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SPECTRAL AND COERCIVITY ANALYSIS OF A TIME-SPACE FRACTIONAL ADVECTION-DIFFUSION SYSTEM

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Abstract

In this paper, we consider a time–space fractional advection–diffusion equation that models complex transport phenomena in heterogeneous media. The equation involves a Caputo fractional derivative and a fractional Laplacian. A detailed mathematical analysis of the proposed model is presented. The spectral properties of the corresponding operator are examined and a uniform coercivity condition is obtained under certain assumptions. It is also shown that the operator is sectorial, which allows using semigroup theory to prove existence and uniqueness of mild solutions. In contrast to most existing works that mainly focus on numerical approximations or particular cases, we provide a unified functional analytic framework for the fractional advection–diffusion model, clarifying its stability and solvability. The proposed approach gives us strong theoretical guarantees but may involve challenges for numerical implementation due to the nonlocal nature of the operators.

Keywords: Time-space fractional advection-diffusion equation, fractional Laplacian, coercivity estimate, spectral analysis

1. Introduction

Fractional partial differential equations (FPDEs) offer a powerful framework for modeling anomalous transport phenomena that deviate from classical Fickian behavior. These equations are particularly well suited to capture long-range temporal memory and spatial nonlocality, which are common in heterogeneous media and complex physical systems [6, 2, 7, 5, 8].

A representative example is the time–space fractional advection–diffusion equation, which extends the classical advection–diffusion model by incorporating a Caputo fractional derivative of order $\alpha \in (0,1)$ in time and a fractional Laplacian of order $\beta \in (0,1]$ in space [1, 3]. These nonlocal operators allow for the modeling of subdiffusive processes and Lévy-type spatial jumps, but they also introduce substantial analytical challenges, particularly concerning stability and spectral properties.

Recent studies have addressed fractional advection–diffusion and related equationsfrom both numerical and analytical perspectives. Arshad et al. [14] proposed a finite scheme for the equation and provided rigorous stability and convergence analysis. Hrizi et al. [4] considered the inverse reconstruction of a singular time-dependent source in a fractional subdiffusion model. Arfaoui and Ben Makhlouf [12] proved novel stability results for fractional advection diffusion systems by employing Fourier decomposition and Mittag-Leffler functions. Albritton et al. [15] introduced a variational framework for the kinetic Fokker–Planck equation and established new regularity and dissipation results. Lin et al. [9] studied a Calderón-type inverse problem for a nonlocal diffusion equation with time-dependent coefficients and established global uniqueness theorems. These works, while valuable, often emphasize numerical approximations or problem-specific settings.

In contrast, the present study develops a unified analytical framework for the continuous operator associated with the time–space fractional advection–diffusion equation. A rigorous study of the associated space–time fractional operator is provided, including spectral characterization and a uniform coercivity estimate. The operator is shown to be sectorial under suitable assumptions, which allows the application of semigroup theory for the existence and uniqueness of mild solutions [11, 13]. The novelty of this work lies in establishing both sectoriality and uniform coercivity in a general setting, thereby providing the first comprehensive theoretical foundation that guarantees stability and well-posedness for this class of fractional equations. This fills an important gap in the mathematical understanding of anomalous transport processes.

The remainder of this paper is organized as follows. Section 2 presents the mathematical preliminaries, including definitions of the Caputo derivative, fractional Laplacian, and relevant function spaces. Section 3 introduces the fractional advection–diffusion equation together with its initial and boundary conditions. In Section 4, a detailed theoretical analysis of the spectral and stability properties of the associated operator is provided. Finally, Section 5 concludes the paper.

2. Preliminaries

This section introduces the fundamental mathematical definitions used throughout the paper.

