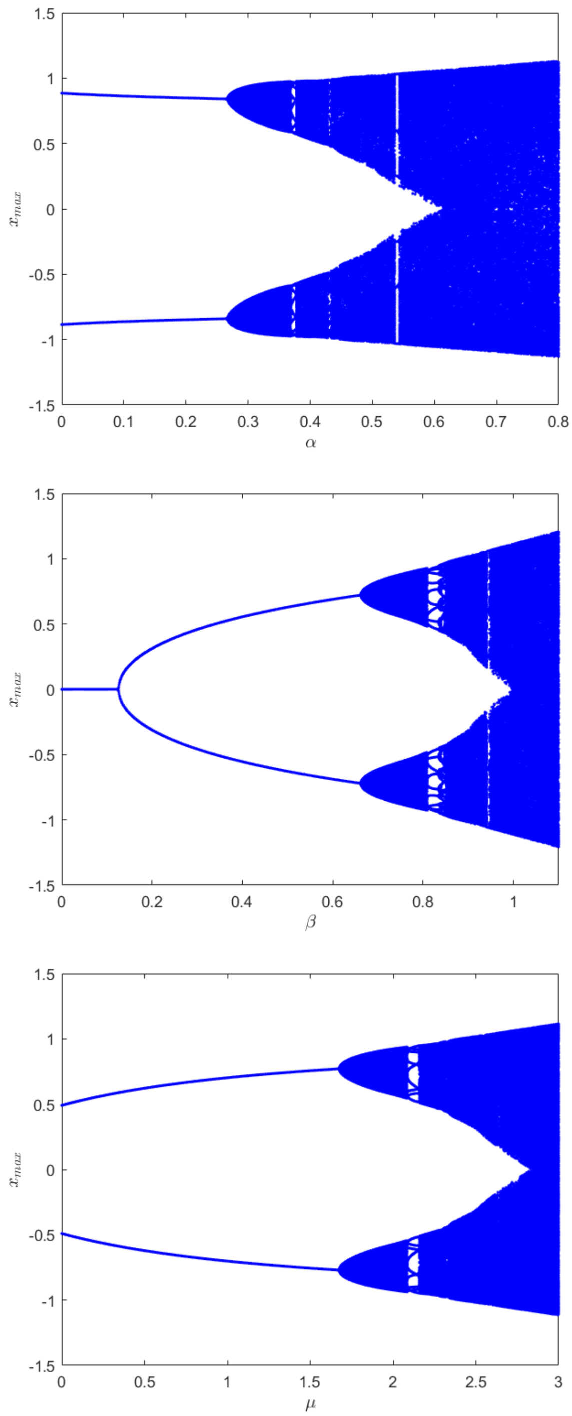

Figure 2. Bifurcation diagrams of the system (6) showing the maximum values of x_n (x_{max}) as functions of the bifurcation parameters (from top to bottom): α, β, and μ. The diagrams are generated over the intervals $\alpha \in [0, 0.8]$, $\beta \in [0, 1.1]$, and $\mu \in [0, 3]$, respectively. The remaining parameters are fixed at $\alpha = 0.5$, $\beta = 0.9$, $\mu = 2.5$, unless varied, with initial conditions $x_0 = 0.1, y_0 = 0.1$.

Figure 2 presents the bifurcation diagrams of system (6), which plot the maximum values of x_n (x_{max}) as functions of the bifurcation parameters α, β, and μ, under the initial conditions $x_0 = 0.1$ and $y_0 = 0.1$.

To analyze the bifurcation behavior of the two-dimensional memristor-based cubic map, the following three cases were considered:

(i) $\beta = 0.9$, $\mu = 2.5$ fixed while α varied over $[0, 0.8]$
(ii) $\alpha = 0.5$, $\mu = 2.5$ fixed while β varied over $[0, 1.1]$
(iii) $\alpha = 0.5$, $\beta = 0.9$ fixed while μ varied over $[0, 3]$

The bifurcation diagrams illustrate the transition from periodic to chaotic dynamics, showing that both the onset of chaos and the complexity of the bifurcation structures increase with larger values of the control parameters. The corresponding Lyapunov exponents (LEs) of system (6) were computed for $\beta = 0.9$ and $\mu = 2.5$, with α varied over the interval $[0, 0.8]$, as shown in Fig.3. The LE results are consistent with the corresponding bifurcation diagram, which confirms that transitions from periodic to chaotic behavior occur in the system as α increases.

