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THE APPLICATION OF A NEW TYPE OF CONNECTION ON 3-DIMENSIONAL QUASI-SASAKIAN MANIFOLDS

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Abstract

We first introduce a new type of quarter-symmetric non-metric connection on almost contact metric manifolds. Using this connection, we study certain curvature conditions on 3-dimensional quasi-Sasakian manifolds e.g. $(\tilde{R}(X,\xi).\tilde{R})(Y,V)W=0$, $(\tilde{P}(X,\xi).\tilde{H})(Y,V)W=0$, $(\tilde{R}(X,\xi).\tilde{S})(Y,Z)=0$, $(\tilde{H}(X,\xi).\tilde{S})(Y,Z)=0, and$ $(\tilde{P}(X,\xi).\tilde{S})(Y,Z)$=0. Finally we give an example of 3-dimensional quasi-Sasakian manifold satisfies this new type of connection.

Keywords: Quarter-Symmetric Non-Metric Connection, 3-dimensional quasi-Sasakian manifolds, η-parallel

1. Introduction

In Riemannian geometry, the most commonly used connection is the Levi-Civita connection, which is uniquely characterized by being torsion-free and metric-compatible. Connections can be generalized to broader contexts, such as affine, semi-symmetric, or quarter-symmetric connections, each of which introduces distinct geometric features through their torsion or non-metric properties.

The concept of a semi-symmetric connection was first introduced by K. Yano in 1970 [33] to generalize the Levi-Civita connection by allowing torsion in a specific form. A linear connection $\tilde{\nabla}$ on $n=2m+1$ -dimensional Riemannian manifold M is said to be a semi-symmetric connection if its torsion tensor $\tilde{T}$

$$\tilde{T}(X,Y)=\tilde{\nabla}_X Y-\tilde{\nabla}_Y X-[X,Y], \tag{1.1}$$

satisfies

$$\tilde{T}(X,Y)=\eta(Y)X-\eta(X)Y,$$

where η is a non-zero 1-form. Moreover if $\tilde{\nabla} g = 0$, then the connection is called a metric connection, otherwise it is non-metric connection defined by H.A. Hayden [15]. This type of connection has been extensively studied in various geometric settings, due to its applications in Riemannian geometry and theoretical physics [1], [8], {12], [16], [18], [27]. To further generalize the concept of a semi-symmetric connection, S. Golab introduced the notion of quarter-symmetric connection [13]. A linear connection $\tilde{\nabla}$ on M is called a quarter-symmetric connection if its torsion tensor $\tilde{T}$ satisfies the following condition

$$\bar{T}(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y,$$

where η is a non-zero 1-form and φ is a tensor of type (1,1). Just like the semi-symmetric connection, the quarter-symmetric connection has also been studied by many researchers [17, 30, 32, 36].

In the context of contact geometry, one important class of manifolds is the almost contact metric manifold. There are $n = 2m+1$-dimensional differentiable manifolds endowed with an almost contact metric structure (φ, ξ, η, g), where φ, ξ, η are tensor fields on M of types (1,1), (1,0), (0,1), respectively, such that [4, 5, 35]:

$$\varphi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \ \ X, Y \in T(M) \qquad (1.2)$$

where $T(M)$ is the Lie algebra of vector fields of the manifold M. Also

$$\varphi\xi = 0, \eta \circ \varphi = 0, \eta(X) = g(X,\xi). \qquad (1.3)$$

This structure serves as a generalization of almost complex structures to odd-dimensional manifolds and provides a natural setting to study various contact metric geometries.

A significant subclass of almost contact metric manifolds is formed by quasi-Sasakian manifolds. Quasi-Sasakian manifolds were first studied by Blair [6] using almost contact metric manifolds. They have been extensively investigated by many researchers, including [9, 10, 14, 19, 20, 21, 22, 23, 24, 26]. Following basic concepts to be used in this paper have been obtained in previous works.

The manifold M is said to be quasi-Sasakian if the almost contact structure (φ, ξ, η) is normal and the fundamental 2-form Φ is closed, that is,

$$[\varphi, \varphi](X,Y) + d\eta(X,Y)\xi = 0, \ \ and \ \ d\Phi = 0, \ \ \Phi(X,Y) = g(X, \varphi Y). \qquad (1.4)$$

An almost contact metric manifold M is a 3-dimensional quasi-Sasakian manifold if and only if [25]

$$\nabla_X \xi = -\beta\varphi X, \qquad (1.5)$$

for a certain function β on M. Since in a 3-dimensional quasi-Sasakian manifold (1.5) holds, it can written [25]

$$(\nabla_X \varphi)Y = \beta[g(X,Y)\xi - \eta(Y)X], \qquad (1.6)$$

for a certain function β on M, such that $\xi\beta = 0$ and ∇ being the operator of the covariant differentiation with respect to the Riemannian connection of M. Such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$.

Also in 3-dimensional quasi-Sasakian manifolds [25],

$$R(X,Y)\xi = -(X\beta)\varphi Y + (Y\beta)\varphi X + \beta^2[\eta(Y)X - \eta(X)Y], \quad (1.7)$$
$$R(Y,\xi)Z = -R(\xi,Y)Z = (Z\beta)\varphi Y + g(Y,\varphi Z)\mathrm{grad}\beta + \beta^2[g(Y,Z)\xi - \eta(Z)Y], \quad (1.8)$$
$$R(Y,\xi)\xi = \beta^2[Y - \eta(Y)\xi], \quad (1.9)$$
$$R(\xi,\xi)\xi = 0, \quad (1.10)$$
$$S(Y,Z) = \left(\frac{\tau}{2} - \beta^2\right) g(Y,Z) + \left(3\beta^2 - \frac{\tau}{2}\right) \eta(Y)\eta(Z) - (\varphi Z\beta)\eta(Y) - (\varphi Y\beta)\eta(Z), \quad (1.11)$$
$$S(X,\xi) = 2\beta^2\eta(X) - (\varphi X\beta), \quad (1.12)$$
$$QX = \left(\frac{\tau}{2} - \beta^2\right) X + \left(3\beta^2 - \frac{\tau}{2}\right) \eta(X)\xi + \eta(X)(\varphi\mathrm{grad}\beta) - (\varphi X\beta)\xi, \quad (1.13)$$
$$\begin{aligned} R(X,Y)Z = {} & \left(\frac{r}{2} - \beta^{\,2}\right) [g(Y,Z)X - g(X,Z)Y] + (\varphi X\beta)\eta(Z)Y - (\varphi Y\beta)\eta(Z)X \\ & + \left(3\beta^2 - \frac{\tau}{2}\right) [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ & + g(Y,Z)[\eta(X)\varphi\mathrm{grad}\beta - (\varphi X\beta)\xi] + g(X,Z)[(a\varphi Y\beta)\xi - \eta(Y)\varphi\mathrm{grad}\beta] \\ & + (\varphi Z\beta)[\eta(X)Y - \eta(Y)X], \end{aligned} \quad (1.14)$$

where R is Riemannian curvature tensor, S is Ricci tensor, Q is the Ricci operator, that is, $S(X,Y) = g(QX,Y)$, τ is the scalar curvature of the manifold and the gradient of a function β is related to by the formula $(X\beta) = g(\mathrm{grad}\beta, X)$.

In this paper, we consider the projective curvature tensor P and the concircular curvature tensor H are defined by [34], [37]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \quad (1.15)$$

and

$$H(X,Y)Z = R(X,Y)Z - \frac{\tau}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \quad (1.16)$$

for all vector fields X, Y, Z, both of which have been extensively studied by various researchers [11], [2], [3].

Moreover the projective tensor and concircular tensor on 3-dimensional quasi-Sasakian manifold are given by [25]

$$P(X,Y)\xi = -(X\beta)\varphi Y + (Y\beta)\varphi X + \frac{1}{2}[(\varphi Y\beta)X - (\varphi X\beta)X], \quad (1.17)$$
$$\begin{aligned} -P(\xi,X)Z = P(X,\xi)Z = {} & -(Z\beta)\varphi X - g(X,\varphi Z)\mathrm{grad}\,\beta - \frac{1}{2}\begin{bmatrix} -(\varphi Z\beta)X + (\varphi X\beta)Z \\ +(\varphi Z\beta)\eta(X)\xi \end{bmatrix} \\ & + \left[\frac{\tau}{4} - \frac{3\beta^2}{2}\right](g(X,Z)\xi - \eta(X)\eta(Z)\xi) \end{aligned} \quad (1.18)$$
$$H(X,Y)\xi = -(X\beta)\varphi Y + (Y\beta)\varphi X + \left[\beta^2 - \frac{\tau}{6}\right](\eta(Y)X - \eta(X)Y), \quad (1.19)$$
$$H(X,\xi)\xi = \left[\beta^2 - \frac{\tau}{6}\right](X - \eta(X)\xi), \quad (1.20)$$
$$H(X,\xi)Z = -(Z\beta)\varphi X - g(X,\varphi Z)\mathrm{grad}\beta + \left[\beta^2 - \frac{\tau}{6}\right](\eta(Z)X - g(X,Z)\xi), \quad (1.21)$$

The present paper is organized as follows: In Section 2 we establish a new type of a quarter-symmetric non-metric connection in an almost contact manifolds and prove its existence. Also we find Riemannian curvature tensor with respect to this connection. In Section 3, we classify the connection defined in Section 2. In Section 4, we investigate some curvature conditions on 3-dimensional quasi-Sasakian manifolds e.g. $(\tilde{R}(X,\xi).\tilde{R})(Y,V)W = 0$, $(\tilde{P}(X,\xi).\tilde{H})(Y,V)W = 0$, $(\tilde{R}(X,\xi).\tilde{S})(Y,Z) = 0$, $(\tilde{H}(X,\xi).\tilde{S})(Y,Z) = 0$, and $(\tilde{P}(X,\xi).\tilde{S})(Y,Z) = 0$. Finally, we find an example of a 3-dimensional Quasi-Sasakian manifold admitting the quarter symmetric non- metric connection.

2. A new type of quarter-symmetric connection

In this section, we define a new type of connection in an almost contact metric manifold as follows:

Theorem 2.1. Let M be an $n = 2m + 1$- dimensional almost contact metric manifold with the Levi-Civita connection ∇ of its Riemannian metric g. Let Φ_1, Φ_2 and Φ_3 are skew-symmetric parts of the (0,2) tensor Φ such that

$$g(\varphi X, Y) \equiv \Phi(X,Y) = \Phi_1(X,Y) + \Phi_2(X,Y) + \Phi_3(X,Y). \tag{2.1}$$

Also let φ_1, φ_2 and φ_3 are (1,1)-type tensor fields, η is a 1-form and ξ is a vector field which satisfy

$$g(\varphi_1 X, Y) + g(\varphi_2 X, Y) + g(\varphi_3 X, Y) = \Phi_1(X,Y) + \Phi_2(X,Y) + \Phi_3(X,Y), \tag{2.2}$$

$$\eta(X) = g(X,\xi),\ \ \varphi_1\xi = \varphi_2\xi = \varphi_3\xi = \varphi\xi = 0, \tag{2.3}$$

and

$$\eta(\varphi_1 X) = \eta(\varphi_2 X) = \eta(\varphi_3 X) = 0. \tag{2.4}$$

Then there exists a quarter-symmetric non-metric connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\varphi_1 Y + \eta(Y)\varphi_2 X + \frac{1}{2}[\eta(Y)\varphi_3 X - \eta(X)\varphi_3 Y] - f_1 g(X,Y)\xi, \tag{2.5}$$

satisfying the conditions

$$\tilde{T}(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y, \tag{2.6}$$

and

$$(\tilde{\nabla}_X g)(Y,Z) = f_1[g(X,Y)\eta(Z) + g(X,Z)\eta(Y)] - \frac{1}{2}[\eta(Y)g(\varphi_3 X,Z) + \eta(Z)g(\varphi_3 X,Y)]$$
$$-\eta(Y)g(\varphi_2 X,Z) - \eta(Z)g(\varphi_2 X,Y) \neq 0, \tag{2.7}$$

where f_1 is a differentiable function on M.

