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SMARANDACHE CURVES ACCORDING TO SABBAN FRAME BELONGING TO MANNHEIM CURVES PAIR

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Abstract. In this paper, we investigate special Smarandache curves according to Sabban frame, using the idea of Mannheim partner curve of spherical indicatrix. We created Sabban frame by means of Mannheim partner curve of spherical indicatrix. We explained that position vectors of a Smarandache curves are consisted by Sabban frame given above. Then we calculated geodesic curvatures of this Smarandache curve and the results which we found after calculations are expressed by means of Mannheim partner curve of spherical indicatrix.

1. Introduction and Preliminaries

In differential geometry special curves play an important role. One of these special curves is Mannheim partner curve. Mannheim curve first was defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\lambda = \frac{\kappa}{\kappa^2+\tau^2}$ where κ is the curvature of the curve and τ is the torsion of the curve. Later Mannheim curve is redefined by Lui and Wang. According to this new definition, take two curves and if the principal vector of the first curve is linearly dependent to the binormal vector of the second curve then these curves are said to be Mannheim partner curves,[5]. There are lots of papers related to Mannheim curves,[6]. A regular curve in Minkowski spacetime case, Frenet vectors of a curve are taken as the position vectors and then the regular curve drawn by the new vector, in this way we get a new curve called as Smarandache curve, [9]. Smarandache curves also have been studied by, [1, 2]. Taşköprü and Tosun studied special Smarandache curves by means of Sabban frame on S^2, [8]. Şenyurt and Çalışkan researched special Smarandache curves according to Sabban frame using spherical indicatrix of a curve and they gave some characterizations of Smarandache curves, [3]. We investigated special Smarandache curves according to Sabban frame drawn on the

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surface of a sphere by Darboux vectors of involute and Bertrand partner curves, [7].

Let $\alpha : I \to E^3$ be a unit speed curve, we can define the Frenet frame and Frenet formulae, as follows

$$T(s) = \alpha'(s), \;\; N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \;\; B(s) = T(s) \wedge N(s), \tag{1.1}$$

$$T'(s) = \kappa(s)N(s), \;\; N'(s) = -\kappa(s)T(s) + \tau(s)B(s), \;\; B'(s) = -\tau(s)N(s). \tag{1.2}$$

Let α, α^* be Mannheim pair curve and $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ and $\{T_1(s), N_1(s), B_1(s), \kappa_1(s), \tau_1(s)\}$ are the Frenet apparatus respectively. The relation between the Frenet apparatus are given as,

$$T_1 = \sin\phi T + \cos\phi B, \;\; N_1 = -\cos\phi T + \sin\phi B, \;\; B_1 = N. \tag{1.3}$$

and

$$\kappa_1 = \frac{\kappa\phi'}{\lambda\tau\|W\|}, \; \tau_1 = \frac{\kappa}{\lambda\tau}. \tag{1.4}$$

Let $\gamma : I \to S^2$ be a unit speed spherical curve. We symbolize s as the arc-length parameter of γ. Let's give

$$\begin{cases} \gamma(s) = \gamma(s), \\ t(s) = \gamma'(s), \\ d(s) = \gamma(s) \wedge t(s) \end{cases} \tag{1.5}$$

$\{\gamma(s), t(s), d(s)\}$ frame is denominated the Sabban frame of γ on S^2. The spherical Frenet formulae of γ is given as,

$$\begin{cases} \gamma'(s) = t(s), \\ t'(s) = -\gamma(s) + \kappa_g(s)d(s), \\ d'(s) = -\kappa_g(s)t(s) \end{cases} \tag{1.6}$$

where κ_g is the geodesic curvature of the curve γ on S^2 , [8]

$$\kappa_g(s) = \langle t'(s), d(s)\rangle. \tag{1.7}$$

2. Results

In this section, we investigated special Smarandache curves created by Sabban frame, $\{T_1, T_{T_1}, T_1 \wedge T_{T_1}\}$, $\{N_1, T_{N_1}, N_1 \wedge T_{N_1}\}$ and $\{B_1, T_{B_1}, B_1 \wedge T_{B_1}\}$. We find some results. These results can be expressed depending on the Mannheim curve. Let's write results on these Smarandache curves.

Let

$$\alpha_{T_1}(s) = T_1(s), \;\; \alpha_{N_1}(s) = N_1(s) \quad \text{and} \;\; \alpha_{B_1}(s) = B_1(s)$$

be a regular spherical curve on S^2. The Sabban frame of spherical indicatrix according to Mannheim partner curve can be written as,

$$T_1 = T_1,\ \ T_{T_1} = N_1,\ \ T_1 \wedge T_{T_1} = B_1, \tag{2.1}$$

$$\begin{cases} N_1 = N_1 \\ T_{N_1} = -\cos\phi_1 T_1 + \sin\phi_1 B_1 \\ N_1 \wedge T_{N_1} = \sin\phi_1 T_1 + \cos\phi_1 B_1, \end{cases} \tag{2.2}$$

and

$$B_1 = B_1,\ \ T_{B_1} = -N_1,\ \ B_1 \wedge T_{B_1} = T_1. \tag{2.3}$$

From the equation (1.6), the spherical Frenet formulae of (T_1), (N_1) and (B_1) are