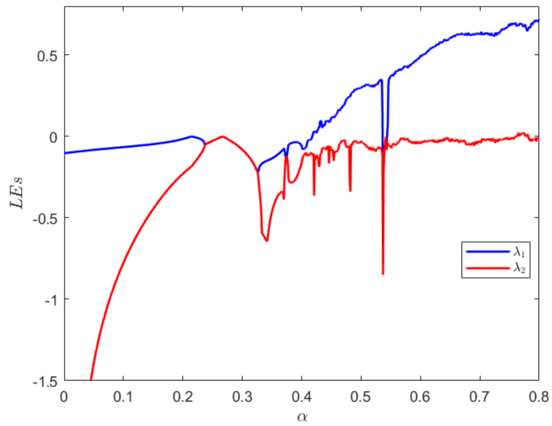

Figure 3. Lyapunov exponents λ_1 and λ_2 of system (6) as functions of the bifurcation parameter α, computed over the interval $\alpha \in [0, 0.8]$ with a step size of 0.001. The system parameters are set as $\mu = 2.5$, $\beta = 0.9$ with initial conditions $x_0 = 0.1$ and $y_0 = 0.1$.

Phase portraits, bifurcation diagrams, and Lyapunov exponents provide key insights into the dynamics of the two-dimensional memristor-based cubic map. They show how parameter variations induce transitions between periodic and chaotic behavior. However, individual trajectories alone cannot capture the system's sensitivity and multistability. Studying basins of attraction is therefore essential.

Basins of attraction illustrate how different initial conditions lead to distinct long-term behaviors. They are particularly informative in systems with fractal basin boundaries, where small variations in initial conditions can result in completely different outcomes. Such behavior highlights the system's sensitivity and inherent unpredictability.

To investigate the multistability and sensitivity of the 2D-MCM, the basins of attraction of system (6) were computed. A uniform 400×400 grid of initial conditions was sampled over the domain $[-2, 2] \times [-2, 2]$. For each grid point, the long-term behavior was determined numerically, and the corresponding attractors were identified using the recurrence-based approach described in [22], which enables automatic attractor classification without requiring prior knowledge of their structure.