Proof. Let $\tilde{\nabla}$ be a linear connection in M given by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{2.8}$$

where $B(X,Y)$ is a (1,1) tensor field in M.
We can find the tensor field $B(X,Y)$ such that $\tilde{\nabla}$ ensures (2.6) and (2.7). Using (2.8) in (1.1), we obtain

$$\tilde{T}(X,Y) = B(X,Y) - B(Y,X). \tag{2.9}$$

Denote

$$G(X,Y,Z) = (\tilde{\nabla}_X g)(Y,Z). \tag{2.10}$$

From (2.8) it follows that

$$G(X,Y,Z) = -g(B(X,Y),Z) - g\big(Y,B(X,Z)\big). \qquad (2.11)$$

Now using (2.9), we have

$$g\big(\tilde{T}(X,Y),Z\big) + g\big(\tilde{T}(Z,Y),X\big) + g\big(\tilde{T}(Z,X),Y\big)$$
$$= g(B(X,Y),Z) - g(B(Y,X),Z) + g(B(Z,Y),X) - g(B(Y,Z),X)$$
$$+ g(B(Z,X),Y) - g(B(X,Z),Y).$$

In the above equation taking (2.11), we write

$$g\big(\tilde{T}(X,Y),Z\big) + g\big(\tilde{T}(Z,Y),X\big) + g\big(\tilde{T}(Z,X),Y\big)$$
$$= g(B(X,Y),Z) + G(Y,X,Z) - G(Z,X,Y) - g(B(X,Z),Y)$$
$$= 2g(B(X,Y),Z) + G(X,Y,Z) + G(Y,X,Z) - G(Z,X,Y). \qquad (2.12)$$

If we put (2.7) in (2.12), we get

$$g(\bar{T}(X,Y),Z) + g(\bar{T}(Z,X),Y) + g(\bar{T}(Z,Y),X)$$
$$= 2g(B(X,Y),Z) - \frac{1}{2}\eta(Y)g(\varphi_3X,Z) - \frac{1}{2}\eta(Z)g(\varphi_3X,Y) + f_1g(X,Y)\eta(Z)$$
$$+ f_1g(X,Z)\eta(Y) - \eta(Y)g(\varphi_2X,Z) + \eta(Y)g(\varphi_2Z,X) - \frac{1}{2}\eta(X)g(\varphi_3Y,Z)$$
$$- \frac{1}{2}\eta(Z)g(\varphi_3Y,X) + f_1g(Y,Z)\eta(X) + f_1g(X,Y)\eta(Z) - \eta(X)g(\varphi_2Y,Z)$$
$$- \eta(Z)g(X,\varphi_2Y) + \frac{1}{2}\eta(X)g(\varphi_3Z,Y) + \frac{1}{2}\eta(Y)g(\varphi_3X,Z) - f_1g(Z,X)\eta(Y)$$
$$- f_1g(Z,Y)\eta(X) + \eta(X)g(\varphi_2Z,Y) + \eta(Y)g(X,\varphi_2Z). \qquad (2.13)$$

Since φ_1, φ_2 *and* φ_3 are skew-symmetric therefore (2.13) reduces to

$$g\big(\tilde{T}(X,Y),Z\big) + g\big(\tilde{T}(Z,Y),X\big) + g\big(\tilde{T}(Z,X),Y\big)$$
$$= 2g(B(X,Y),Z) + \eta(Y)g(\varphi_3Z,X) - \eta(X)g(\varphi_3Y,Z) + 2f_1g(X,Y)\eta(Z)$$
$$- 2\eta(Y)g(\varphi_2X,Z) - 2\eta(X)g(\varphi_2Y,Z). \qquad (2.14)$$

By (2.14), we obtain

$$g(B(X,Y),Z) = \frac{1}{2}\{g\big(\tilde{T}(X,Y),Z\big) + g\big(\tilde{T}(Z,X),Y\big) + g\big(\tilde{T}(Z,Y),X\big)\} - f_1g(X,Y)\eta(Z)$$
$$- \frac{1}{2}\{\eta(Y)g(\varphi_3Z,X) - \eta(X)g(\varphi_3Y,Z)\} + \eta(Y)g(\varphi_2X,Z)$$
$$+ \eta(X)g(\varphi_2Y,Z) \qquad (2.15)$$

On the other hand (2.6) implies the following relations

$$g\big(\tilde{T}(X,Y),Z\big) = \eta(Y)\sum_{i=1}^{3} g(\varphi_iX,Z) - \eta(X)\sum_{i=1}^{3} g(\varphi_iY,Z),$$
$$g\big(\tilde{T}(Z,X),Y\big) = \eta(X)\sum_{i=1}^{3} g(\varphi_iZ,Y) - \eta(Z)\sum_{i=1}^{3} g(\varphi_iY,X),$$
$$g\big(\tilde{T}(Z,Y),X\big) = \eta(Y)\sum_{i=1}^{3} g(\varphi_iZ,X) - \eta(Z)\sum_{i=1}^{3} g(\varphi_iY,X).$$

Combining the above equations with (2.15), we have

$$g(B(X,Y),Z)=\frac{1}{2}\Bigg[\eta(Y)\sum_{i=1}^{3}g(\varphi_iX,Z)-\eta(X)\sum_{i=1}^{3}g(\varphi_iY,Z)+\eta(X)\sum_{i=1}^{3}g(\varphi_iZ,Y)$$
$$-\eta(Z)\sum_{i=1}^{3}g(\varphi_iY,X)+\eta(Y)\sum_{i=1}^{3}g(\varphi_iZ,X)-\eta(Z)\sum_{i=1}^{3}g(\varphi_iY,X)\Bigg]$$
$$-f_1g(X,Y)\eta(Z)-\frac{1}{2}[\eta(Y)g(\varphi_3Z,X)-\eta(X)g(\varphi_3Y,Z)]+\eta(Y)g(\varphi_2X,Z)$$
$$+\eta(X)g(\varphi_2Y,Z),$$

from which

$$B(X,Y)=-\eta(X)\varphi_1Y+\eta(Y)\varphi_2X+\frac{1}{2}[\eta(Y)\varphi_3X-\eta(X)\varphi_3Y]-f_1g(X,Y)\xi, \qquad (2.16)$$

for $\forall Z\in\chi(M)$. This completes the proof.

Lemma 2.2. Let M be an n = 2m+1-dimensional almost contact metric manifold. Then we can give a relation between Riemannian curvature tensors R and $\widetilde{R}$ respectively Levi-Civita connection ∇ and quarter-symmetric non-metric connection $\widetilde{\nabla}$ which defined by (2.5) on M such that

$$\widetilde{R}(X,Y)Z=R(X,Y)Z-g(\nabla_X\xi,Y)\varphi_1Z-\eta(Y)(\nabla_X\varphi_1)Z+\eta(Y)f_1g(X,\varphi_1Z)\xi+g(\nabla_X\xi,Z)\varphi_2Y$$
$$+\eta(Z)(\nabla_X\varphi_2)Y-\frac{1}{2}\eta(X)\eta(Z)\varphi_3\varphi_2Y-\eta(X)\eta(Z)\varphi_1\varphi_2Y-\eta(Z)f_1g(X,\varphi_2Y)\xi$$
$$+\frac{1}{2}g(\nabla_X\xi,Z)\varphi_3Y+\frac{1}{2}\eta(Z)(\nabla_X\varphi_3)Y-\frac{1}{4}\eta(X)\eta(Z)\varphi_3\varphi_3Y$$
$$-\frac{1}{2}\eta(X)\eta(Z)\varphi_1\varphi_3Y-\frac{1}{2}f_1\eta(Z)g(X,\varphi_3Y)\xi-\frac{1}{2}g(\nabla_X\xi,Y)\varphi_3Z$$
$$-\frac{1}{2}\eta(Y)(\nabla_X\varphi_3)Z+\frac{1}{2}f_1\eta(Y)g(X,\varphi_3Z)\xi-(Xf_1)g(Y,Z)\xi-f_1g(Y,Z)\nabla_X\xi$$
$$-\frac{1}{2}f_1g(Y,Z)\varphi_3X-f_1g(Y,Z)\varphi_2X+f_1{}^2g(Y,Z)\eta(X)\xi+g(\nabla_Y\xi,X)\varphi_1Z$$
$$+\eta(X)(\nabla_Y\varphi_1)Z-\eta(X)f_1g(Y,\varphi_1Z)\xi-g(\nabla_Y\xi,Z)\varphi_2X-\eta(Z)(\nabla_Y\varphi_2)X$$
$$+\frac{1}{2}\eta(Y)\eta(Z)\varphi_3\varphi_2X+\eta(Y)\eta(Z)\varphi_1\varphi_2X+\eta(Z)f_1g(Y,\varphi_2X)\xi$$
$$-\frac{1}{2}g(\nabla_Y\xi,Z)\varphi_3X-\frac{1}{2}\eta(Z)(\nabla_Y\varphi_3)X+\frac{1}{4}\eta(Y)\eta(Z)\varphi_3\varphi_3X$$
$$+\frac{1}{2}\eta(Y)\eta(Z)\varphi_1\varphi_3X+\frac{1}{2}\eta(Z)f_1g(Y,\varphi_3X)\xi+\frac{1}{2}g(\nabla_Y\xi,X)\varphi_3Z$$
$$+\frac{1}{2}\eta(X)(\nabla_Y\varphi_3)Z-\eta(X)\frac{1}{2}f_1g(Y,\varphi_3Z)\xi+(Yf_1)g(X,Z)+f_1g(X,Z)\nabla_Y\xi$$
$$+\frac{1}{2}f_1g(X,Z)\varphi_3Y+f_1g(X,Z)\varphi_2Y-f_1{}^2g(Y,Z)\eta(X)\xi(2.17)$$

for $X,Y,Z\in\chi(M)$.

Proof. The Riemannian curvature tensor $\tilde{R}$ is defined by

$$\tilde{R}(X,Y)Z=\widetilde{\nabla}_X\widetilde{\nabla}_YZ-\widetilde{\nabla}_Y\widetilde{\nabla}_XZ-\widetilde{\nabla}_{[X,Y]}Z, \qquad (2.18)$$

where $\widetilde{\nabla}$ is quarter-symmetric non-metric connection for $X,Y,Z\in\chi(M)$. With the helping of (2.5), the relation (1.18) converts to

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\left(\nabla_Y Z - \eta(Y)\varphi_1 Z + \eta(Z)\varphi_2 Y + \frac{1}{2}(\eta(Z)\varphi_3 Y - \eta(Y)\varphi_3 Z) - f_1 g(Y,Z)\xi\right)$$
$$- \tilde{\nabla}_Y\left(\nabla_X Z - \eta(X)\varphi_1 Z + \eta(Z)\varphi_2 X + \frac{1}{2}(\eta(Z)\varphi_3 X - \eta(X)\varphi_3 Z)\right.$$
$$\left. - f_1 g(X,Z)\xi\right) - (\nabla_{[X,Y]}Z - \eta([X,Y])\varphi_1 Z + \eta(Z)\varphi_2[X,Y]$$
$$+ \frac{1}{2}(\eta(Z)\varphi_3[X,Y] - \eta([X,Y])\varphi_3 Z) - f_1 g([X,Y],Z)\xi).$$

From linearity of $\tilde{\nabla}$ and (2.5), it follows that

$\tilde{R}(X,Y)Z$=$\nabla_X\nabla_Y Z + \frac{1}{2}\eta(\nabla_Y Z)\varphi_3 X - \frac{1}{2}\eta(X)\varphi_3\nabla_Y Z + \eta(\nabla_Y Z)\varphi_2 X - \eta(X)\varphi_1\nabla_Y Z -$
$f_1 g(X,\nabla_Y Z)\xi - \nabla_X\eta(Y)\varphi_1 Z - \frac{1}{2}\eta(\eta(Y)\varphi_1 Z)\varphi_3 X + \frac{1}{2}\eta(X)\eta(Y)\varphi_1\varphi_3 Z -$
$\eta(\eta(Y)\varphi_1 Z)\varphi_2 X + \eta(X)\eta(Y)\varphi_1\varphi_1 Z + f_1 g(X,\eta(Y)\varphi_1 Z)\xi + \nabla_X\eta(Z)\varphi_2 Y +$
$\frac{1}{2}\eta(\eta(Z)\varphi_2 Y)\varphi_3 X - \frac{1}{2}\eta(X)\eta(Z)\varphi_3\varphi_2 Y - \eta(X)\eta(Z)\varphi_1\varphi_2 Y + \eta(\eta(Z)\varphi_2 Y)\varphi_2 X -$
$f_1 g(X,\eta(Z)\varphi_2 Y)\xi + \frac{1}{2}\nabla_X\eta(Z)\varphi_3 Y + \frac{1}{4}\eta(\eta(Z)\varphi_3 Y)\varphi_3 X - \frac{1}{4}\eta(X)\eta(Z)\varphi_3\varphi_3 Y +$
$\frac{1}{2}\eta(\eta(Z)\varphi_3 Y)\varphi_2 X - \frac{1}{2}\eta(X)\eta(Z)\varphi_1\varphi_3 Y - \frac{1}{2}f_1 g(X,\eta(Z)\varphi_3 Y)\xi - \frac{1}{2}\nabla_X\eta(Y)\varphi_3 Z -$
$\frac{1}{4}\eta(\eta(Y)\varphi_3 Z)\varphi_3 X + \frac{1}{4}\eta(X)\eta(Y)\varphi_3\varphi_3 Z - \frac{1}{2}\eta(\eta(Y)\varphi_3 Z)\varphi_2 X + \frac{1}{2}\eta(X)\eta(Y)\varphi_1\varphi_3 Z +$
$\frac{1}{2}f_1 g(X,\eta(Y)\varphi_3 Z)\xi - \nabla_X f_1 g(Y,Z)\xi - \frac{1}{2}\eta(f_1 g(Y,Z)\xi)\varphi_3 X + \frac{1}{2}\eta(X)f_1 g(Y,Z)\varphi_3\xi -$
$\eta(f_1 g(Y,Z)\xi)\varphi_2 X + \eta(X)f_1 g(Y,Z)\varphi_1\xi + {f_1}^2 g(Y,Z)\eta(X)\xi - \nabla_Y\nabla_X Z - \frac{1}{2}\eta(\nabla_X Z)\varphi_3 Y +$
$\frac{1}{2}\eta(Y)\varphi_3\nabla_X Z - \eta(\nabla_X Z)\varphi_2 Y + \eta(Y)\varphi_1\nabla_X Z + f_1 g(Y,\nabla_X Z)\xi + \nabla_Y\eta(X)\varphi_1 Z +$
$\frac{1}{2}\eta(\eta(X)\varphi_1 Z)\varphi_3 Y - \frac{1}{2}\eta(Y)\eta(X)\varphi_1\varphi_3 Z + \eta(\eta(X)\varphi_1 Z)\varphi_2 Y - \eta(Y)\eta(X)\varphi_1\varphi_1 Z -$
$f_1 g(Y,\eta(X)\varphi_1 Z)\xi - \nabla_Y\eta(Z)\varphi_2 X - \frac{1}{2}\eta(\eta(Z)\varphi_2 X)\varphi_3 Y + \frac{1}{2}\eta(Y)\eta(Z)\varphi_3\varphi_2 X +$
$\eta(Y)\eta(Z)\varphi_1\varphi_2 X - \eta(\eta(Z)\varphi_2 X)\varphi_2 Y + f_1 g(Y,\eta(Z)\varphi_2 X)\xi - \frac{1}{2}\nabla_Y\eta(Z)\varphi_3 X -$
$\frac{1}{4}\eta(\eta(Z)\varphi_3 X)\varphi_3 Y + \frac{1}{4}\eta(Y)\eta(Z)\varphi_3\varphi_3 X - \frac{1}{2}\eta(\eta(Z)\varphi_3 X)\varphi_2 Y + \frac{1}{2}\eta(Y)\eta(Z)\varphi_1\varphi_3 X +$
$\frac{1}{2}f_1 g(Y,\eta(Z)\varphi_3 X)\xi + \frac{1}{2}\nabla_Y\eta(X)\varphi_3 Z + \frac{1}{4}\eta(\eta(X)\varphi_3 Z)\varphi_3 Y - \frac{1}{4}\eta(Y)\eta(X)\varphi_3\varphi_3 Z +$
$\frac{1}{2}\eta(\eta(X)\varphi_3 Z)\varphi_2 Y - \frac{1}{2}\eta(Y)\eta(X)\varphi_1\varphi_3 Z - \frac{1}{2}f_1 g(Y,\eta(X)\varphi_3 Z)\xi + \nabla_Y f_1 g(X,Z)\xi +$
$\frac{1}{2}\eta(f_1 g(X,Z)\xi)\varphi_3 Y - \frac{1}{2}\eta(Y)f_1 g(X,Z)\varphi_3\xi + \eta(f_1 g(X,Z)\xi)\varphi_2 Y - \eta(Y)f_1 g(X,Z)\varphi_1\xi -$
${f_1}^2 g(Y,Z)\eta(X)\xi - \nabla_{[X,Y\}}Z + \eta(\nabla_X Y - \nabla_Y X)\varphi_1 Z - \eta(Z)\varphi_2\nabla_X Y + \eta(Z)\varphi_2\nabla_Y X -$
$\frac{1}{2}\{\eta(Z)\varphi_3\nabla_X Y - \eta(Z)\varphi_3\nabla_Y X - \eta(\nabla_X Y - \nabla_Y X)\varphi_3 Z\} + f_1 g(\nabla_X Y - \nabla_Y X, Z$