$$T_1{}' = T_{T_1},\ \ T_{T_1}{}' = -T_1 + \frac{\tau_1}{\kappa_1} T_1 \wedge T_{T_1},\ \ (T_1 \wedge T_{T_1})' = -\frac{\tau_1}{\kappa_1} T_{T_1}, \tag{2.4}$$

$$N_1{}' = T_{N_1},\ \ T_{N_1}{}' = -N_1 + \frac{\phi_1{}'}{\|W_1\|} N_1 \wedge T_{N_1},\ \ (N_1 \wedge T_{N_1})' = -\frac{\phi_1{}'}{\|W_1\|} T_{N_1}, \tag{2.5}$$

and

$$B_1{}' = T_{B_1},\ \ T_{B_1}{}' = -B_1 + \frac{\kappa_1}{\tau_1} B_1 \wedge T_{B_1},\ \ (B_1 \wedge T_{B_1})' = -\frac{\kappa_1}{\tau_1} T_{B_1}. \tag{2.6}$$

respectively. Using the equation (1.7), the geodesic curvatures of (T_1), (N_1) and (B_1) are

$$\kappa_g^{T_1} = \frac{\tau_1}{\kappa_1},\ \ \kappa_g^{N_1} = \frac{\phi_1{}'}{\|W_1\|} \ \text{ and } \ \kappa_g^{B_1} = \frac{\kappa_1}{\tau_1}. \tag{2.7}$$

Definition 1. *Let (T_1) be a spherical curve of α_1, T_1 and T_{T_1} be Sabban vectors of (T_1). Then β_1-Smarandache curve can be identified as*

$$\beta_1(s) = \frac{1}{\sqrt{2}}\big(T_1 + T_{T_1}\big). \tag{2.8}$$

Theorem 1. *The geodesic curvature according to β_1-Smarandache curve of the Mannheim curve is*

$$\kappa_g^{\beta_1} = \frac{1}{\big(2 + \big(\frac{\|W\|}{\phi'}\big)^2\big)^{\frac{5}{2}}} \Big(\frac{\|W\|}{\phi'}\overline{\lambda}_1 - \frac{\|W\|}{\phi'}\overline{\lambda}_2 + 2\overline{\lambda}_3\Big), \tag{2.9}$$

where

$$\begin{cases} \overline{\lambda}_1 = -2 - \big(\frac{\|W\|}{\phi'}\big)^2 + \big(\frac{\|W\|}{\phi'}\big)'\big(\frac{\|W\|}{\phi'}\big) \\ \overline{\lambda}_2 = -2 - 3\big(\frac{\|W\|}{\phi'}\big)^2 - \big(\frac{\|W\|}{\phi'}\big)^4 - \big(\frac{\|W\|}{\phi'}\big)'\big(\frac{\|W\|}{\phi'}\big) \\ \overline{\lambda}_3 = 2\big(\frac{\|W\|}{\phi'}\big) + \big(\frac{\|W\|}{\phi'}\big)^3 + \big(\frac{\|W\|}{\phi'}\big)' \end{cases} \tag{2.10}$$

Proof.

$$\beta_1(s_{T_1}) = \frac{1}{\sqrt{2}}\big(T_1 + T_{T_1}\big), \tag{2.11}$$

or from the equation (2.1), we can write

$$\beta_1(s_1) = \frac{1}{\sqrt{2}}\big(T_1 + N_1\big). \tag{2.12}$$

If we take the derivative of the equation (2.11), then T_{β_1} vector is

$$T_{\beta_1}(s_1) = \frac{1}{\sqrt{2 + \big(\frac{\tau_1}{\kappa_1}\big)^2}}(-T_1 + N_1 + \frac{\tau_1}{\kappa_1}B_1). \tag{2.13}$$

Considering the equations (2.12) and (2.13), we have

$$\big(\beta_1 \wedge T_{\beta_1}\big)(s_1) = \frac{1}{\sqrt{4 + 2\big(\frac{\tau_1}{\kappa_1}\big)^2}}(\frac{\tau_1}{\kappa_1}T_1 - \frac{\tau_1}{\kappa_1}N_1 + 2B_1). \tag{2.14}$$

If we take the derivative of the equation (2.13), then T'_{β_1} vector is

$$T'_{\beta_1}(s_1) = \frac{\sqrt{2}}{\Big(2 + \big(\frac{\tau_1}{\kappa_1}\big)^2\Big)^2}(\lambda_1 T_1 + \lambda_2 N_1 + \lambda_3 B_1). \tag{2.15}$$

where the coefficients are,

$$\begin{cases} \lambda_1 = -2 - \big(\frac{\tau_1}{\kappa_1}\big)^2 + \big(\frac{\tau_1}{\kappa_1}\big)'\big(\frac{\tau_1}{\kappa_1}\big), \\ \lambda_2 = -2 - 3\big(\frac{\tau_1}{\kappa_1}\big)^2 - \big(\frac{\tau_1}{\kappa_1}\big)^4 - \big(\frac{\tau_1}{\kappa_1}\big)'\big(\frac{\tau_1}{\kappa_1}\big), \\ \lambda_3 = 2\big(\frac{\tau_1}{\kappa_1}\big) + \big(\frac{\tau_1}{\kappa_1}\big)^3 + \big(\frac{\tau_1}{\kappa_1}\big)'. \end{cases} \tag{2.16}$$