Definition 2.1 [7] Let $u: [a,b] \to \mathbb{R}$ be a function such that $u \in C^n[a,b]$, where $n \in \mathbb{N}$ satisfies $n-1 \leq \alpha < n$, and let Γ denote the Euler gamma function. The left-sided Caputo fractional derivative of order α is defined by

$$ {}_a^C D_t^{\alpha} u(t) \coloneqq \frac{1}{\Gamma(n-\alpha)} \int_a^x (t-s)^{n-\alpha-1} u^{(n)}(t)\, dt. \tag{2.1}$$

Definition 2.2 [3] Let $0 < \beta < 1$ and let $T(x,t)$ be defined on a bounded domain $\Omega \subset \mathbb{R}^d$. The fractional Laplacian $(-\Delta)^{\beta}$ is defined via the singular integral

$$ (-\Delta)^{\beta} T(x,t) \coloneqq C_{\beta} \int_{\Omega} \frac{T(y,t) - T(x,t)}{|x-y|^{d+2\beta}}\, dy, \tag{2.2}$$

where C_{β} is a positive constant depending on β, the spatial dimension d . It is defined as:

$$C_\beta = \frac{2^{2\beta}\Gamma(\beta + d/2)}{\pi^{d/2}|\Gamma(-\beta)|}.$$

Definition 2.3 [16] For a domain $(\Omega \subset \mathbb{R}^d)$, the Lebesgue space $L^p(\Omega), 1 \le p < \infty$, is defined by

$$\|T\|_{L^p(\Omega)} := \left(\int_\Omega |T(x,t)|^p dx\right)^{1/p}.$$

The fractional Sobolev space $H^s(\Omega)$, for $s > 0$, is given by the norm

$$\|T\|_{H^s(\Omega)} := \left(\|T\|^2_{L^2(\Omega)} + \int_\Omega \int_\Omega \frac{|T(x,t) - T(y,t)|^2}{|x-y|^{d+2s}} dx\, dy\right)^{1/2}.$$

Definition 2.4 [10] Let $T(x,t) \in L^1(\mathbb{R}^d)$. Its Fourier transform with respect to the spatial variable x is defined as

$$\hat{T}(\xi, t) := \mathcal{F}[T(x,t)] = \int_{\mathbb{R}^d} T(x,t) e^{-2\pi i x \cdot \xi} dx.$$

Moreover, the fractional Laplacian satisfies

$$\mathcal{F}\left[(-\Delta)^\beta T\right](\xi, t) = |\xi|^{2\beta} \hat{T}(\xi, t).$$

We now proceed to formulate the governing fractional advection-diffusion model incorporating the above concepts.

3. Mathematical model

We consider the time-space fractional advection-diffusion equation defined on a bounded domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. The model takes the form

$$\tau^{1-\alpha}\, {}_0^C D_t^\alpha T(x,t) + v(x,t) \cdot \nabla T(x,t) - \kappa(-\Delta)^\beta T(x,t) = Q(x,t), \quad x \in \Omega,\ t > 0, \tag{3.1}$$

where $T(x,t)$ denotes the unknown scalar field, $v(x,t)$ is a prescribed velocity field, and $Q(x,t)$ is a given source term. The parameters satisfy $0 < \alpha < 1$, $0 < \beta \le 1$, $\tau > 0$, and $\kappa > 0$. The operator $(-\Delta)^\beta$ is defined as in Section 2.

The corresponding differential operator in the homogeneous case is defined by

$$\mathcal{L}T := \tau^{1-\alpha}\, {}_0^C D_t^\alpha T + v(x,t) \cdot \nabla T - \kappa(-\Delta)^\beta T. \tag{3.2}$$

The Caputo fractional derivative is given by

$${}_0^C D_t^\alpha T(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial T(x,\xi)}{\partial \xi} (t-\xi)^{-\alpha} d\xi, \quad 0 < \alpha < 1. \tag{3.3}$$

The system is equipped with the initial and boundary conditions

$T(x, 0) = T_0(x), \quad x \in \Omega,$ (3.4)
$T(x, t) = g(x, t), \quad x \in \partial\Omega,\ t > 0.$ (3.5)

The operator $\mathcal{L}$ defines a nonlocal space-time evolution equation, which under suitable assumptions generates an analytic semigroup on appropriate Banach spaces ([11]). The spectral and stability properties of $\mathcal{L}$ are investigated in the subsequent section.

4. Stability analysis

We analyze the stability properties of the operator $\mathcal{L}$ defined in Section 3. We first establish the sectoriality of $\mathcal{L}$ and then derive a uniform coercivity estimate.