Applying properties of Levi-Civita connection ∇ and using (2.3) and (2.4), we arrive at (2.17) which proves assertion.

3. Classification of the connection given with (2.5)

In this section, we will give some special connection types depending on the connection defined with (2.5).

i) If $\varphi_1 = \varphi_2 = 0$, then (2.5) becomes

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}[\eta(Y)\varphi X - \eta(X)\varphi Y] - f_1 g(X,Y)\xi. \quad (3.1)$$

This connection is called a quarter symmetric non-metric connection on almost contact metric manifold. We call this connection as (M, φ_3, f_1).

Let's find Riemannian curvature tensor $\tilde{R}$ due to (3.1) applying $\varphi_1 = \varphi_2 = 0$ in (2.17).

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z &+ \frac{1}{2}g(\nabla_X\xi, Z)\varphi Y + \frac{1}{2}\eta(Z)(\nabla_X\varphi)Y \\ &- \frac{1}{4}\eta(X)\eta(Z)\varphi^2 Y - \frac{1}{2}f_1\eta(Z)g(X,\varphi Y)\xi - \frac{1}{2}g(\nabla_X\xi, Y)\varphi Z \\ &- \frac{1}{2}\eta(Y)(\nabla_X\varphi)Z + \frac{1}{2}f_1\eta(Y)g(X,\varphi Z)\xi - (Xf_1)g(Y,Z)\xi - f_1 g(Y,Z)\nabla_X\xi \\ &- \frac{1}{2}f_1 g(Y,Z)\varphi X + {f_1}^2 g(Y,Z)\eta(X)\xi - \frac{1}{2}g(\nabla_Y\xi, Z)\varphi X - \frac{1}{2}\eta(Z)(\nabla_Y\varphi)X \\ &+ \frac{1}{4}\eta(Y)\eta(Z)\varphi^2 X + \frac{1}{2}\eta(Z)f_1 g(Y,\varphi X)\xi + \frac{1}{2}g(\nabla_Y\xi, X)\varphi Z \\ &+ \frac{1}{2}\eta(X)(\nabla_Y\varphi)Z - \eta(X)\frac{1}{2}f_1 g(Y,\varphi Z)\xi + (Yf_1)g(X,Z)\xi + f_1 g(X,Z)\nabla_Y\xi \\ &+ \frac{1}{2}f_1 g(X,Z)\varphi Y - {f_1}^2 g(Y,Z)\eta(X)\xi. \qquad (3.2)\end{aligned}$$

ii) Using $\varphi_2 = \varphi_3 = 0$ and $f_1 = 0$ in (3.1), we obtain

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\varphi Y. \qquad (3.3)$$

This connection is called quarter symmetric metric connection [36]. It is easily seen that, the Riemannian curvature tensor is

$$\tilde{R}(X,Y)Z = R(X,Y)Z - g(\nabla_X\xi, Y)\varphi Z - \eta(Y)(\nabla_X\varphi)Z + g(\nabla_Y\xi, X)\varphi Z + \eta(X)(\nabla_Y\varphi)Z \quad (3.4)$$

with respect to (3.3).

iii) Setting $\varphi_1 = \varphi_3 = 0$ and $f_1 = 0$ in (2.5), the connection (2.5) reduces to

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi X. \qquad (3.4)$$

This connection is called quarter symmetric non-metric connection [30]. If we take $\varphi_1 = \varphi_3 = 0$ and $f_1 = 0$ in (2.17), then we obtain

$$\tilde{R}(X,Y)Z = R(X,Y)Z + g(\nabla_X\xi, Z)\varphi Y + \eta(Z)(\nabla_X\varphi)Y - g(\nabla_Y\xi, Z)\varphi X - \eta(Z)(\nabla_Y\varphi)X. \quad (3.5)$$

iv) Taking $\varphi_1 = \varphi_3 = 0, \varphi_2 X = X$ and $f_1 = 0$ in (2.5), then it becomes

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \qquad (3.6)$$

This connection is called semi-symmetric non-metric connection [1]. Riemannian curvature tensor $\tilde{R}$ is

$$\tilde{R}(X,Y)Z = R(X,Y)Z + g(Z, \nabla_X\xi)Y - g(Z, \nabla_Y\xi)X,$$

due to (3.6).

Table 1. Classification of the Connection

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\varphi_1 Y + \eta(Y)\varphi_2 X + \frac{1}{2}[\eta(Y)\varphi_3 X - \eta(X)\varphi_3 Y] - f_1 g(X,Y)\xi$$

Connection Name	Special Case	Connection
(M,φ_3,f_1) Quarter Symmetric Non-Metric Connection	$\varphi_1 = \varphi_2 = 0$	$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\eta(Y)\varphi X - \frac{1}{2}\eta(X)\varphi Y - f_1 g(X,Y)\xi$
Quarter Symmetric Metric Connection	$\varphi_2 = 0$ $\varphi_3 = 0$ and $f_1 = 0$	$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\varphi Y$
Quarter Symmetric Non-Metric Connection	$\varphi_1 = 0$, $\varphi_3 = 0$ and $f_1 = 0$	$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi X$
Semi Symmetric Non-Metric Connection	$\varphi_1 = 0$, $\varphi_2 X = X$, $\varphi_3 = 0$ *and* $f_1 = 0$	$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X$

4. 3-Dimensional quasi-Sasakian manifolds given with quarter symmetric non-metric connection (M,φ_3,f_1)

In this section, some curvature types will be studied in the 3-dimensional quasi-Sasakian manifolds given with (M,φ_3,f_1).

Lemma 4.1. Let M be a 3-dimensional quasi-Sasakian manifold. The relation between Riemannian curvature tensors R and $\tilde{R}$ with respect to the connections ∇ and quarter-symmetric non-metric connection (M,φ_3,f_1) defined by $\tilde{\nabla}$, is given by

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & R(X,Y)Z - (Xf_1)g(Y,Z)\xi + (Yf_1)g(X,Z)\xi + \frac{1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\ & + \left(f_1{}^2 + \frac{\beta}{2}\right)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] - \frac{1}{2}f_1\eta(X)g(\varphi Z,Y)\xi \\ & + \left(f_1\beta - \frac{f_1}{2}\right)[g(Y,Z)\varphi X - g(X,Z)\varphi Y] + \frac{1}{2}f_1\eta(Y)g(\varphi Z,X)\xi + \frac{\beta}{2}(2g(\varphi X,Y)\varphi Z \\ & + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y) \\ & + f_1\eta(Z)g(\varphi X,Y)\xi, \qquad (4.1)\end{aligned}$$

for $\forall X,Y,Z \in \chi(M)$.

Proof. Since M is a 3-dimensional quasi-Sasakian manifold, using the expressions (1.6), (1.5) and $\varphi_1 = \varphi_2 = 0$, (3.2) reduces to

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & R(X,Y)Z-(Xf_1)g(Y,Z)\xi+(Yf_1)g(X,Z)\xi+{f_1}^2[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi] \\ & +f_1\beta g(Y,Z)\varphi X-f_1\beta g(X,Z)\varphi X \\ & +\frac{1}{2}\Big[\beta\eta(Z)g(X,Y)\xi-\beta\eta(Z)\eta(Y)X-\beta\eta(Y)g(X,Z)\xi+\beta\eta(Z)\eta(Y)X \\ & +\beta\eta(X)g(Y,Z)\xi-\beta\eta(X)\eta(Z)Y+\beta\eta(X)g(Y,Z)\xi-\beta\eta(X)\eta(Z)Y \\ & -\beta\eta(Z)g(Y,X)\xi+\beta\eta(Z)\eta(X)Y+\beta g(\varphi X,Y)\varphi Z-\beta g(\varphi Y,X)\varphi Z \\ & +\beta g(\varphi Y,Z)\varphi X-\beta g(\varphi X,Z)\varphi Y+2f_1\eta(Z)g(\varphi X,Y)\xi-f_1\eta(X)g(\varphi Z,Y)\xi \\ & +f_1\eta(Y)g(\varphi Z,X)\xi-f_1g(Y,Z)\varphi X+f_1g(X,Z)\varphi Y+\frac{1}{2}\eta(X)\eta(Z)Y \\ & -\frac{1}{2}\eta(Y)\eta(Z)X\Big],\end{aligned}$$

which proves (4.1).

Lemma 4.2. In 3-dimensional quasi-Sasakian manifold given with (M,φ_3,f_1),

$$\begin{aligned}\tilde{R}(X,Y)\xi = {} & -(X\beta)\varphi Y+(Y\beta)\varphi X-(Xf_1)\eta(Y)\xi+(Yf_1)\eta(X)\xi \\ & +\left(f_1\beta-\frac{f_1}{2}\right)[\eta(Y)\varphi X-\eta(X)\varphi Y]+\left(\frac{1}{4}-\beta^2\right)[\eta(X)Y-\eta(Y)X] \\ & +f_1g(\varphi X,Y)\xi, \end{aligned} \tag{4.2}$$

$$\begin{aligned}\tilde{R}(X,\xi)Z = {} & -(Z\beta)\varphi X-g(X,\varphi Z)\mathrm{grad}\beta-\frac{f_1}{2}g(\varphi X,Z)\xi+\left(f_1\beta-\frac{f_1}{2}\right)\eta(Z)\varphi X-(Xf_1)\eta(Z)\xi \\ & +\left((\xi f_1)-\beta^2-{f_1}^2-\frac{\beta}{2}\right)g(X,Z)\xi+(\left(\frac{\beta}{2}+{f_1}^2+\frac{1}{4}\right)]\eta(X)\eta(Z)\xi \\ & +\left(\beta^2-\frac{1}{4}\right)\eta(Z)X, \end{aligned} \tag{4.3}$$

$$\tilde{R}(X,\xi)\xi = \left(f_1\beta-\frac{f_1}{2}\right)\varphi X-(Xf_1)\xi+\left[\frac{1}{4}-\beta^2\right](\eta(X)\xi-X)+(\xi f_1)\eta(X)\xi, \tag{4.4}$$

$$\tilde{R}(\xi,\xi)\xi=0, \tag{4.5}$$

$$\tilde{R}(X,Y)Z=-\tilde{R}(Y,X)Z, \tag{4.6}$$

$$\begin{aligned}\tilde{R}(X,Y)Z+\tilde{R}(Y,Z)X+\tilde{R}(Z,X)Y = {} & \frac{\beta}{2}\{4g(\varphi X,Y)\varphi Z+4g(\varphi Y,Z)\varphi X+4g(\varphi Z,X)\varphi Y\} \\ & +2f_1\eta(Z)g(\varphi X,Y)\xi. \end{aligned} \tag{4.7}$$

with the help of (4.7), we can prove the following lemma and remark:

Lemma 4.3. Let M be a 3-dimensional quasi-Sasakian manifold. Manifold M satisfies first Bianchi identity with respect to (M,φ_3,f_1) if and only if $\beta=0$, $f_1=0$, for $\forall X,Y,Z\in\chi(M)$. Notice that first Bianchi Identity with respect to ∇ is given by [35]

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0. \tag{4.8}$$

Remark 4.4. Let M be a 3-dimensional quasi-Sasakian manifold. Manifold satisfies first Bianchi identity with respect to (M,φ_3,f_1) if and only if $f_1=0$ and M is cosymplectic.