Using the equations (1.7), (2.14) and (2.15), $\kappa_g^{\beta_1}$ geodesic curvature of $\beta_1(s_1)$ is

$$\kappa_g^{\beta_1} = \frac{1}{\big(2 + \big(\frac{\tau_1}{\kappa_1}\big)^2\big)^{\frac{5}{2}}}\big(\frac{\tau_1}{\kappa_1}\lambda_1 + \frac{\tau_1}{\kappa_1}\lambda_2 + 2\lambda_3\big).$$

Substituting the equations (1.3) and (1.4) into equations (2.12), (2.13), (2.14) and (2.15), Sabban apparatus of the β_1-Smarandache curve for Mannheim curve are

$$\beta_1(s) = \frac{1}{\sqrt{2}}\big((\sin\phi-\cos\phi)T+(\cos\phi+\sin\phi)B\big),$$

$$T_{\beta_1}(s) = \frac{\phi'(-\sin\phi-\cos\phi)}{\sqrt{\|W\|^2+2\phi'^2}}T+\frac{\|W\|}{\sqrt{\|W\|^2+2\phi'^2}}N+\frac{\phi'(\sin\phi-\cos\phi)}{\sqrt{\|W\|^2+2\phi'^2}}B,$$

$$(\beta_1\wedge T_{\beta_1})(s) = \frac{\|W\|(\sin\phi-\cos\phi)}{\sqrt{2\|W\|^2+4\phi'^2}}T+\frac{2\phi'}{\sqrt{2\|W\|^2+4\phi'^2}}N+\frac{\|W\|(\cos\phi+\sin\phi)}{\sqrt{2\|W\|^2+4\phi'^2}}B,$$

$$T'_{\beta_1}(s) = \frac{(\phi'^4\sqrt{2}(\overline{\lambda}_1\sin\phi-\overline{\lambda}_2\cos\phi)}{(\|W\|^2+2\phi'^2)^2}T+\frac{(\phi'^4\sqrt{2}\overline{\lambda}_3}{(\|W\|^2+2\phi'^2)^2}N+\frac{(\phi'^4\sqrt{2}(\overline{\lambda}_2\sin\phi+\overline{\lambda}_1\cos\phi)}{(\|W\|^2+2\phi'^2)^2}B$$

and

$$\kappa_g^{\beta_1} = \frac{1}{\left(2+\left(\frac{\|W\|}{\phi'}\right)^2\right)^{\frac{5}{2}}}\big(\frac{\|W\|}{\phi'}\overline{\lambda}_1-\frac{\|W\|}{\phi'}\overline{\lambda}_2+2\overline{\lambda}_3\big),$$

where

$$\begin{cases}\overline{\lambda}_1 = -2-\big(\frac{\|W\|}{\phi'}\big)^2+\big(\frac{\|W\|}{\phi'}\big)'\big(\frac{\|W\|}{\phi'}\big)\\ \overline{\lambda}_2 = -2-3\big(\frac{\|W\|}{\phi'}\big)^2-\big(\frac{\|W\|}{\phi'}\big)^4-\big(\frac{\|W\|}{\phi'}\big)'\big(\frac{\|W\|}{\phi'}\big)\\ \overline{\lambda}_3 = 2\big(\frac{\|W\|}{\phi'}\big)+\big(\frac{\|W\|}{\phi'}\big)^3+\big(\frac{\|W\|}{\phi'}\big)'\end{cases}$$

□

Corollary 1. *The Sabban frame according to β_1-Smarandache curve of Mannheim curve is*

$$\beta_1(s) = \frac{1}{\sqrt{2}}\big((\sin\phi-\cos\phi)T+(\cos\phi+\sin\phi)B\big),$$

$$T_{\beta_1}(s) = \frac{\phi'(-\sin\phi-\cos\phi)}{\sqrt{\|W\|^2+2\phi'^2}}T+\frac{\|W\|}{\sqrt{\|W\|^2+2\phi'^2}}N+\frac{\phi'(\sin\phi-\cos\phi)}{\sqrt{\|W\|^2+2\phi'^2}}B,$$

$$(\beta_1\wedge T_{\beta_1})(s) = \frac{\|W\|(\sin\phi-\cos\phi)}{\sqrt{2\|W\|^2+4\phi'^2}}T+\frac{2\phi'}{\sqrt{2\|W\|^2+4\phi'^2}}N+\frac{\|W\|(\cos\phi+\sin\phi)}{\sqrt{2\|W\|^2+4\phi'^2}}B,$$

The proofs of the subsequent theorems according to β_2, β_3, β_4, β_5, β_6, β_7, β_8 and β_9- Smarandache curves will be similar to the Theorem 2.1.