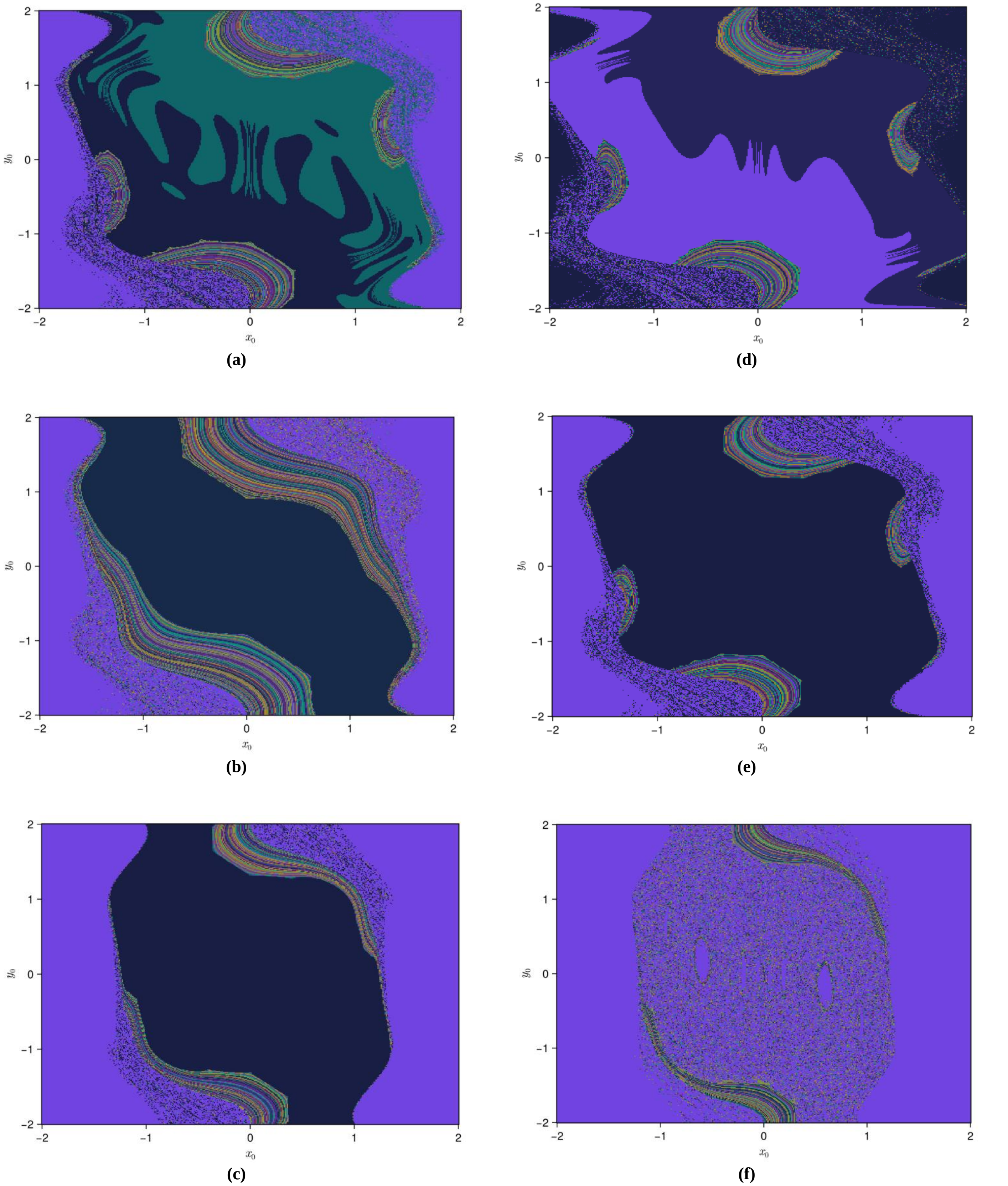

Figure 4. Basins of attraction of the 2D memristive cubic map for varying values of parameters α, β, and μ over the initial condition space $(x_0, y_0) \in [-2, 2] \times [-2, 2]$. Each subplot displays the distribution of final states (attractors) for a dense grid of initial conditions, colored according to their asymptotic behavior. Parameter settings: **(a)** $\alpha = 0.2, \beta = 0.9, \mu = 2.5$; **(b)** $\alpha = 0.2, \beta = 0.6, \mu = 2.5$; **(c)** $\alpha = 0.5, \beta = 0.9, \mu = 2.5$; **(d)** $\alpha = 0.1, \beta = 1.0, \mu = 3.0$; **(e)** $\alpha = 0.2, \beta = 1.0, \mu = 3.0$; **(f)** $\alpha = 0.6, \beta = 1.0, \mu = 3.0$.

Figures 4(a)–4(f) display the resulting basins of attraction for various combinations of the parameters α, β, and μ. Each point in the grid is colour-coded according to the attractor to which it converges. Points that do not settle into any attractor are marked in purple, indicating regions of instability where trajectories either diverge or fail to exhibit stable long-term behavior. These visualizations illustrate how basin geometry evolves in response to changes in parameters.

To quantify the unpredictability of final states in nonlinear dynamical systems, the basin entropy framework introduced in [23] was utilized. This method provides a global, quantitative measure of complexity and uncertainty associated with basins of attraction in multistable systems.

According to the procedure outlined in [23], the phase space was discretized into N small cells, referred to as ε-boxes. Within each cell i, the probability $p_{i,j}$ was computed that an initial condition would lead to attractor j, where $j = 1, 2, \dots, N_A$ and N_A is the total number of distinct attractors in the system.

The basin entropy S_b is defined as the mean Shannon entropy calculated across all boxes, as follows:

$$S_b = -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{m_i} p_{i,j}\log(p_{i,j}) \tag{7}$$

where $m_i \in [1, N_A]$ represents the number of distinct attractors observed in box i, and $p_{i,j}$ corresponds to the probability of convergence to attractor j, determined by the fraction of trajectories that settle into that attractor within the box.

The value of S_b lies within the interval $[0, \log(N_A)]$. A lower S_b indicates more predictable, well-separated basin structures, whereas higher values, approaching $\log(N_A)$, reflect greater uncertainty and increased complexity in the basin geometry.