Lemma 4.5. Let M be a 3-dimensional quasi-Sasakian manifold. Ricci tensor $\tilde{S}$, Ricci operator $\tilde{Q}$ and scalar curvature $\tilde{\tau}$ with respect to (M,φ_3,f_1) are given by

$$\tilde{S}(Y,Z) = S(Y,Z) + (Yf_1)\eta(Z) + f_1(\beta-1)g(Y,\varphi Z) - \left(\frac{1}{2}+f_1^2\right)\eta(Y)$$
$$+ \left(f_1^2 - (\xi f_1)\right)g(Y,Z)$$
$$= (Yf_1)\eta(Z) + f_1(\beta-1)g(Y,\varphi Z) - (\varphi Z\beta)\eta(Y) - (\varphi Y\beta)\eta(Z)$$
$$+ \left[\frac{\tau}{2} - \beta^2 + f_1^2 - (\xi f_1)\right]g(Y,Z) + \left[3\beta^2 - \frac{\tau}{2} - \frac{1}{2} - f_1^2\right]\eta(Y)\eta(Z), \qquad (4.9)$$
$$\tilde{Q}Y = QY + (Yf_1)\xi - f_1(\beta-1)\varphi Y - \left(\frac{1}{2}+f_1^2\right)\eta(Y)\xi + \left(f_1^2 - (\xi f_1)\right)Y, \qquad (4.10)$$
$$\tilde{\tau} = \tau + 2\left(f_1^2 - (\xi f_1)\right) - \frac{1}{2}. \qquad (4.11)$$

Proof. Taking the inner product of both sides of the equation (4.1) with W, we obtain

$$g\left(\tilde{R}(X,Y)Z,W\right)$$
$$= g(R(X,Y)Z,W) - (Xf_1)g(Y,Z)\eta(W) + (Yf_1)g(X,Z)\eta(W)$$
$$+ \left(f_1^2 + \frac{\beta}{2}\right)[g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)]$$
$$+ \left(f_1\beta - \frac{f_1}{2}\right)[g(Y,Z)g(\varphi X,W) - g(X,Z)g(\varphi Y,W)]$$
$$+ \frac{\beta}{2}[2g(\varphi X,Y)g(\varphi Z,W) + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W)]$$
$$+ \frac{1}{4}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)] + f_1\eta(Z)g(\varphi X,Y)\eta(W)$$
$$- \frac{1}{2}f_1\eta(X)g(\varphi Z,Y)\eta(W) + \frac{1}{2}f_1\eta(Y)g(\varphi Z,X)\eta(W).$$

Putting $X = W = e_i$, $i = 1,2,3$ in the above equation, we have

$$\tilde{S}(Y,Z) = S(Y,Z) - (\xi f_1)g(Y,Z) + (Yf_1)\eta(Z) + \left(f_1^2 + \frac{\beta}{2}\right)[g(Y,Z) - \eta(Y)\eta(Z)]$$
$$- \left(f_1\beta - \frac{f_1}{2}\right)g(\varphi Y,Z) + \frac{\beta}{2}[-2g(\varphi Z,\varphi Y) + g(\varphi Y,\varphi Z)]$$
$$+ \frac{1}{4}[\eta(Y)\eta(Z) - 3\eta(Y)\eta(Z)] - \frac{1}{2}f_1 g(\varphi Z,Y),$$

from which we get (4.9). Also (4.9) yields

$$\tilde{Q}Y = QY + (Yf_1)\xi - f_1(\beta-1)\varphi Y - \left(\frac{1}{2}+f_1^2\right)\eta(Y)\xi + \left(f_1^2 - (\xi f_1)\right),$$

for every $Z \in \chi(M)$. Contracting Y and Z in (4.9), we get

$$\tilde{\tau} = \tau + (\xi f_1) - \left(\frac{1}{2}+f_1^2\right) + 3\left(f_1^2 - (\xi f_1)\right).$$

Using (4.9), we can prove following lemma:

Lemma 4.6 For 3-dimensional quasi-Sasakian manifold with respect to (M, φ_3, f_1)

$$\tilde{S}(Y,\xi) = (Yf_1) - (\varphi Y\beta) + \left[2\beta^2 - \frac{1}{2} - (\xi f_1)\right]\eta(Y), \qquad (4.12)$$
$$\tilde{S}(\xi,Z) = -(\varphi Z\beta) + \left[2\beta^2 - \frac{1}{2}\right]\eta(Z), \qquad (4.13)$$
$$\tilde{S}(\xi,\xi) = 2\beta^2 - \frac{1}{2}. \qquad (4.14)$$

Proof. By (4.9), putting ξ for Y, Z respectively it is obvious that (4.12), (4.13) and (4.14) are true.

Theorem 4.7 Let M be a 3-dimensional quasi-Sasakian manifold. The Ricci tensor of M is symmetric with respect to (M, φ_3, f_1) if and only if f_1 is constant and $\beta = 1$.

Proof. Assume that the Ricci tensor of M is symmetric with respect to (M, φ_3, f_1), that is

$$\tilde{S}(Y,Z) = \tilde{S}(Z,Y), \tag{4.15}$$

for $\forall Y, Z \in \chi(M)$. First of all, in order to find the expression $\tilde{S}(Z,Y)$, in (4.9) substitute Z instead of Y and Y instead of Z.

$$\tilde{S}(Z,Y). = S(Z,Y) + (Zf_1)\eta(Y) + f_1(\beta - 1)g(Z, \varphi Y) - \left[\frac{1}{2} + f_1^2\right]\eta(Z)\eta(Y)$$
$$+[f_1^2 - (\xi f_1)]g(Y,Z). \tag{4.16}$$

With the help of (4.9), (4.15) and (4.16), we can write

$$S(Y,Z) + (Yf_1)\eta(Z) + f_1(\beta - 1)g(Y, \varphi Z) - \left[\frac{1}{2} + f_1^2\right]\eta(Y)\eta(Z) + [f_1^2 - (\xi f_1)]g(Y,Z)$$
$$= S(Z,Y) + (Zf_1)\eta(Y) + f_1(\beta - 1)g(Z, \varphi Y) - \left[\frac{1}{2} + f_1^2\right]\eta(Z)\eta(Y)$$
$$+ [f_1^2 - (\xi f_1)]g(Y,Z).$$

After required calculations in the above equation, we get

$$(Yf_1)\eta(Z) + f_1(\beta - 1)g(Y, \varphi Z) = (Zf_1)\eta(Y) + f_1(\beta - 1)g(Z, \varphi Y),$$

or

$$(Yf_1)\eta(Z) + 2f_1(\beta - 1)g(Y, \varphi Z) - (Zf_1)\eta(Y) = 0.$$

From here we can easily see that f_1 is constant and $\beta = 1$. Conversely, suppose that f_1 is constant and $\beta = 1$. Using these values in (4.9), we have

$$\tilde{S}(Y,Z) = S(Y,Z) - \left[\frac{1}{2} + f_1^2\right]\eta(Y)\eta(Z) + f_1^2 g(Y,Z). \tag{4.17}$$

In (4.17), using symmetry property of Ricci tensor S and metric g respect to ∇, we obtain (4.15). This completes the proof.

With the help of (4.9), (4.1), (1.15), (1.16) and (4.11), we can give the following propositions.

Proposition 4.8. Let M be a $3-$dimensional quasi-Sasakian manifold. Projective curvature tensor $\tilde{P}$ and concircular curvature tensor $\tilde{H}$ with respect to (M, φ_3, f_1) are

$$\tilde{P}(X,Y)Z = P(X,Y)Z - (Xf_1)g(Y,Z)\xi + (Yf_1)g(X,Z)\xi + ({f_1}^2 + \frac{\beta}{2})[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] + (f_1\beta - \frac{f_1}{2})[g(Y,Z)\varphi X - g(X,Z)\varphi Y] + \frac{\beta}{2}[2g(\varphi X,Y)\varphi Z + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y] + \frac{{f_1}^2}{2}\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] + f_1\,\eta(Z)g(\varphi X,Y)\xi - \frac{1}{2}\,f_1\,\eta(X)g(\varphi Z,Y)\xi + \frac{1}{2}\,f_1\,\eta(Y)g(\varphi Z,X)\xi - \frac{1}{2}[(Yf_1)\eta(Z)X - (Xf_1)\eta(Z)Y] - \frac{1}{2}f_1(\beta - 1)[g(Y,\varphi Z)X - g(X,\varphi Z)Y] - \frac{1}{2}({f_1}^2 - (\xi f_1))[g(Y,Z)X - g(X,Z)Y)], \quad (4.18)$$

and

$$\widetilde{H}(X,Y)Z = H(X,Y)Z - (Xf_1)g(Y,Z)\xi + (Yf_1)g(X,Z)\xi + ({f_1}^2 + \frac{\beta}{2})[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] + (f_1\beta - \frac{f_1}{2})[g(Y,Z)\varphi X - g(X,Z)\varphi Y] + \frac{\beta}{2}[2g(\varphi X,Y)\varphi Z + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y] + \frac{1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] + f_1\,\eta(Z)g(\varphi X,Y)\xi - \frac{1}{2}f_1\,\eta(X)g(\varphi Z,Y)\xi + \frac{1}{2}f_1\,\eta(Y)g(\varphi Z,X)\xi - (\frac{{f_1}^2 - (\xi f_1)}{3} - \frac{1}{12})[g(Y,Z)X - g(X,Z)Y)]. \quad (4.19)$$

Proposition 4.9 In $3-$dimensional quasi-Sasakian manifold given with $(\mathrm{M}, \varphi_3, \mathrm{f}_1)$, the following relations hold:

$$-\widetilde{H}(\xi,X)Z = \widetilde{H}(X,\xi)Z = -(Z\beta)\varphi X - g(X,\varphi Z)grad\beta - (Xf_1)\eta(Z)\xi + \left(f_1\beta - \frac{f_1}{2}\right)\eta(Z)\varphi X + \left[f_1^2 + \frac{\beta}{2} + \frac{1}{4}\right]\eta(X)\eta(Z)\xi + \left[\beta^2 - \frac{\tau}{6} + \frac{(\xi f_1)}{3} - \frac{1}{6} - \frac{f_1^2}{3}\right]\eta(Z)X + \left[-\beta^2 - \frac{\beta}{2} + \frac{\tau}{6} - \frac{1}{12} + \frac{2(\xi f_1)}{3} - \frac{2f_1^2}{3}\right]g(Y,Z)\xi + \frac{f_1}{2}g(X,\varphi Z)\xi, \quad (4.20)$$

$$\widetilde{H}(X,Y)\xi = -(X\beta)\varphi Y + (Y\beta)\varphi X - (Xf_1)\eta(Y)\xi + (Yf_1)\eta(X)\xi + \left(f_1\beta - \frac{f_1}{2}\right)[\eta(Y)\varphi X - \eta(X)\varphi Y] + f_1 g(\varphi X,Y)\xi + \left[\beta^2 - \frac{\tau+1}{6} + \frac{(\xi f_1) - f_1^2}{3}\right][\eta(Y)X - \eta(X)Y], \quad (4.21)$$

$$\widetilde{H}(\xi,\xi)\xi = 0, \quad (4.22)$$

$$\tilde{P}(X,Y)\xi = -(X\beta)\varphi Y + (Y\beta)\varphi X + \frac{1}{2}[(\varphi Y\beta)X - (\varphi X\beta)Y - (Yf_1)X + (Xf_1)Y] - (Xf_1)\eta(Y)\xi + (Yf_1)\eta(X)\xi + \left(f_1\beta - \frac{f_1}{2}\right)[\eta(Y)\varphi X - \eta(X)\varphi Y] + f_1 g(\varphi X,Y)\xi + \{1\}\{2\}(\xi f\_\{1\})[\eta(Y)X - \eta(X)Y)], \quad (4.23)$$

$$-\tilde{P}(\xi, X)Z = \tilde{P}(X, \xi)Z$$
$$= -(Z\beta)\varphi X - g(X, \varphi Z)\text{grad}\beta + \left(f_1\beta - \frac{f_1}{2}\right)\eta(Z)\varphi X + \frac{1}{2}[(\varphi Z\beta)X$$
$$- (Xf_1\})\eta(Z)\xi - \eta(X)(\varphi Z\beta)\xi - \eta(Z)(\varphi X\beta)\xi + f_1\beta g(X, \varphi Z)\xi]$$
$$+ \left[\frac{\tau}{4} + \frac{(\xi f_1) - 3\beta^2 - \beta - f_1^2}{2}\right] g(X, Z)\xi$$
$$+ \left[-\frac{\tau}{4} + \frac{3\beta^2 + \beta + f_1^2}{2}\right] \eta(X)\eta(Z)\xi. \quad (4.24)$$

We assume that $f_1 = \beta$ for the following theorems.