Definition 2. *Let (T_1) be a spherical curve of α_1, T_{T_1} and $T_1 \wedge T_{T_1}$ be Sabban vectors of (T_1). Then β_2-Smarandache curve can be identified as*

$$\beta_2(s) = \frac{1}{\sqrt{2}}(T_{T_1} + T_1 \wedge T_{T_1}). \tag{2.17}$$

Theorem 2. *Let α_1 be a Mannheim partner of α. Then Sabban apparatus according to β_2-Smarandache curve of Mannheim curve are,*

$$\begin{aligned}
\beta_2(s) &= \frac{1}{\sqrt{2}}\big(-\cos\phi\, T + N + \sin\phi\, B\big),\\
T_{\beta_2}(s) &= \frac{\|W\|\cos\phi - \phi'\sin\phi}{\sqrt{2\|W\|^2 + {\phi'}^2}}T + \frac{\|W\|}{\sqrt{2\|W\|^2 + {\phi'}^2}}N - \frac{\phi'\cos\phi + \|W\|\sin\phi}{\sqrt{2\|W\|^2 + {\phi'}^2}}B,\\
\big(\beta_2 \wedge T_{\beta_2}\big)(s) &= \frac{2\|W\|\sin\phi + \phi'\cos\phi}{\sqrt{4\|W\|^2 + 2{\phi'}^2}}T + \frac{\phi'}{\sqrt{4\|W\|^2 + 2{\phi'}^2}}N + \frac{2\|W\|\cos\phi - \phi'\sin\phi}{\sqrt{4\|W\|^2 + 2{\phi'}^2}}B.
\end{aligned}$$

and

$$\kappa_g^{\beta_2} = \frac{1}{\Big(1 + 2\big(\frac{\|W\|}{\phi'}\big)^2\Big)^{\frac{5}{2}}}\big(2\frac{\|W\|}{\phi'}\varepsilon_1 - \varepsilon_2 + \varepsilon_3\big).$$

where the coefficients are

$$\begin{cases}
\varepsilon_1 = \big(\frac{\|W\|}{\phi'}\big) + 2\big(\frac{\|W\|}{\phi'}\big)^3 + 2\big(\frac{\|W\|}{\phi'}\big)'\big(\frac{\|W\|}{\phi'}\big)\\
\varepsilon_2 = -1 - 3\big(\frac{\|W\|}{\phi'}\big)^2 - 2\big(\frac{\|W\|}{\phi'}\big)^4 - \big(\frac{\|W\|}{\phi'}\big)'\\
\varepsilon_3 = -\big(\frac{\|W\|}{\phi'}\big)^2 - 2\big(\frac{\|W\|}{\phi'}\big)^4 + \big(\frac{\|W\|}{\phi'}\big)'
\end{cases}$$

Proof. The proof is the similar case as in proof of Theorem 2.1.

Definition 3. *Let (T_1) be a spherical curve be α_1, T_1, T_{T_1} and $T_1 \wedge T_{T_1}$ be Sabban vectors of (T_1). Then β_3-Smarandache curve can be identified as*

$$\beta_3(s) = \frac{1}{\sqrt{3}}\big(T_1 + T_{T_1} + T_1 \wedge T_{T_1}\big)$$

Theorem 3. *Let α_1 be a Mannheim partner of α. Then Sabban apparatus according to β_3-Smarandache curve of Mannheim curve are,*

$$\begin{aligned}
\beta_3(s) &= \frac{1}{\sqrt{3}}\big((\sin\phi-\cos\phi)T+N+(\cos\phi+\sin\phi)B\big),\\
T_{\beta_3}(s) &= \frac{\phi'\sin\phi+\big(\|W\|-\phi'\big)\cos\phi}{\sqrt{2(\|W\|^2-\|W\|\phi'+{\phi'}^2)}}T+\frac{\|W\|}{\sqrt{2(\|W\|^2-\|W\|\phi'+{\phi'}^2)}}N\\
&\quad+\frac{\phi'\cos\phi-(\|W\|-\phi')\sin\phi}{\sqrt{2(\|W\|^2-\|W\|\phi'+{\phi'}^2)}}B,\\
\big(\beta_3\wedge T_{\beta_3}\big)(s) &= \frac{(2\|W\|-\phi')\sin\phi+(\|W\|+\phi')\cos\phi}{\sqrt{6\|W\|^2-6\|W\|\phi'+6{\phi'}^2}}T+\frac{2\phi'-\|W\|}{\sqrt{6\|W\|^2-6\|W\|\phi'+6{\phi'}^2}}N\\
&\quad+\frac{(2\|W\|-\phi')\cos\phi-(\|W\|+\phi')\sin\phi}{\sqrt{6\|W\|^2-6\|W\|\phi'+6{\phi'}^2}}B.
\end{aligned}$$

and

$$\kappa_g^{\beta_3}=\frac{1}{4\sqrt{2}\Big(1+\frac{\|W\|}{\phi'}+\big(\frac{\|W\|}{\phi'}\big)^2\Big)^{\frac{5}{2}}}\Big(\big(2\frac{\|W\|}{\phi'}-1\big)\Im_1-\big(1+\frac{\|W\|}{\phi'}\big)\Im_2+\big(2-\frac{\|W\|}{\phi'}\big)\Im_3\Big).$$