TABLE I
BASIN ENTROPY VALUES FOR DIFFERENT COMBINATIONS OF THE SYSTEM PARAMETERS α, β, AND μ IN THE 2D MEMRISTIVE CUBIC MAP

α	β	μ	S_b
0.1	1.0	3.0	0.7346
0.2	1.0	3.0	0.5222
0.2	0.9	2.5	0.8347
0.2	0.6	2.5	1.1681
0.5	0.9	2.5	0.4613
0.6	1.0	3.0	1.2597

Table 1 presents the basin entropy S_b of the 2D memristive cubic map for different parameter sets (α, β, μ), while Figures 4(a)–4(f) illustrate how variations in these parameters influence the basin geometry and predictability. The results demonstrate that the unpredictability of the system varies significantly with parameter changes. Higher S_b values correspond to more complex and fragmented basins, indicating increased sensitivity to initial conditions, while lower values reflect simpler and more predictable basin structures. The system exhibits the highest basin entropy for $\alpha = 0.6$, $\beta = 1.0$, and $\mu = 3.0$, highlighting a highly unpredictable regime, whereas $\alpha = 0.5$, $\beta = 0.9$, and $\mu = 2.5$ shows the lowest entropy, corresponding to a more regular basin structure.

4. CONCLUSION

In this study, the dynamic behavior of a two-dimensional memristor-based cubic map was thoroughly investigated. By varying system parameters, the map exhibited diverse dynamical regimes, including periodic, chaotic, and hyperchaotic behavior. These results show that the system is highly sensitive to parameter changes. A recurrence-based method was employed to detect and classify attractors without prior knowledge of their locations. The corresponding basins of attraction displayed intricate structures, such as fractal boundaries and regions of trajectory divergence. This highlights the system's strong sensitivity to initial conditions and its potential for unpredictable outcomes. The incorporation of memristive elements significantly enriched the complexity of the dynamics. It was also shown that small variations in α, β, and μ can dramatically alter basin structures and predictability. These findings indicate that the memristor-based map is suitable for applications that exploit unpredictability and sensitivity, such as secure communications, encryption, and pseudorandom number generation.

It should be noted, however, that this study focused on the fundamental dynamics of the memristor-based system under ideal conditions. The possible effects of external disturbances, such as additive Gaussian noise or parameter fluctuations, were not considered. This limitation is acknowledged, and future research will focus on the robustness and stability of the system under noisy conditions. Such an analysis will provide a more comprehensive assessment of its practical applicability.

REFERENCES

[1] L. O. Chua, "Memristor—the missing circuit element" IEEE Transactions on Circuit Theory, vol. 18, no. 5, pp. 507–519, 1971. https://doi.org/10.1109/TCT.1971.1083337

[2] L. O. Chua, "Memristors on 'edge of chaos'", Nature Reviews Electrical Engineering vol.1, no. 9, pp. 614–627, 2024. https://doi.org/10.1038/s44287-024-00082-1

[3] Y. Li et al., "Memristor-type chaotic mapping" Chaos: An Interdisciplinary Journal of Nonlinear Science, vol. 32, no. 2, 2022. https://doi.org/10.1063/5.0082983

[4] L. Huang et al., "Design and analysis of a three-dimensional discrete memristive chaotic map with infinitely wide parameter range" Physica Scripta, vol. 97, no. 6, 065210, 2022. https://doi.org/10.1088/1402-4896/ac6e99

[5] W. A. Yihyis et al., "A class of discrete memristor chaotic maps based on the internal perturbation" Symmetry, vol. 15, no. 8, 1574, 2023. https://doi.org/10.3390/sym15081574

[6] L. Guodong et al., "Two modified chaotic maps based on discrete memristor model" Symmetry, vol. 14, no.4, 800, 2022. https://doi.org/10.3390/sym14040800

[7] M. T. Shatnawi et al., "A multistable discrete memristor and its application to discrete-time FitzHugh–Nagumo model" Electronics, vol. 12, no. 13, 2929, 2023. https://doi.org/10.3390/electronics12132929

[8] L. Fu et al., "A memristive Hénon map based on the state variable difference and its analog circuit implementation" IEEE Transactions on Industrial Electronics, vol. 71, no. 8, pp. 9668-9676, 2023. https://doi.org/10.1109/TIE.2023.3292857