Remark 4.10. A contact manifold M of dimension $n = 2m + 1$ with contact form η and associated metric g is called an η-Einstein manifold if the Ricci tensor S is given by [29]

$$S = ag + b\eta \otimes \eta. \quad (4.25)$$

Remark 4.11. For a $(0, k)$-tensor field, $k \geq 1$ and a $(0,4)$-tensor R, the $(0, k + 2)$ tensor fields $R.T$ and $Q(B, T)$ are defined by [7],

$$(R.T)(X_1, X_2, \ldots, X_k; X, Y) = (R(X, Y).T)(X_1, X_2, \ldots, X_k) = -T(R(X, Y)X_1, X_2, \ldots, X_k)$$
$$-, \ldots . -T(X_1, X_2, \ldots, X_{k-1}, R(X, Y)X_k), \quad (4.26)$$

and

$$Q(B, T)(X_1, X_2, \ldots, X_k; X, Y) = \big((X \wedge_B Y).T\big)(X_1, X_2, \ldots, X_k)$$
$$= -T((X \wedge_B Y)X_1, X_2, \ldots, X_k) - \ldots . -T((X \wedge_B Y)X_1, X_2, \ldots (X \wedge_B Y)X_{k-1}, X_k), \quad (4.27)$$

where $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the corresponding (1,3)-tensor of R and

$$(X \wedge_B Y)Z = B(Y, Z)X - B(X, Z)Y,$$

is and endomorphism for a symmetric (0,2)-tensor field B on M for $X, Y, Z \in \chi(M)$.

Theorem 4.12. Let M be a 3-dimensional quasi-Sasakian manifold satisfying the condition $(\tilde{R}(X, \xi).\tilde{S})(Y, Z) = 0$, where $\tilde{R}$ is Riemannian curvature tensor and $\tilde{S}$ is Ricci tensor with respect to quarter symmetric non-metric connection (M, φ_3, f_1). Then the manifold M is η $-$Einstein manifold with respect to ∇.

Proof. We suppose that a 3-dimensional quasi-Sasakian manifold M satisfies

$$\big(\tilde{R}(X, \xi).\tilde{S}\big)(Y, Z) = 0, \quad (4.28)$$

where $\tilde{R}$ is Riemannian curvature tensor respect to quarter symmetric non-metric connection $\tilde{\nabla}$. From the relation (4.26), this implies that

$$-\tilde{S}\big(\tilde{R}(X, \xi)Y, Z\big) - \tilde{S}\big(Y, \tilde{R}(X, \xi)Z\big) = 0,$$

or

$$\tilde{S}\big(\tilde{R}(X, \xi)Y, Z\big) + \tilde{S}\big(Y, \tilde{R}(X, \xi)Z\big) = 0, \quad (4.29)$$

for all $X, Y, Z \in \chi(M)$. In view of (4.3), it follows from (4.29) that

$$-(Y\beta)\tilde{S}(\varphi X,Z)-g(X,\varphi Y)\tilde{S}(grad\beta,Z)-\frac{\beta}{2}g(\varphi X,Y)\tilde{S}(\xi,Z)+\left(\beta^2-\frac{\beta}{2}\right)\eta(Y)\tilde{S}(\varphi X,Z)$$
$$-(X\beta)\eta(Y)\tilde{S}(\xi,Z)-\left(2\beta^2+\frac{\beta}{2}\right)g(X,Y)\tilde{S}(\xi,Z)-(Z\beta)\tilde{S}(Y,\varphi X)$$
$$+\left(\beta^2+\frac{\beta}{2}+\frac{1}{4}\right)\eta(X)\eta(Y)\tilde{S}(\xi,Z)+\left(\beta^2-\frac{1}{4}\right)\eta(Y)\tilde{S}(X,Z)$$
$$-g(X,\varphi Z)\tilde{S}(Y,\mathrm{grad}\beta)-\frac{\beta}{2}g(\varphi X,Z)\tilde{S}(Y,\xi)+\left(\beta^2-\frac{\beta}{2}\right)\eta(Z)\tilde{S}(Y,\varphi X)$$
$$-(X\beta)\eta(Z)\tilde{S}(Y,\xi)-\left(2\beta^2+\frac{\beta}{2}\right)g(X,Z)\tilde{S}(Y,\xi)+\left(\beta^2-\frac{1}{4}\right)\eta(Z)\tilde{S}(Y,X)$$
$$+\left(\beta^2+\frac{\beta}{2}+\frac{1}{4}\right)\eta(X)\eta(Z)\tilde{S}(Y,\xi)=0. \tag{4.30}$$

Helping of (4.9), (4.12) and (4.13), (4.30) yields that

$$-(Y\beta)((\varphi X\beta)\eta(Z)+\eta(Z)(X\beta)+\beta(\beta-1)g(\varphi X,\varphi Z)+\frac{\tau}{2}g(\varphi X,Z))-(Y\beta)((\varphi X\beta)\eta(Z)$$
$$+\eta(Z)(X\beta)+\beta(\beta-1)g(\varphi X,\varphi Z)+\frac{\tau}{2}g(\varphi X,Z))-g(X,\varphi Y)(\|grad\beta\|^2\eta(Z)+\frac{\tau}{2}(Z\beta)$$
$$+\beta(\beta-1)(\varphi Z\beta))-\frac{\beta}{2}g(\varphi X,Y)(-(\varphi Z\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Z))$$
$$+\left(\beta^2-\frac{\beta}{2}\right)\eta(Y)\left[(\varphi X\beta)\eta(Z)+\eta(Z)(X\beta)+\beta(\beta-1)g(\varphi X,\varphi Z)+\frac{\tau}{2}g(\varphi X,Z)\right]$$
$$-(X\beta)\eta(Y)\left[-(\varphi Z\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Z)\right]-\left(2\beta^2+\frac{\beta}{2}\right)g(X,Y)\left[-(\varphi Z\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Z)\right]$$
$$+\left(\beta^2-\frac{1}{4}\right)\eta(Y)\left[(X\beta)\eta(Z)-(\varphi Z\beta)\eta(X)-(\varphi X\beta)\eta(Z)+\beta(1-\beta)g(\varphi X,Z)+\frac{\tau}{2}g(X,Z)\right.$$
$$\left.+\left(2\beta^2-\frac{1}{2}-\frac{\tau}{2}\right)\eta(X)\eta(Z)\right]+\left(\beta^2+\frac{\beta}{2}+\frac{1}{4}\right)\eta(X)\eta(Y)\left[-(\varphi Z\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Z)\right]$$
$$-(Z\beta)\left[(X\beta)\eta(Y)-\beta(\beta-1)g(\varphi X,\varphi Y)+\frac{\tau}{2}g(\varphi X,Y)\right]$$
$$-g(X,\varphi Z)\left[\frac{\tau}{2}(Y\beta)-\beta(\beta-1)(\varphi Y\beta)\right]-\frac{\beta}{2}g(\varphi X,Z)\left[(Y\beta)-(\varphi Y\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Y)\right]$$
$$+\left(\beta^2-\frac{\beta}{2}\right)\eta(Z)\left[\eta(Y)(X\beta)-\beta(\beta-1)g(\varphi X,\varphi Y)+\frac{\tau}{2}g(\varphi X,Y)\right]$$
$$-(X\beta)\eta(Z)\left[(Y\beta)-(\varphi Y\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Y)\right]$$
$$-\left(2\beta^2+\frac{\beta}{2}\right)g(X,Z)\left[(Y\beta)-(\varphi Y\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Y)\right]$$
$$+\left(\beta^2-\frac{1}{4}\right)\eta(Z)\left[(Y\beta)\eta(X)-(\varphi X\beta)\eta(Y)-(\varphi Y\beta)\eta(X)+\beta(\beta-1)g(\varphi X,Y)+\frac{\tau}{2}g(X,Y)\right.$$
$$\left.+\left(2\beta^2-\frac{1}{2}-\frac{\tau}{2}\right)\eta(Y)\eta(X)\right]+\left(\beta^2+\frac{\beta}{2}+\frac{1}{4}\right)\eta(X)\eta(Z)\left[(Y\beta)-(\varphi Y\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Y)\right]$$
$$=0. \tag{4.31}$$

Putting $X=Z=e_i$ in (4.31), taking summation over i, we have

$$\left[\left(\beta^2-\frac{1}{4}\right)(\tau-2\beta-8\beta^2)+2\beta^2(\beta-1)\left(\beta-\frac{1}{2}\right)-\|\mathrm{grad}\beta\|^2\right]\eta(Y)+\left[-4\beta^2-\frac{\beta}{2}\right](Y\beta)$$
$$+[6\beta^2+\frac{3\beta}{2}](\varphi Y\beta)=0. \tag{4.32}$$

The relation (4.32) satisfies

$$[(\beta^2 - \frac{1}{4})(\tau - 2\beta - 8\beta^2) + 2\beta^2(\beta - 1)(\beta - \frac{1}{2}) - \| \text{grad}\beta \|^2]\xi + [-4\beta^2 - \frac{\beta}{2}]\text{grad}\beta - [6\beta^2 + \frac{3\beta}{2}]\varphi\text{grad}\beta = 0, \tag{4.33}$$

for $\forall Y \in \chi(M)$. Applying φ on both side of (4.33), we get

$$\left[-4\beta^2 - \frac{\beta}{2}\right]\varphi\text{grad}\beta + \left[6\beta^2 + \frac{3\beta}{2}\right]\text{grad}\beta = 0. \tag{4.34}$$

Using (4.33) and (4.34), we can write

$$(X\beta) = \frac{[(\beta^2 - \frac{1}{4})(\tau - 2\beta - 8\beta^2) + 2\beta^2(\beta - 1)(\beta - \frac{1}{2}) - \| \text{grad}\beta \|^2][4\beta^2 + \frac{\beta}{2}]}{[4\beta^2 + \frac{\beta}{2}]^2 + [6\beta^2 + \frac{3\beta}{2}]^2}\eta(X).$$

Thus the manifold is η −Einstein with respect to ∇ because of (1.11). This completes the proof.

Theorem 4.13. Let M be a 3-dimensional quasi-Sasakian manifold satisfying the condition $(\tilde{P}(X,\xi).\tilde{S})(Y,Z) = 0$, where $\tilde{P}$ is projective curvature tensor and $\tilde{S}$ is Ricci tensor respect to quarter symmetric non-metric connection (M, φ_3, f_1). Then the manifold M is η −Einstein manifold with respect to ∇.

Proof. Let us consider a 3-dimensional quasi-Sasakian manifold M satisfying condition

$$\left(\tilde{P}(X,\xi).\tilde{S}\right)(Y,Z) = 0. \tag{4.35}$$

Using (4.26), we can write

$$-\tilde{S}\left(\tilde{P}(X,\xi)Y,Z\right) - \tilde{S}\left(Y,\tilde{P}(X,\xi)Z\right) = 0,$$

or

$$\tilde{S}\left(\tilde{P}(X,\xi)Y,Z\right) + \tilde{S}\left(Y,\tilde{P}(X,\xi)Z\right) = 0, \tag{4.36}$$

for all $X, Y, Z \in \chi(M)$. It follows from (4.24) that

$$\begin{aligned} -(Y\beta)\tilde{S}(\varphi X,Z) - g(X,\varphi Y)\tilde{S}(\text{grad}\beta,Z) + \left(\beta^2 - \frac{\beta}{2}\right)\left[\eta(Y)\tilde{S}(\varphi X,Z) + \eta(Z)\tilde{S}(Y,\varphi X)\right] \\ + \left[2\beta^2 + \frac{\beta}{2} - \frac{\tau}{4}\right)\left[\eta(X)\eta(Y)\tilde{S}(\xi,Z) - g(X,Y)\tilde{S}(\xi,Z) + \eta(X)\eta(Z)\tilde{S}(Y,\xi)\right. \\ \left. - g(X,Z)\tilde{S}(Y,\xi)\right] - (Z\beta)\tilde{S}(Y,\varphi X) - g(X,\varphi Z)\tilde{S}(Y,\text{grad}\beta) \\ + \frac{1}{2}\left[(\varphi Y\beta)\tilde{S}(X,Z) - (\varphi X\beta)\eta(Y)\tilde{S}(\xi,Z) - \eta(X)(\varphi Y\beta)\tilde{S}(\xi,Z)\right. \\ + \beta^2 g(X,\varphi Y)\tilde{S}(\xi,Z) - \eta(Y)(X\beta)\tilde{S}(\xi,Z) + (\varphi Z\beta)\tilde{S}(Y,X) \\ - (\varphi X\beta)\eta(Z)\tilde{S}(Y,\xi) - \eta(X)(\varphi Z\beta)\tilde{S}(Y,\xi) + \beta^2 g(X,\varphi Z)\tilde{S}(Y,\xi) \\ \left. - \eta(Z)(X\beta)\tilde{S}(Y,\xi)\right] = 0. \end{aligned} \tag{4.37}$$