where the coefficients are

$$\begin{cases}
\Im_1=-2+4\big(\frac{\|W\|}{\phi'}\big)-4\big(\frac{\|W\|}{\phi'}\big)^2+2\big(\frac{\|W\|}{\phi'}\big)^3+\big(\frac{\|W\|}{\phi'}\big)'\big(2\frac{\|W\|}{\phi'}-1\big)\\
\Im_2=-2+2\big(\frac{\|W\|}{\phi'}\big)-4\big(\frac{\|W\|}{\phi'}\big)^2+\big(\frac{\|W\|}{\phi'}\big)^3-2\big(\frac{\|W\|}{\phi'}\big)^4-\big(\frac{\|W\|}{\phi'}\big)'\big(1+\frac{\|W\|}{\phi'}\big)\\
\Im_3=2\big(\frac{\|W\|}{\phi'}\big)-4\big(\frac{\|W\|}{\phi'}\big)^2+4\big(\frac{\|W\|}{\phi'}\big)^3-2\big(\frac{\|W\|}{\phi'}\big)^4+\big(\frac{\|W\|}{\phi'}\big)'\big(2-\frac{\|W\|}{\phi'}\big)
\end{cases}$$

Proof. The proof is the same as in proof of Theorem 2.1.

Definition 4. *Let (N_1) be a spherical curve of α_1, N_1, T_{N_1} be Sabban vectors of (N_1). Then β_4-Smarandache curve can be identified as*

$$\beta_4(s)=\frac{1}{\sqrt{2}}\big(N_1+T_{N_1}\big)$$

Theorem 4. *Let α_1 be a Mannheim partner of α. Then Sabban apparatus according to β_4-Smarandache curve of Mannheim curve are,*

$$\begin{aligned}
\beta_4(s) &= \frac{-\sqrt{\phi'^2+\|W\|^2}\cos\phi-\phi'\sin\phi}{\sqrt{2\phi'^2+2\|W\|^2}}T+\frac{\|W\|}{\sqrt{2\phi'^2+2\|W\|^2}}N \\
&\quad +\frac{\sqrt{\phi'^2+\|W\|^2}\sin\phi-\phi'\cos\phi}{\sqrt{2\phi'^2+2\|W\|^2}}B,
\end{aligned}$$

$$\begin{aligned}
T_{\beta_4}(s) &= \frac{(\overline{F}\|W\|-\phi')\sin\phi+\sqrt{\|W\|^2+\phi'^2}\cos\phi}{\sqrt{\|W\|^2+\phi'^2}\sqrt{2+\overline{F}^2}}T+\frac{\overline{F}\phi'+\|W\|}{\sqrt{\|W\|^2+\phi'^2}\sqrt{2+\overline{F}^2}}N \\
&\quad -\frac{(\overline{F}\|W\|-\phi')\cos\phi-\sqrt{\|W\|^2+\phi'^2}\sin\phi}{\sqrt{\|W\|^2+\phi'^2}\sqrt{2+\overline{F}^2}}B,
\end{aligned}$$

$$\begin{aligned}
\left(\beta_4\wedge T_{\beta_4}\right)(s) &= \frac{(2\|W\|-\overline{F}\phi')\sin\phi+\overline{F}\sqrt{\|W\|^2+\phi'^2}\cos\phi}{\sqrt{4+2\overline{F}^2}\sqrt{\|W\|^2+\phi'^2}}T \\
&\quad +\frac{2\phi'-\overline{F}\|W\|}{\sqrt{4+2\overline{F}^2}\sqrt{\|W\|^2+\phi'^2}}N \\
&\quad +\frac{(2\|W\|-\overline{F}\phi')\cos\phi-\overline{F}\sqrt{\|W\|^2+\phi'^2}\sin\phi}{\sqrt{4+2\overline{F}^2}\sqrt{\|W\|^2+\phi'^2}}B.
\end{aligned}$$

and

$$\kappa_g^{\beta_4}=\frac{1}{\left(2+\overline{F}^2\right)^{\frac{5}{2}}}\left(\overline{F}\,\underline{\widetilde{\varpi}}_1-\overline{F}\,\underline{\widetilde{\varpi}}_2+2\underline{\widetilde{\varpi}}_3\right).$$

where the coefficients are

$$\overline{F}=\Big(\frac{\|W\|}{\sqrt{(\phi'^2+\|W\|^2}}\Big)'\frac{\lambda\tau}{\phi'}\sec\phi$$

it follows,

$$\begin{cases}
\underline{\widetilde{\varpi}}_1=-2-\overline{F}^2+\overline{F}'\overline{F}\\
\underline{\widetilde{\varpi}}_2=-2-3\overline{F}^2-\overline{F}^4-\overline{F}'\overline{F}\\
\underline{\widetilde{\varpi}}_3=2\overline{F}+\overline{F}^3+\overline{F}'
\end{cases}$$

Proof. The proof is similar as in proof of Theorem 2.1.