[9] J. Mou et al., "Discrete second-order memristor and its application to chaotic map" IEEE Transactions on Circuits and Systems II: Express Briefs, vol. 71, no. 5, pp. 2824-2828, 2024. https://doi.org/10.1109/TCSII.2023.3348998

[10] A. R. Nieto et. al., "A simple method to enlarge a basin of attraction using a memristive function" Journal of Difference Equations and Applications, vol. 1, no. 8, 2025. https://doi.org/10.1080/10236198.2025.2510551

[11] S. Gao et al., "Design, hardware implementation, and application in video encryption of the 2D memristive cubic map" IEEE Internet of Things Journal, vol. 11, no. 12, pp. 21807 -21815, 2024. https://doi.org/10.1109/JIOT.2024.3376572

[12] Q. Lai, L. Yang, and Y. Liu, "Design and realization of discrete memristive hyperchaotic map with application in image encryption" Chaos, Solitons & Fractals, vol. 165, 112781, 2022. https://doi.org/10.1016/j.chaos.2022.112781

[13] M. Wang et al., "Two-dimensional memristive hyperchaotic maps with different coupling frames and its hardware implementation" Chaos: An Interdisciplinary Journal of Nonlinear Science, vol. 33, no. 7, 2023. https://doi.org/10.1063/5.0154516

[14] H. Li et al., "Two-dimensional memristive hyperchaotic maps and application in secure communication" IEEE transactions on industrial electronics, vol. 68, no. 10, pp. 9931-9940, 2020. https://doi.org/10.1109/TIE.2020.3022539

[15] B. Karakaya, A. Gülten, and M. Frasca, "A true random bit generator based on a memristive chaotic circuit: Analysis, design and FPGA implementation." Chaos, Solitons & Fractals 119 (2019): 143-149. https://doi.org/10.1016/j.chaos.2018.12.021

[16] M. Şahin, H. Guler, and S. Hamamcı, "Design and realization of a hyperchaotic memristive system for communication system on FPGA" Traitement du Signal, vol. 37, no. 6, 2020. https://doi.org/10.18280/ts.370607

[17] M. E. Sahin et al., "Application and modeling of a novel 4D memristive chaotic system for communication systems" Circuits, Systems, and Signal Processing, vol. 39, no. 7, pp. 3320-3349, 2020. https://doi.org/10.1007/s00034-019-01332-6

[18] M. E. Sahin et al., "Design of a hyperchaotic memristive circuit based on wien bridge oscillator" Computers & Electrical Engineering, vol. 88, 106826, 2020. https://doi.org/10.1016/j.compeleceng.2020.106826

[19] S. Gürsul and S. E. Hamamcı, "Investigation of power consumption effect of various memristor emulators on a logic gate" European Journal of Technique (EJT), vol. 11, no. 2, pp. 200-208, 2021. https://doi.org/10.36222/ejt.931338

[20] S. M. Mohamed et al., "An encryption application and FPGA realization of a fractional memristive chaotic system" Electronics, vol. 12, no. 5, 1219, 2023. https://doi.org/10.3390/electronics12051219

[21] S. Gao, et al. "A 3D memristive cubic map with dual discrete memristors: design, implementation, and application in image encryption" IEEE Transactions on Circuits and Systems for Video Technology, vol. 35, no. 8, pp. 7706 -7718, 2025. https://doi.org/10.1109/TCSVT.2025.3545868

[22] G. Datseris and A. Wagemakers, "Effortless estimation of basins of attraction" Chaos: An Interdisciplinary Journal of Nonlinear Science, vol. 32, no.2, 2022. https://doi.org/10.1063/5.0076568

[23] A. Daza et. al., “Basin entropy: a new tool to analyze uncertainty in dynamical systems” Scientific reports, vol. 6, no. 1, 31416, 2016. https://doi.org/10.1038/srep31416

BIOGRAPHY

Serpil Yılmaz Kutluay is currently working as an Assistant Professor in the Department of Computer Engineering at İzmir Katip Çelebi University. She received her B.Sc. degree in Electrical and Electronics Engineering from Gaziantep University in 2009. She completed her M.Sc. and Ph.D. degrees in Electronics and Communication Engineering at İzmir Institute of Technology in 2012 and 2019, respectively. Her research interests include chaotic systems, stochastic modeling, and embedded systems.