In view of (4.9), (4.12) and (4.13), the above equation converts to

$$-(Y\beta)\Big[(\varphi X\beta)\eta(Z)+\eta(Z)(X\beta)+\beta(\beta-1)g(\varphi X,\varphi Z)+\frac{\tau}{2}g(\varphi X,Z)\Big]$$
$$-g(X,\varphi Y)\Big[|grad\beta|^2\eta(Z)+\frac{\tau}{2}(Z\beta)+\beta(\beta-1)(\varphi Z\beta)\Big]$$
$$+\Big(\beta^2-\frac{\beta}{2}\Big)\eta(Y)\Big[(\varphi X\beta)\eta(Z)+\eta(Z)(X\beta)+\beta(\beta-1)g(\varphi X,\varphi Z)$$
$$+\frac{\tau}{2}g(\varphi X,Z)\Big]-g(X,\varphi Z)\Big[-\beta(\beta-1)(\varphi Y\beta)+\frac{\tau}{2}(Y\beta)\Big]$$
$$+\Big(\beta^2-\frac{\beta}{2}\Big)\eta(Z)\Big[\eta(Y)(X\beta)-\beta(\beta-1)g(\varphi X,\varphi Y)+\frac{\tau}{2}g(\varphi X,Y)\Big]$$
$$+\Big[2\beta^2+\frac{\beta}{2}-\frac{\tau}{4}\Big)\Big[\big(\eta(X)\eta(Y)-g(X,Y)\big)\Big(-(\varphi Z\beta)+\Big(2\beta^2-\frac{1}{2}\Big)\eta(Z)\Big)$$
$$+\big(\eta(X)\eta(Z)-g(X,Z)\big)\Big((Y\beta)-(\varphi Y\beta)+\Big(2\beta^2-\frac{1}{2}\Big)\eta(Y)\Big)\Big]$$
$$+\frac{1}{2}\Big[(\varphi Y\beta)\Big(\beta(\beta-1)g(X,\varphi Z)+(X\beta)\eta(Z)-(\varphi Z\beta)\eta(X)-(\varphi X\beta)\eta(Z)$$
$$+\frac{\tau}{2}g(X,Z)+\Big(2\beta^2-\frac{1}{2}-\frac{\tau}{2}\Big)\eta(X)\eta(Z)\Big)$$
$$+(\varphi Z\beta)\Big(\beta(\beta-1)g(Y,\varphi X)+(Y\beta)\eta(X)-(\varphi X\beta)\eta(Y)-(\varphi Y\beta)\eta(X)$$
$$+\frac{\tau}{2}g(Y,X)+\Big(2\beta^2-\frac{1}{2}-\frac{\tau}{2}\Big)\eta(X)\eta(Y)\Big)$$
$$+\big(-(\varphi X\beta)\eta(Y)-\eta(X)(\varphi Y\beta)+\beta^2g(X,\varphi Y)-\eta(Y)(X\beta)\big)\Big(-(\varphi Z\beta)$$
$$+\Big(2\beta^2-\frac{1}{2}\Big)\eta(Z)\Big)$$
$$+\big(-(\varphi X\beta)\eta(Z)-\eta(X)(\varphi Z\beta)+\beta^2g(X,\varphi Z)-\eta(Z)(X\beta)\big)\Big((Y\beta)-(\varphi Y\beta)$$
$$+\Big(2\beta^2-\frac{1}{2}\Big)\eta(Y)\Big)\Big]-(Z\beta)\Big[\beta(\beta-1)g(Y,\varphi^2X)+(X\beta)\eta(Y)+\frac{\tau}{2}g(Y,\varphi X)\Big]$$
$$=0. \tag{4.38}$$

Putting $X=Z=e_i$ in (4.38), taking summation over i, we get

$$[-(4\beta^2-\frac{\tau}{2}+\beta)(2\beta^2-\frac{1}{2})+2\beta^2(\beta-1)(6\beta-\frac{1}{2})-$$
$$\parallel grad\beta\parallel^2]\eta(Y)+[\frac{\tau}{2}-\frac{6\beta^2}{2}-\frac{3\beta}{2}](Y\beta)+[6\beta^2+\frac{3\beta}{2}](\varphi Y\beta)=0,$$

which gives us

$$[-(4\beta^2-\frac{\tau}{2}+\beta)(2\beta^2-\frac{1}{2})+2\beta^2(\beta-1)(6\beta-\frac{1}{2})-$$
$$\parallel grad\beta\parallel^2]\xi+[\frac{\tau}{2}-\frac{6\beta^2}{2}-\frac{3\beta}{2}]grad\beta-[6\beta^2+\frac{3\beta}{2}]\varphi grad\beta=0, \tag{4.39}$$

for $\forall Y\in\chi(M)$. Applying φ to both side of the equation (4.39), we get

$$\Big[\frac{\tau}{2}-\frac{6\beta^2}{2}-\frac{3\beta}{2}\Big]\varphi\mathrm{grad}\beta+\Big[6\beta^2+\frac{3\beta}{2}\Big]\mathrm{grad}\beta=0. \tag{4.40}$$

From (4.39) and (4.40), it follows that

$$(X\beta) = -\frac{[-(4\beta^2-\frac{\tau}{2}+\beta)(2\beta^2-\frac{1}{2})+2\beta^2(\beta-1)(6\beta-\frac{1}{2})-\| \operatorname{grad}\beta \|^2][[\frac{\tau}{2}-\frac{6\beta^2}{2}-\frac{3\beta}{2}]]}{[\frac{\tau}{2}-\frac{6\beta^2}{2}-\frac{3\beta}{2}]^2+[6\beta^2+\frac{3\beta}{2}]^2}\eta(X),$$

which implies that the manifold is η −Einstein with respect to ∇. This completes the proof.

Theorem 4.14. Let M be a 3-dimensional quasi-Sasakian manifold satisfying the condition $(\widetilde{H}(X,\xi).\tilde{S})(Y,Z)=0$, where $\widetilde{H}$ is concircular curvature tensor and $\tilde{S}$ is Ricci tensor respect to quarter symmetric non-metric connection (M,φ_3,f_1). Then the manifold M is η −Einstein manifold with respect to ∇.

Proof. Assume that a 3-dimensional quasi-Sasakian manifold M yields that

$$\left(\widetilde{H}(X,\xi).\tilde{S}\right)(Y,Z)=0, \tag{4.41}$$

where $\widetilde{H}$ is concircular curvature tensor respect to quarter symmetric non-metric connection $\widetilde{\nabla}$. In the view of (4.26), we obtain

$$\tilde{S}(\widetilde{H}(X,\xi)Y,Z)+\tilde{S}(Y,\widetilde{H}(X,\xi)Z)=0,$$

for all $X,Y,Z\in\chi(M)$. Using (4.20), we get

$$\begin{aligned}
-(Y\beta)\tilde{S}(\varphi X,Z)-g(X,\varphi Y)\tilde{S}(\operatorname{grad}\beta,Z)-\frac{\beta}{2}g(\varphi X,Y)\tilde{S}(\xi,Z)\\
+\left(\beta^2-\frac{\beta}{2}\right)\left[\eta(Y)\tilde{S}(\varphi X,Z)+\eta(Z)\tilde{S}(Y,\varphi X)\right]-(X\beta)\eta(Y)\tilde{S}(\xi,Z)\\
+\left(\beta^2+\frac{\beta}{2}+\frac{1}{4}\right)\left[\eta(X)\eta(Y)\tilde{S}(\xi,Z)+\eta(X)\eta(Z)\tilde{S}(Y,\xi)\right]\\
-\left(\frac{5\beta^2}{3}+\frac{\beta}{2}-\frac{\tau}{6}+\frac{1}{12}\right)\left[g(X,Y)\tilde{S}(\xi,Z)+g(X,Z)\tilde{S}(Y,\xi)\right]\\
+\left(\frac{2\beta^2}{3}-\frac{1}{6}-\frac{\tau}{6}\right)\left[\eta(Y)\tilde{S}(X,Z)+\eta(Z)\tilde{S}(Y,X)\right]-(Z\beta)\tilde{S}(Y,\varphi X)\\
-g(X,\varphi Z)\tilde{S}(Y,\operatorname{grad}\beta)-\frac{\beta}{2}g(\varphi X,Z)\tilde{S}(Y,\xi)-(X\beta)\eta(Z)\tilde{S}(Y,\xi)=0
\end{aligned} \tag{4.42}$$

Applying (4.9), (4.12) and (4.13), (4.42) transforms that

$$-(Y\beta)\left[(\varphi X\beta)\eta(Z)+\eta(Z)(X\beta)+\beta(\beta-1)g(\varphi X,\varphi Z)+\frac{\tau}{2}g(\varphi X,Z)\right]$$
$$-g(X,\varphi Y)\left[|\mathrm{grad}\beta|^2\eta(Z)+\frac{\tau}{2}(Z\beta)+\beta(\beta-1)(\varphi Z\beta)\right]$$
$$-\frac{\beta}{2}g(\varphi X,Y)\left[-(\varphi Z\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Z)\right]$$
$$+\left(\beta^2-\frac{\beta}{2}\right)\eta(Y)\left[(\varphi X\beta)\eta(Z)+\eta(Z)(X\beta)+\beta(\beta-1)g(\varphi X,\varphi Z)\right.$$
$$\left.+\frac{\tau}{2}g(\varphi X,Z)\right]$$
$$+\left(\beta^2-\frac{\beta}{2}\right)\eta(Z)\left[(X\beta)\eta(Y)-\beta(\beta-1)g(\varphi X,\varphi Y)+\frac{\tau}{2}g(\varphi X,Y)\right]$$
$$-(X\beta)\eta(Y)\left[-(\varphi Z\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Z)\right]$$
$$-(X\beta)\eta(Z)\left[(Y\beta)-(\varphi Y\beta)+\left(2\beta^2-\frac{1}{2}\right)\eta(Y)\right]$$
$$-(Z\beta)\left[\eta(Y)(X\beta)-\beta(\beta-1)g(\varphi X,\varphi Y)+\frac{\tau}{2}g(\varphi X,Y)\right]-g(X,\phi Z)[-\beta(\beta$$
$$-1)(\phi Y\beta)+\frac{\tau}{2}(Y\beta)]+\left(\beta^2+\frac{\beta}{2}+\frac{1}{4}\right)[-(\phi Z\beta)\eta(X)\eta(Y)$$
$$+(4\beta^2-1)\eta(X)\eta(Y)\eta(Z)+(Y\beta)\eta(X)\eta(Z)-(\phi Y\beta)\eta(X)\eta(Z))+(\frac{2\beta^2}{3}$$
$$-\frac{\tau+1}{6})(\eta(Y)\eta(Z)(X\beta)+\beta(\beta-1)g(X,\phi Z)\eta(Y)-(\phi Z\beta)\eta(X)\eta(Y)$$
$$-(\phi X\beta)\eta(Y)\eta(Z)+\frac{\tau}{2}g(X,Z)\eta(Y)+2\left(2\beta^2-\frac{\tau}{2}-\frac{1}{2}\right)\eta(Y)\eta(X)\eta(Z)$$
$$+\eta(X)\eta(Z)(Y\beta)+\beta(\beta-1)\eta(Z)g(\phi X,Y)\eta(Z)-(\phi X\beta)\eta(Z)\eta(Y)$$
$$-(\phi Y\beta)\eta(Z)\eta(X)+\frac{\tau}{2}\eta(Z)g(Y,X)]$$
$$+\left(-\frac{5\beta^2}{3}+\frac{\tau}{6}-\frac{1}{12}-\frac{\beta}{2}\right)\left[-g(X,Y)(\phi Z\beta)+\left(2\beta^2-\frac{1}{2}\right)g(X,Y)\eta(Z)\right.$$
$$\left.+g(X,Z)(Y\beta)-g(X,Z)(\phi Y\beta)+(2\beta^2-\frac{1}{2})g(X,Z)\eta(Y)\right]=0. \qquad (4.43)$$

Putting $X=Z=e_i$ in (4.43), taking summation over i, we have

$$\left[2\beta\left(\beta^2-\frac{\beta}{2}\right)(\beta-1)+2\left(2\beta^2-\frac{1}{2}\right)\left(\frac{-\beta}{2}+\frac{1}{12}+\frac{\tau}{3}-\frac{7\beta^2}{3}\right)-\|\mathrm{grad}\beta\|^2\right]\eta(Y)$$
$$+\left[\frac{\tau}{3}-\frac{\beta}{2}-\frac{1}{6}-\frac{10\beta^2}{3}\right](Y\beta)+\left[\frac{1}{4}-\frac{\tau}{2}+7\beta^2+\frac{3\beta}{2}\right](\varphi Y\beta)=0. \qquad (4.44)$$

The relation (4.48) satisfies

$$\left[2\beta\left(\beta^2-\frac{\beta}{2}\right)(\beta-1)+2\left(2\beta^2-\frac{1}{2}\right)\left(\frac{-\beta}{2}+\frac{1}{12}+\frac{\tau}{3}-\frac{7\beta^2}{3}\right)-\|\mathrm{grad}\beta\|^2\right]\xi$$
$$+\left[\frac{\tau}{3}-\frac{\beta}{2}-\frac{1}{6}-\frac{10\beta^2}{3}\right]\mathrm{grad}\beta-\left[\frac{1}{4}-\frac{\tau}{2}+7\beta^2+\frac{3\beta}{2}\right]\varphi\mathrm{grad}\beta=0, \qquad (4.45)$$

for $\forall Y\in\chi(M)$. Operating φ on both side of (4.45), we get

$$\left[\frac{\tau}{3}-\frac{\beta}{2}-\frac{1}{6}-\frac{10\beta^2}{3}\right]\varphi\mathrm{grad}\beta+\left[\frac{1}{4}-\frac{\tau}{2}+7\beta^2+\frac{3\beta}{2}\right]\mathrm{grad}\beta=0. \qquad (4.46)$$

Using (4.45) and (4.46), we have

$$(X\beta)=\frac{\begin{pmatrix}2\beta(\beta^2-\frac{\beta}{2})(\beta-1)+2(2\beta^2-\frac{1}{2})(\frac{-\beta}{2}+\frac{1}{12}+\frac{\tau}{3}-\frac{7\beta^2}{3})\\ -\|\operatorname{grad}\beta\|^2][\frac{\tau}{3}-\frac{\beta}{2}-\frac{1}{6}-\frac{10\beta^2}{3}]\end{pmatrix}}{[\frac{1}{4}-\frac{\tau}{2}+7\beta^2+\frac{3\beta}{2}]^2+[\frac{\tau}{3}-\frac{\beta}{2}-\frac{1}{6}-\frac{10\beta^2}{3}]^2}\eta(X).$$

It is obvious that the manifold is η −Einstein respect to ∇ because of (1.11). This completes the proof.