Definition 5. *Let (N_1) be a spherical curve of α_1, T_{N_1} and $N_1 \wedge T_{N_1}$ be Sabban vectors of (N_1). Then β_5-Smarandache curve can be identified as*

$$\beta_5(s) = \frac{1}{\sqrt{2}}(T_{N_1} + N_1 \wedge T_{N_1})$$

Theorem 5. *Let α_1 be a Mannheim partner of α. Then Sabban apparatus according to β_5-Smarandache curve of Mannheim curve are,*

$$\beta_5(s) = \frac{(\|W\| + \phi')\sin\phi}{\sqrt{2{\phi'}^2 + 2\|W\|^2}}T + \frac{\phi' + \|W\|}{\sqrt{2{\phi'}^2 + 2\|W\|^2}}N + \frac{(\|W\| - \phi')\cos\phi}{\sqrt{2{\phi'}^2 + 2\|W\|^2}}B,$$

$$\begin{aligned} T_{\beta_5}(s) = {} & \frac{\overline{F}(\|W\| + \phi')\sin\phi + \sqrt{\|W\|^2 + {\phi'}^2}\cos\phi}{\sqrt{1 + 2\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}T + \frac{\overline{F}(\phi' - \|W\|)}{\sqrt{1 + 2\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}N \\ & + \frac{\overline{F}(\|W\| + \phi')\cos\phi - \sqrt{\|W\|^2 + {\phi'}^2}\sin\phi}{\sqrt{1 + 2\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}B, \end{aligned}$$

$$\begin{aligned} (\beta_5 \wedge T_{\beta_5})(s) = {} & \frac{(\|W\| + \phi')\sin\phi - 2\overline{F}\sqrt{\|W\|^2 + {\phi'}^2}\cos\phi}{\sqrt{2 + 4\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}T + \frac{\phi' - \|W\|}{\sqrt{2 + 4\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}N \\ & + \frac{2\overline{F}\sqrt{\|W\|^2 + {\phi'}^2}\sin\phi + (\|W\| + \phi')\cos\phi}{\sqrt{2 + 4\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}B. \end{aligned}$$

and

$$\kappa_g^{\beta_5} = \frac{1}{\left(2 + \overline{F}^2\right)^{\frac{5}{2}}}\left(2\overline{F}\,\underline{\widetilde{\delta}}_1 - \underline{\widetilde{\delta}}_2 + \underline{\widetilde{\delta}}_3\right).$$

where the coefficients are

$$\begin{cases} \underline{\widetilde{\delta}}_1 = \overline{F} + 2\overline{F}^3 + 2\overline{F}'\overline{F} \\ \underline{\widetilde{\delta}}_2 = -1 - 3\overline{F}^2 - 2\overline{F}^4 - \overline{F}' \\ \underline{\widetilde{\delta}}_3 = -\overline{F}^2 - 2\overline{F}^4 + \overline{F}' \end{cases}$$

Proof. The proof is the similar case as in proof of Theorem 2.1.

Definition 6. *Let (N_1) be a spherical curve of α_1, N_1, T_{N_1} and $N_1 \wedge T_{N_1}$ be Sabban vectors of (N_1). Then β_6-Smarandache curve can be identified as*

$$\beta_6(s) = \frac{1}{\sqrt{3}}\big(N_1 + T_{N_1} + N_1 \wedge T_{N_1}\big)$$

Theorem 6. *Let α_1 be a Mannheim partner of α. Then Sabban apparatus according to β_6-Smarandache curve of Mannheim curve are,*

$$\begin{aligned}
\beta_6(s) \quad = \quad & \frac{(\|W\| - \phi')\sin\phi - \sqrt{{\phi'}^2 + \|W\|^2}\cos\phi}{\sqrt{3{\phi'}^2 + 3\|W\|^2}}T + \frac{\phi' + \|W\|}{\sqrt{3{\phi'}^2 + 3\|W\|^2}}N \\
& + \frac{(\|W\| - \phi')\cos\phi + \sqrt{{\phi'}^2 + \|W\|^2}\sin\phi}{\sqrt{3{\phi'}^2 + 3\|W\|^2}}B,
\end{aligned}$$

$$\begin{aligned}
T_{\beta_6}(s) \quad = \quad & \frac{\big(\overline{F}\|W\| - (1 - \overline{F})\phi'\big)\sin\phi + \sqrt{\|W\|^2 + {\phi'}^2}\cos\phi}{\sqrt{2(1 - \overline{F} + \overline{F}^2)}\sqrt{\|W\|^2 + {\phi'}^2}}T \\
& + \frac{\overline{F}\phi' + (1 - \overline{F})\|W\|}{\sqrt{2(1 - \overline{F} + \overline{F}^2)}\sqrt{\|W\|^2 + {\phi'}^2}}N \\
& + \frac{\big(\overline{F}\|W\| - (1 - \overline{F})\phi'\big)\cos\phi - \sqrt{\|W\|^2 + {\phi'}^2}\sin\phi}{\sqrt{2(1 - \overline{F} + \overline{F}^2)}\sqrt{\|W\|^2 + {\phi'}^2}}B,
\end{aligned}$$

$$\begin{aligned}
\big(\beta_6 \wedge T_{\beta_6}\big)(s) \quad = \quad & \frac{\big((2 - \overline{F})\|W\| + (1 + \overline{F})\phi'\big)\sin\phi - (2\overline{F} - 1)\sqrt{\|W\|^2 + {\phi'}^2}\cos\phi}{\sqrt{6 - 6\overline{F} + 6\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}T \\
& + \frac{(2 - \overline{F})\phi' - (1 + \overline{F})\|W\|}{\sqrt{6 - 6\overline{F} + 6\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}N \\
& + \frac{(2\overline{F} - 1)\sqrt{\|W\|^2 + {\phi'}^2}\sin\phi + \big((2 - \overline{F})\|W\| + (1 + \overline{F})\phi'\big)\cos\phi}{\sqrt{6 - 6\overline{F} + 6\overline{F}^2}\sqrt{\|W\|^2 + {\phi'}^2}}B.
\end{aligned}$$