Theorem 4.15 Let M be a 3-dimensional quasi-Sasakian manifold satisfying the condition $(\widetilde{R}(X,\xi).\widetilde{R})(Y,V)W=0$, where $\widetilde{R}$ is Riemannian curvature tensor with respect to quarter symmetric non-metric connection (M,φ_3,f_1). Then the manifold M is η −Einstein manifold with respect to ∇.

Proof. Consider a 3-dimensional quasi-Sasakian manifold M satisfying condition

$$(\tilde{R}(X,\xi).\tilde{R})(Y,V)W=0,$$

where $\tilde{R}$ is Riemannian curvature tensor respect to quarter symmetric non-metric connection $\widetilde{\nabla}$. Then from the relation (4.27), it follows that

$$\tilde{R}(X,\xi)\tilde{R}(Y,V)W-\tilde{R}(\tilde{R}(X,\xi)Y,V)W-\tilde{R}(Y,\tilde{R}(X,\xi)V)W-\tilde{R}(Y,V)\tilde{R}(X,\xi)W=0, \quad (4.47)$$

for all $X,Y,V,W\in\chi(M)$. Putting $V=W=\xi$ in (4.51), we obtain

$$\tilde{R}(X,\xi)\tilde{R}(Y,\xi)\xi-\tilde{R}(\tilde{R}(X,\xi)Y,\xi)\xi-\tilde{R}(Y,\tilde{R}(X,\xi)\xi)\xi-\tilde{R}(Y,\xi)\tilde{R}(X,\xi)\xi=0.$$

In view of (4.2) and (4.3), the above equation converts to

$$\begin{aligned}
&-(\tilde{R}(Y,\xi)\xi\beta)\varphi X-g(X,\varphi\tilde{R}(Y,\xi)\xi)\operatorname{grad}\beta-\frac{\beta}{2}g(\varphi X,\tilde{R}(Y,\xi)\xi)\xi+(X\beta)\eta(Y)\tilde{R}(\xi,\xi)\xi\\
&\quad+\left(\beta^2-\frac{\beta}{2}\right)\left[\eta(\tilde{R}(Y,\xi)\xi)\varphi X-\eta(Y)\tilde{R}(\varphi X,\xi)\xi-\tilde{R}(Y,\varphi X)\xi-\tilde{R}(Y,\xi)\varphi X\right]\\
&\quad-(X\beta)\eta(\tilde{R}(Y,\xi)\xi)\xi-\left(2\beta^2+\frac{\beta}{2}\right)\left[g(X,\tilde{R}(Y,\xi)\xi)\xi-g(X,Y)\tilde{R}(\xi,\xi)\xi\right]\\
&\quad+\left(\beta^2-\frac{1}{4}\right)\left[\eta(\tilde{R}(Y,\xi)\xi)X-\eta(Y)\tilde{R}(X,\xi)\xi+\eta(X)\tilde{R}(Y,\xi)\xi-\tilde{R}(Y,X)\xi\right.\\
&\quad\left.-\tilde{R}(Y,\xi)X+\eta(X)\tilde{R}(Y,\xi)\xi\right]+(Y\beta)\tilde{R}(\varphi X,\xi)\xi+g(X,\varphi Y)\tilde{R}(\operatorname{grad}\beta,\xi)\xi\\
&\quad+\left(\beta^2+\frac{\beta}{2}+\frac{1}{4}\right)\left[\eta(\tilde{R}(Y,\xi)\xi)\eta(X)\xi-\eta(Y)\eta(X)\tilde{R}(\xi,\xi)\xi\right]+\frac{\beta}{2}g(\varphi X,Y)\tilde{R}(\xi,\xi)\xi\\
&\quad+2(X\beta)\tilde{R}(Y,\xi)\xi=0,
\end{aligned}$$

which implies that

$$-\left(\tilde{R}(Y,\xi)\xi\beta\right)g(\varphi X,U)-g\left(X,\varphi\tilde{R}(Y,\xi)\xi\right)g(grad\beta,U)-\frac{\beta}{2}g\left(\varphi X,\tilde{R}(Y,\xi)\xi\right)\eta(U)$$
$$+\left(\beta^2-\frac{\beta}{2}\right)\left[\eta\left(\tilde{R}(Y,\xi)\xi\right)g(\varphi X,U)-\eta(Y)g\left(\tilde{R}(\varphi X,\xi)\xi,U\right)\right.$$
$$\left.-g\left(\tilde{R}(Y,\varphi X)\xi,U\right)-g\left(\tilde{R}(Y,\xi)\varphi X,U\right)\right]-(X\beta)\eta\left(\tilde{R}(Y,\xi)\xi\right)\eta(U)$$
$$-\left(2\beta^2+\frac{\beta}{2}\right)\left[g\left(X,\tilde{R}(Y,\xi)\xi\right)\eta(U)-g(X,Y)g\left(\tilde{R}(\xi,\xi)\xi,U\right)\right]$$
$$+\left(\beta^2-\frac{1}{4}\right)\left[\eta\left(\tilde{R}(Y,\xi)\xi\right)g(X,U)-\eta(Y)g\left(\tilde{R}(X,\xi)\xi,U\right)+\eta(X)g\left(\tilde{R}(Y,\xi)\xi,U\right)\right.$$
$$\left.-g\left(\tilde{R}(Y,X)\xi,U\right)-g\left(\tilde{R}(Y,\xi)X,U\right)+\eta(X)g\left(\tilde{R}(Y,\xi)\xi,U\right)\right]$$
$$+\left(\beta^2+\frac{\beta}{2}+\frac{1}{4}\right)\left[\eta\left(\tilde{R}(Y,\xi)\xi\right)\eta(X)\eta(U)-\eta(Y)\eta(X)g\left(\tilde{R}(\xi,\xi)\xi,U\right)\right]$$
$$+(Y\beta)g\left(\tilde{R}(\varphi X,\xi)\xi,U\right)+g(X,\varphi Y)g\left(\tilde{R}(grad\beta,\xi)\xi,U\right)$$
$$+\frac{\beta}{2}g(\varphi X,Y)g\left(\tilde{R}(\xi,\xi)\xi,U\right)+(X\beta)\eta(Y)g\left(\tilde{R}(\xi,\xi)\xi,U\right)$$
$$+2(X\beta)g\left(\tilde{R}(Y,\xi)\xi,U\right)=0. \qquad (4.48)$$

Putting $X=U=e_i$ in (4.48), taking summation over i and then using (4.3), (4.2) we obtain

$$\left(-3\beta^2+\frac{\beta}{2}+\frac{1}{2}\right)(\varphi Y\beta)+\left(-6\beta^2+\frac{5\beta}{2}+\frac{1}{2}\right)(Y\beta)=0. \qquad (4.49)$$

The relation (4.49) gives us

$$-\left(-3\beta^2+\frac{\beta}{2}+\frac{1}{2}\right)\varphi\text{grad}\beta+\left(-6\beta^2+\frac{5\beta}{2}+\frac{1}{2}\right)\text{grad}\beta=0, \qquad (4.50)$$

for $\forall Y\in\chi(M)$. Applying φ both side of the equation (4.50), we get

$$\left(-3\beta^2+\frac{\beta}{2}+\frac{1}{2}\right)\text{grad}\beta+\left(-6\beta^2+\frac{5\beta}{2}+\frac{1}{2}\right)\varphi\text{grad}\beta=0. \qquad (4.51)$$

From (4.50) and (4.51), it follows that

$$\left[(-3\beta^2+\frac{\beta}{2}+\frac{1}{2})^2+(-6\beta^2+\frac{5\beta}{2}+\frac{1}{2})^2\right]\text{grad}\beta=0,$$

which implies that $\text{grad}\beta=0$ and hence β is constant. If β is constant, then because of (1.11) the manifold is η −Einstein with respect to ∇. This completes the proof.

Theorem 4.16 Let M be a 3-dimensional quasi-Sasakian manifold satisfying the condition $(\tilde{P}(X,\xi).\tilde{H})(Y,V)W=0$, where $\tilde{P}$ is Projective curvature tensor and $\tilde{H}$ is concircular curvature tensor with respect to quarter symmetric non-metric connection (M,φ_3,f_1). Then the manifold M is η −Einstein manifold with respect to ∇.

Proof. It is known that from the relation (4.27)

$$\left(\tilde{P}(X,\xi)\tilde{H}\right)(Y,V)W$$
$$=\tilde{P}(X,\xi)\tilde{H}(Y,V)W-\tilde{H}\left(\tilde{P}(X,\xi)Y,V\right)W-\tilde{H}\left(Y,\tilde{P}(X,\xi)V\right)W$$
$$-\tilde{H}(Y,V)\tilde{P}(X,\xi)W,$$

holds for all $X,Y,V,W\in\chi(M)$. Since $(\tilde{P}(X,\xi)\tilde{H})(Y,V)W=0$, we have

$$\tilde{P}(X,\xi)\tilde{H}(Y,V)W-\tilde{H}\left(\tilde{P}(X,\xi)Y,V\right)W-\tilde{H}\left(Y,\tilde{P}(X,\xi)V\right)W-\tilde{H}(Y,V)\tilde{P}(X,\xi)W=0, \qquad (4.52)$$

which yields

$$\tilde{P}(X,\xi)\tilde{H}(Y,\xi)\xi-\tilde{H}\left(\tilde{P}(X,\xi)Y,\xi\right)\xi-\tilde{H}\left(Y,\tilde{P}(X,\xi)\xi\right)\xi-\tilde{H}(Y,\xi)\tilde{P}(X,\xi)\xi=0, \qquad (4.53)$$

using $V=W=\xi$ in (4.56). The relation (4.53) yields by virtue of (4.24) that

$$-(\widetilde{H}(Y,\xi)\xi\beta)g(\varphi X,U)-g(X,\varphi\widetilde{H}(Y,\xi)\xi)g(grad\beta,U)+(Y\beta)g(\widetilde{H}(\varphi X,\xi)\xi,U)$$
$$+\left(\beta^2-\frac{\beta}{2}\right)[\eta(\widetilde{H}(Y,\xi)\xi)g(\varphi X,U)eta(Y)-g(\widetilde{H}(\varphi X,\xi)\xi,U)$$
$$-g(\widetilde{H}(Y,\varphi X)\xi,U)-g(\widetilde{H}(Y,\xi)\varphi X,U)]+g(X,\varphi Y)g(\widetilde{H}(grad\beta,\xi)\xi,U)$$
$$+\frac{1}{2}[(\varphi\widetilde{H}(Y,\xi)\xi\beta)g(X,U)-(X\beta)\eta(\widetilde{H}(Y,\xi)\xi)\eta(U)$$
$$-\eta(X)(\varphi\widetilde{H}(Y,\xi)\xi\beta)\eta(U)-\eta(\widetilde{H}(Y,\xi)\xi)(\varphi X\beta)\eta(U)$$
$$+\beta^2 g(X,\varphi\widetilde{H}(Y,\xi)\xi)\eta(U)-(\varphi Y\beta)g(\widetilde{H}(X,\xi)\xi,U)+(X\beta)\eta(Y)g(\widetilde{H}(\xi,\xi)\xi,U)$$
$$+\eta(X)(\varphi Y\beta)g(\widetilde{H}(\xi,\xi)\xi,U)+\eta(Y)(\varphi X\beta)g(\widetilde{H}(\xi,\xi)\xi,U)$$
$$-\beta^2 g(X,\varphi Y)g(\widetilde{H}(\xi,\xi)\xi,U)+(X\beta)g(\widetilde{H}(Y,\xi)\xi,U)+(\varphi X\beta)g(\widetilde{H}(Y,\xi)\xi,U)$$
$$+(X\beta)g(\widetilde{H}(Y,\xi)\xi,U)+(\varphi X\beta)g(\widetilde{H}(Y,\xi)\xi,U)]$$
$$-\left(2\beta^2+\frac{\beta}{2}-\frac{\tau}{4}\right)[g(X,\widetilde{H}(Y,\xi)\xi)\eta(U)-\eta(X)\eta(\widetilde{H}(Y,\xi)\xi)\eta(U)$$
$$-g(X,Y)g(\widetilde{H}(\xi,\xi)\xi,U)+\eta(X)\eta(Y)g(\widetilde{H}(\xi,\xi)\xi,U)]=0, \tag{4.54}$$

for all $U\in\chi(M)$. Putting $X=U=e_i$ in (4.54), in the view of (4.21) and (4.20) taking summation for $i=1,2,3$ we get

$$(\frac{5\beta^2}{3}-\frac{\tau}{6}-\frac{\beta}{2}-\frac{1}{6})(Y\beta)+(\frac{1}{6}+\frac{\tau}{6}+\frac{\beta^2}{3}-\frac{\beta}{2})(\varphi Y\beta)-4(\beta^2-\frac{\beta}{2})^2\eta(Y)=0,$$

which implies

$$(\frac{5\beta^2}{3}-\frac{\tau}{6}-\frac{\beta}{2}-\frac{1}{6})\mathrm{grad}\beta-(\frac{1}{6}+\frac{\tau}{6}+\frac{\beta^2}{3}-\frac{\beta}{2})\varphi\mathrm{grad}\beta-4(\beta^2-\frac{\beta}{2})^2\xi=0, \tag{4.55}$$

for all vector fields Y. Let's apply φ both sides of the relation (4.55):

$$\left(\frac{5\beta^2}{3}-\frac{\tau}{6}-\frac{\beta}{2}-\frac{1}{6}\right)\varphi\mathrm{grad}\beta+\left(\frac{1}{6}+\frac{\tau}{6}+\frac{\beta^2}{3}-\frac{\beta}{2}\right)\mathrm{grad}\beta=0. \tag{4.56}$$

Hence (4.55) and (4.56) yield

$$(X\beta)=\frac{4(\beta^2-\frac{\beta}{2})^2(\frac{5\beta^2}{3}-\frac{\tau}{6}-\frac{\beta}{2}-\frac{1}{6})}{(\frac{5\beta^2}{3}-\frac{\tau}{6}-\frac{\beta}{2}-\frac{1}{6})^2+(\frac{1}{6}+\frac{\tau}{6}+\frac{\beta^2}{3}-\frac{\beta}{2})^2}\eta(X),$$

i.e., the manifold is η −Einstein with respect to ∇. This proves the theorem.