and

$$\kappa_g^{\beta_6}=\frac{(2\overline{F}-1)\underline{\widetilde{\rho}}_1-(1+\overline{F})\underline{\widetilde{\rho}}_2+(2-\overline{F})\underline{\widetilde{\rho}}_3}{4\sqrt{2}\Big(1-\overline{F}+\overline{F}^2\Big)^{\frac{5}{2}}}.$$

where the coefficients are

$$\begin{cases}\underline{\widetilde{\rho}}_1=-2+4\overline{F}-4\overline{F}^2+2\overline{F}^3+2\overline{F}'(2\overline{F}-1)\\ \underline{\widetilde{\rho}}_2=-2+2\overline{F}-4\overline{F}^2+2\overline{F}^3-2\overline{F}^4-\overline{F}'(1+\overline{F})\\ \underline{\widetilde{\rho}}_3=2\overline{F}-4\overline{F}^2+4\overline{F}^3-2\overline{F}^4+\overline{F}'(2-\overline{F})\end{cases}$$

Proof. The proof is similar as in proof of Theorem 2.1.

Definition 7. *Let (B_1) be a spherical curve of α_1, B_1 and T_{B_1} be Sabban vectors of (B_1). Then β_7-Smarandache curve can be identified as*

$$\beta_7(s)=\frac{1}{\sqrt{2}}\big(B_1+T_{B_1}\big).$$

Theorem 7. *Let α_1 be a Mannheim partner of α. Then Sabban apparatus according to β_7-Smarandache curve of Mannheim curve are,*

$$\begin{aligned}\beta_7(s)&=\frac{1}{\sqrt{2}}\big(\cos\phi\,T+N-\sin\phi\,B\big),\\ T_{\beta_7}(s)&=\frac{\phi'\sin\phi+\|W\|\cos\phi}{\sqrt{2\|W\|^2+\phi'^2}}T-\frac{\|W\|}{\sqrt{2\|W\|^2+\phi'^2}}N+\frac{\phi'\cos\phi-\|W\|\sin\phi}{\sqrt{2\|W\|^2+\phi'^2}}B,\\ \big(\beta_7\wedge T_{\beta_7}\big)(s)&=\frac{2\|W\|\sin\phi-\phi'\cos\phi}{\sqrt{4\|W\|^2+2\phi'^2}}T+\frac{\phi'}{\sqrt{4\|W\|^2+2\phi'^2}}N+\frac{\phi'\sin\phi+2\|W\|\cos\phi}{\sqrt{4\|W\|^2+2\phi'^2}}B,\end{aligned}$$

and

$$\kappa_g^{\beta_7}=\frac{1}{\Big(2+\big(\frac{\phi'}{\|W\|}\big)^2\Big)^{\frac{5}{2}}}\Big(\frac{\phi'}{\|W\|}\underline{\hbar}_1-\frac{\phi'}{\|W\|}\underline{\hbar}_2+2\underline{\hbar}_3\Big).$$

where the coefficients are

$$\begin{cases}\underline{\hbar}_1=-2-\big(\frac{\phi'}{\|W\|}\big)^2+\big(\frac{\phi'}{\|W\|}\big)'\big(\frac{\phi'}{\|W\|}\big)\\ \underline{\hbar}_2=-2-3\big(\frac{\phi'}{\|W\|}\big)^2-\big(\frac{\phi'}{\|W\|}\big)^4-\big(\frac{\phi'}{\|W\|}\big)'\big(\frac{\phi'}{\|W\|}\big)\\ \underline{\hbar}_3=2\big(\frac{\phi'}{\|W\|}\big)+\big(\frac{\phi'}{\|W\|}\big)^3+2\big(\frac{\phi'}{\|W\|}\big)'\end{cases}$$

Proof. The proof is similar as in proof of Theorem 2.1.

Definition 8. *Let (B_1) be a spherical curve of α_1, T_{B_1} and $B_1 \wedge T_{B_1}$ be Sabban vectors of (B_1). Then β_8-Smarandache curve can be identified as*

$$\beta_8(s) = \frac{1}{\sqrt{3}}\big(T_{B_1} + B_1 \wedge T_{B_1}\big).$$

Theorem 8. *Let α_1 be a Mannheim partner of α. Sabban apparatus according to β_8-Smarandache curve of Mannheim curve are,*

$$\begin{aligned}
\beta_8(s) &= \frac{1}{\sqrt{2}}\big((\sin\phi + \cos\phi)T + (\cos\phi - \sin\phi)B\big),\\
T_{\beta_8}(s) &= \frac{\phi'(\sin\phi - \cos\phi)}{\sqrt{\|W\|^2 + 2\phi'^2}}T - \frac{\|W\|}{\sqrt{\|W\|^2 + 2\phi'^2}}N + \frac{\phi'(\sin\phi + \cos\phi)}{\sqrt{\|W\|^2 + 2\phi'^2}}B,\\
\big(\beta_8 \wedge T_{\beta_8}\big)(s) &= \frac{\|W\|(\sin\phi - \cos\phi)}{\sqrt{2\|W\|^2 + 4\phi'^2}}T + \frac{2\phi'}{\sqrt{2\|W\|^2 + 4\phi'^2}}N + \frac{\|W\|(\sin\phi + \cos\phi)}{\sqrt{2\|W\|^2 + 4\phi'^2}}B,
\end{aligned}$$

and

$$\kappa_g^{\beta_8} = \frac{1}{\big(1 + 2\big(\frac{\phi'}{\|W\|}\big)^2\big)^{\frac{5}{2}}}\big(2\frac{\phi'}{\|W\|}\underline{\nabla}_1 - \underline{\nabla}_2 + \underline{\nabla}_3\big).$$

where the coefficients are

$$\begin{cases}
\underline{\nabla}_1 = \big(\frac{\phi'}{\|W\|}\big) + \big(\frac{\phi'}{\|W\|}\big)^3 + 2\big(\frac{\phi'}{\|W\|}\big)'\big(\frac{\phi'}{\|W\|}\big)\\
\underline{\nabla}_2 = -1 - 3\big(\frac{\phi'}{\|W\|}\big)^2 - 2\big(\frac{\phi'}{\|W\|}\big)^4 - \big(\frac{\phi'}{\|W\|}\big)'\\
\underline{\nabla}_3 = -\big(\frac{\phi'}{\|W\|}\big)^2 - 2\big(\frac{\phi'}{\|W\|}\big)^4 + \big(\frac{\phi'}{\|W\|}\big)'
\end{cases}$$

Proof. The proof is similar as in proof of Theorem 2.1.

Definition 9. *Let (B_1) be a spherical curve of α_1, B_1, T_{B_1} and $B_1 \wedge T_{B_1}$ be Sabban vectors of (B_1). Then β_9-Smarandache curve can be identified as*

$$\beta_9(s) = \frac{1}{\sqrt{3}}(B_1 + T_{B_1} + B_1 \wedge T_{B_1}). \tag{2.18}$$

Theorem 9. *Let α_1 be a Mannheim partner of α. Then Sabban apparatus according to β_9-Smarandache curve of Mannheim curve are,*

$$\beta_9(s) = \frac{1}{\sqrt{3}}\big((\sin\phi+\cos\phi)T+N+(\cos\phi-\sin\phi)B\big),$$

$$\begin{aligned} T_{\beta_9}(s) &= \frac{\phi'\sin\phi-\big(\phi'-\|W\|\big)\cos\phi}{\sqrt{2(\|W\|^2-\|W\|\phi'+{\phi'}^2)}}T-\frac{\|W\|}{\sqrt{2(\|W\|^2-\|W\|\phi'+{\phi'}^2)}}N \\ &+\frac{\phi'\cos\phi+(\phi'-\|W\|)\sin\phi}{\sqrt{2(\|W\|^2-\|W\|\phi'+{\phi'}^2)}}B, \end{aligned}$$

$$\begin{aligned} \big(\beta_9\wedge T_{\beta_9}\big)(s) &= \frac{(2\|W\|-\phi')\sin\phi-(\|W\|+\phi')\cos\phi}{\sqrt{6\|W\|^2-6\|W\|\phi'+6{\phi'}^2}}T+\frac{2\phi'-\|W\|}{\sqrt{6\|W\|^2-6\|W\|\phi'+6{\phi'}^2}}N \\ &+\frac{(\|W\|+\phi')\sin\phi+(2\|W\|-\phi')\cos\phi}{\sqrt{6\|W\|^2-6\|W\|\phi'+6{\phi'}^2}}B, \end{aligned}$$

and

$$\kappa_g^{\beta_9}=\frac{1}{4\sqrt{2}\Big(1+\frac{\phi'}{\|W\|}+\big(\frac{\phi'}{\|W\|}\big)^2\Big)^{\frac{5}{2}}}\Big((2\frac{\phi'}{\|W\|}-1)\flat_1-(1+\frac{\phi'}{\|W\|})\flat_2+(2-\frac{\phi'}{\|W\|})\flat_3\Big).$$

where the coefficients are

$$\begin{cases} \flat_1=-2+4\frac{\phi'}{\|W\|}+4\big(\frac{\phi'}{\|W\|}\big)-\big(\frac{\phi'}{\|W\|}\big)^2+2\big(\frac{\phi'}{\|W\|}\big)^3+\big(\frac{\phi'}{\|W\|}\big)'\big(2\frac{\phi'}{\|W\|}-1\big) \\ \flat_2=-2+2\frac{\phi'}{\|W\|}-4\big(\frac{\phi'}{\|W\|}\big)^2+\big(\frac{\phi'}{\|W\|}\big)^3-2\big(\frac{\phi'}{\|W\|}\big)^4-\big(\frac{\phi'}{\|W\|}\big)'\big(1+\frac{\phi'}{\|W\|}\big) \\ \flat_3=2\frac{\phi'}{\|W\|}-4\big(\frac{\phi'}{\|W\|}\big)^2+4\big(\frac{\phi'}{\|W\|}\big)^3-2\big(\frac{\phi'}{\|W\|}\big)^4+\big(\frac{\phi'}{\|W\|}\big)'\big(2-\frac{\phi'}{\|W\|}\big) \end{cases}$$

Proof. The proof is similar as in proof of Theorem 2.1.

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