5. Example

In this section, we generate an example of a 3-dimensional quasi-Sasakian manifold admitting the quarter symmetric non- metric connection satisfies the results of (M,φ_3,f_1).
We consider the 3-dimensional manifold $\{(x,y,z)\in\mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let

$$e_1=G(z)\frac{\partial}{\partial x}+G(z)\frac{\partial}{\partial y}+\frac{c_1-G^2(z)}{G'(z)}\frac{\partial}{\partial z}, \tag{5.1}$$

$$e_2=\sqrt{c_1-G^2(z)}\frac{\partial}{\partial x}+\sqrt{c_1-G^2(z)}\frac{\partial}{\partial y}-G(z)\frac{\sqrt{c_1-G^2(z)}}{G'(z)}\frac{\partial}{\partial z}, \tag{5.2}$$

$$e_3=\sqrt{k_1c_1}\frac{\sqrt{c_1-G^2(z)}}{G'(z)}=\xi, \tag{5.3}$$

which k_1,c_1 are non-zero constants and $G'(z)\neq 0$ on the three dimensional manifold M.
Choosing a Riemannian metric g such that

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1,\quad g(e_1,e_2)=g(e_2,e_3)=g(e_1,e_3)=0, \tag{5.4}$$

and a (1,1) tensor field φ such that

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \tag{5.5}$$

we get $\{e_1, e_2, e_3\}$ is linear independent orthonormal basis and for $\forall X, Y \in \chi(M)$, we have

$$\eta(e_3) = \eta(\xi) = 1, \quad \varphi^2 X = -X + \eta(X)\xi, \tag{5.6}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{5.7}$$

Using (5.1), (5.2) and (5.3), we write

$$[e_1, e_2] = Me_3, \quad [e_1, e_3] = Mk_1 e_2, \quad [e_2, e_3] = -Mk_1 e_1, \tag{5.8}$$

where $M = -\sqrt{\frac{c_1}{k_1}}$ is a non-zero constant. Additionally we obtain

$$\nabla_{e_1} e_3 = -\frac{M}{2} e_2, \quad \nabla_{e_2} e_3 = \frac{M}{2} e_1, \quad \nabla_{e_3} e_3 = 0, \tag{5.9}$$

$$\nabla_{e_1} e_2 = \frac{M}{2} e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_2 = \frac{M(2k_1 + 1)}{2} e_1, \tag{5.10}$$

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_1 = -\frac{M}{2} e_3, \quad \nabla_{e_3} e_1 = -\frac{M(2k_1 + 1)}{2} e_2, \tag{5.11}$$

helping of Koszul formula [33]

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g([Y, X], Z) - g([X, Z], Y) - g([Y, Z], X).$$

Because of (5.9), (5.10), (5.11) and (5.5), we see that

$$\nabla_X e_3 = \nabla_X \xi = -\frac{M}{2} \varphi X = -\beta \varphi X, \quad (\nabla_X \varphi)Y = \frac{M}{2} [g(X, Y)\xi - \eta(Y)X]. \tag{5.12}$$

In the view of (5.4), (5.5), (5.6), (5.7), (5.9) and (5.12), the manifold M with the structure (ϕ, ξ, η, g) is a 3-dimensional quasi-Sasakian manifold such that $\beta = \frac{M}{2} = constant$. Notice that $(e_3\beta) = (\xi\beta) = 0$.

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In the view of (5.9), (5.10), (5.11) and the above equation, we can easily show that

$$R(e_1, e_2)e_2 = \left(\frac{\tau}{2} - 2\beta^2\right) e_1 = -\beta^2(3 + 4k_1)e_1, \quad R(e_1, e_3)e_2 = 0, \tag{5.13}$$

$$R(e_2, e_1)e_1 = \left(\frac{\tau}{2} - 2\beta^2\right) e_2 = -\beta^2(3 + 4k_1)e_2, \quad R(e_3, e_1)e_1 = \beta^2 e_3, \tag{5.14}$$

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_1 = 0. \tag{5.15}$$

Also from (1.11), (1.15) and (1.16), we get

$$S(e_1, e_1) = \frac{\tau}{2} - \beta^2, \quad S(e_2, e_2) = \frac{\tau}{2} - \beta^2, \quad S(e_3, e_3) = 2\beta^2, \tag{5.16}$$

$$S(e_2, e_3) = 0, \quad S(e_1, e_3) = 0, \quad S(e_1, e_2) = 0, \tag{5.17}$$

$$P(e_1, e_2)e_1 = \left[\frac{3\beta^2}{2} - \frac{\tau}{4}\right] e_2, \quad P(e_1, e_2)e_2 = \left[\frac{\tau}{4} - \frac{3\beta^2}{2}\right] e_1, \quad P(e_1, e_2)e_3 = 0, \tag{5.18}$$

$$P(e_1, e_3)e_1 = \left[\frac{\tau}{4} - \frac{3\beta^2}{2}\right] e_3, \quad P(e_1, e_3)e_2 = 0, \quad P(e_1, e_3)e_3 = 0, \tag{5.19}$$

$$P(e_2, e_3)e_1 = 0, \quad P(e_2, e_3)e_2 = \left[\frac{\tau}{4} - \frac{3\beta^2}{2}\right] e_3, \quad P(e_2, e_3)e_3 = 0, \tag{5.20}$$

$$P(e_2, e_3)e_1 = 0, \quad P(e_2, e_3)e_2 = \left[\frac{\tau}{4} - \frac{3\beta^2}{2}\right] e_3, \quad P(e_2, e_3)e_3 = 0, \tag{5.21}$$

$$H(e_1,e_2)e_2 = 2\left[\frac{\tau}{6}-\beta^2\right]e_1,\ \ H(e_1,e_2)e_3 = 0,\ \ H(e_2,e_3)e_1 = 0, \tag{5.22}$$

$$H(e_2,e_1)e_1 = 2\left[\frac{\tau}{6}-\beta^2\right]e_2,\ \ H(e_3,e_1)e_1 = -\left[\frac{\tau}{6}-\beta^2\right]e_3,\ \ H(e_1,e_3)e_2 = 0, \tag{5.23}$$

$$H(e_1,e_3)e_3 = -\left[\frac{\tau}{6}-\beta^2\right]e_1, H(e_2,e_3)e_3 = -\left[\frac{\tau}{6}-\beta^2\right]e_2,\ H(e_3,e_2)e_2 = -\left[\frac{\tau}{6}-\beta^2\right]e_3., \tag{5.24}$$

where $\tau = -2\beta^2(1+4k_1)$.

Using (4.18), (4.19), and (4.9), we can find projective curvature tensor $\tilde{P}$, Ricci tensor $\tilde{S}$, concircular curvature tensor $\widetilde{H}$ with respect to $\tilde{\nabla}$ as

$$\tilde{P}(e_1,e_2)e_1 = \left[2\beta^2+\frac{\beta}{2}-\frac{\tau}{4}\right]e_2+\frac{\beta^2}{2}e_1,\ \ \tilde{P}(e_1,e_2)e_2 = -\left[2\beta^2+\frac{\beta}{2}-\frac{\tau}{4}\right]e_1+\frac{\beta^2}{2}e_2$$

$$\tilde{P}(e_1,e_2)e_3 = \beta e_3, \tilde{P}(e_1,e_3)e_1 = \left[\frac{\tau}{4}-\frac{5\beta^2}{2}-\frac{\beta}{2}\right]e_3,$$

$$\tilde{P}(e_1,e_3)e_2 = \frac{-\beta^2}{2}e_3,\ \ \tilde{P}(e_1,e_3)e_3 = \left[\beta^2-\frac{\beta}{2}\right]e_2,$$

$$\tilde{P}(e_2,e_3)e_1 = \frac{\beta^2}{2}e_3,\ \ \tilde{P}(e_2,e_3)e_2 = \left[\frac{\tau}{4}-\frac{5\beta^2}{2}\right]e_3,\ \ \tilde{P}(e_2,e_3)e_3 = -\left[\beta^2-\frac{\beta}{2}\right]e_1,$$

$$\tilde{S}(e_1,e_1) = \frac{\tau}{2},\ \ \tilde{S}(e_2,e_2) = \frac{\tau}{2},\ \ \tilde{S}(e_3,e_3) = 2\beta^2-\frac{1}{2},$$

$$\tilde{S}(e_2,e_3) = 0,\ \ \tilde{S}(e_1,e_3) = 0,\ \ \tilde{S}(e_1,e_2) = -(\beta^2-\beta),$$

$$\widetilde{H}(e_1,e_2)e_2 = \left[\frac{\tau}{3}-\frac{7\beta^2}{3}-\frac{\beta}{2}+\frac{1}{12}\right]e_1+\left[\beta^2-\frac{\beta}{2}\right]e_2,\ \ \widetilde{H}(e_1,e_2)e_3 = \beta e_3,$$

$$\widetilde{H}(e_1,e_3)e_2 = -\frac{\beta}{2}e_3, \widetilde{H}(e_2,e_3)e_1 = \frac{\beta}{2}e_3,$$

$$\widetilde{H}(e_3,e_1)e_1 = \left[-\frac{\tau}{6}+\frac{5\beta^2}{3}+\frac{\beta}{2}-\frac{1}{6}\right]e_3,\ \ \widetilde{H}(e_2,e_1)e_1 = \left[\frac{\tau}{3}-\frac{7\beta^2}{3}-\frac{\beta}{2}+\frac{1}{12}\right]e_2-\left[\beta^2-\frac{\beta}{2}\right]e_1,$$

$$\widetilde{H}(e_1,e_3)e_3 = \left[\frac{2\beta^2}{3}-\frac{\tau}{6}-\frac{1}{6}\right]e_2-\left[\beta^2-\frac{\beta}{2}\right]e_1,\ \ \widetilde{H}(e_2,e_3)e_3 = \left[\frac{2\beta^2}{3}-\frac{\tau}{6}-\frac{1}{6}\right]e_2-\left[\beta^2-\frac{\beta}{2}\right]e_1,$$

$$\widetilde{H}(e_3,e_2)e_2 = \left[-\frac{\tau}{6}+\frac{5\beta^2}{3}+\frac{\beta}{2}-\frac{1}{6}\right]e_3.$$

6. Conclusion

This study considers a new type of quarter-symmetric non-metric connection called (M,φ_3,f_1) on almost contact manifolds. Some curvature properties are presented for this connection. It is noticed that by considering some cases on the connection (M,φ_3,f_1) inverted to quarter symmetric metric connection, quarter symmetric non-metric connection and semi symmetric non-metric connection before are obtained. The connection (M,φ_3,f_1) has been studied on 3-dimensional Quasi-Sasakian manifold M in order to obtain more understandable results, and it has been arrived that the manifold given with (M,φ_3,f_1) transforms into an η −Einstein manifold respect to Riemannian connection ∇ under some curvature conditions i.e. $(\tilde{P}(X,\xi).\widetilde{H})(Y,V)W = 0$, $(\tilde{R}(X,\xi).\tilde{R})(Y,V)W = 0$, $(\widetilde{H}(X,\xi).\tilde{S})(Y,Z) = 0$, $(\tilde{R}(X,\xi).\tilde{S})(Y,Z) = 0$, and $(\tilde{P}(X,\xi).\tilde{S})(Y,Z) = 0$. It has also been proven that the Ricci tensor $\tilde{S}$ of manifold M is not symmetric and the first Bianchi identity has been satisfied only under some special conditions respect to (M,φ_3,f_1). Finally, an example has been found to show that there is a connection (M,φ_3,f_1) on 3-dimensional Quasi-Sasakian manifold. As a direction for future work, the non-metric connection introduced in this study could be further explored to analyze the geometric behavior of Ricci solitons, Yamabe solitons, and Mean Curvature Flow solitons e.g. [28]. Investigating these structures under a generalized connection may offer new perspectives on spacetime geometry and gravity theories.

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