Theorem 4.1 (Sectoriality of $\mathcal{L}$). Let $\mathcal{L}$ be the operator defined in (3.2), acting on $T \in D(\mathcal{L}) \subset L^p(\Omega)$, where $1 \leq p < \infty$. Assume that the velocity field $v(x, t)$ is uniformly bounded, i.e.,

$$|v(x, t)| \leq M$$

for some constant $M > 0$. If

$$2\beta - \alpha \leq 0,$$

then $\mathcal{L}$ is sectorial in $L^p(\Omega)$. Moreover, its spectrum satisfies

$$\sigma(\mathcal{L}) \subset \Sigma_\theta \coloneqq \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda - \omega)| \leq \theta\},$$

for some $\omega \in \mathbb{R}$ and $\theta < \frac{\pi}{2} - \frac{\alpha\pi}{2}$.

Proof. Applying the Fourier transform to $\mathcal{L}T$ yields

$$\widehat{\mathcal{L}T}(\xi) = \left(\tau^{1-\alpha}(i\xi)^\alpha + iv(x, t) \cdot \xi - \kappa|\xi|^{2\beta}\right)\hat{T}(\xi),$$

where $\hat{T}(\xi) = \mathcal{F}[T](\xi)$. We define the Fourier symbol

$$\lambda(\xi) \coloneqq \tau^{1-\alpha}(i\xi)^\alpha + iv(x, t) \cdot \xi - \kappa|\xi|^{2\beta}.$$

Using the identity

$$(i\xi)^\alpha = |\xi|^\alpha e^{i\frac{\alpha\pi}{2}\mathrm{sgn}(\xi)},$$

the real and imaginary parts of $\lambda(\xi)$ are given by

$$\mathrm{Re}\,\lambda(\xi) = \tau^{1-\alpha}|\xi|^\alpha \cos\left(\frac{\alpha\pi}{2}\right) - \kappa|\xi|^{2\beta},$$
$$\mathrm{Im}\,\lambda(\xi) = \tau^{1-\alpha}|\xi|^\alpha \sin\left(\frac{\alpha\pi}{2}\right) + v(x, t) \cdot \xi.$$

As $|\xi| \to \infty$, the term $-\kappa|\xi|^{2\beta}$ dominates in $\mathrm{Re}\,\lambda(\xi)$ due to the condition $2\beta - \alpha \leq 0$, implying

$$\mathrm{Re}\,\lambda(\xi) \to -\infty.$$

Hence, the spectrum is shifted to the left-half plane.

For $|\xi| \to 0$, the boundedness of $v(x,t)$ ensures that the term $\tau^{1-\alpha}|\xi|^{\alpha}$ dominates the advection contribution. Consequently,

$$\arg\lambda(\xi) = \arctan\left(\frac{\mathrm{Im}\lambda(\xi)}{\mathrm{Re}\lambda(\xi)}\right),$$

is uniformly bounded and satisfies

$$|\arg\lambda(\xi)| \leq \theta < \frac{\pi}{2} - \frac{\alpha\pi}{2}.$$

Therefore, $\sigma(\mathcal{L}) \subset \Sigma_{\theta}$, and the operator $\mathcal{L}$ is sectorial.
We conduct the analysis in the framework of $L^p(\Omega)$ and fractional Sobolev spaces $H^s(\Omega)$ introduced in Section 2.

Theorem 4.2 (Uniform Coercivity) Let $\mathcal{L}$ be the operator defined in (3.2), and assume that the hypotheses of Theorem 4.1 are satisfied. Then, there exists a constant $c > 0$ such that

$$c\tau^{1-\alpha}\|T\|_{L^p(\Omega)} \leq \|(\mathcal{L} + \lambda I)T\|_{L^p(\Omega)}, \quad \forall T \in L^p(\Omega),\ \lambda > 0.$$

Proof. Taking the Fourier transform, we obtain

$$\widehat{(\mathcal{L} + \lambda I)}T(\xi) = \Lambda(\xi)\hat{T}(\xi),$$

where $\Lambda(\xi)$ is given by

$$\Lambda(\xi) := \lambda + \tau^{1-\alpha}(i\xi)^{\alpha} + iv(x,t)\cdot\xi - \kappa|\xi|^{2\beta}.$$

We estimate the $L^p - norm$ as

$$\|(\mathcal{L} + \lambda I)T\|_{L^p(\Omega)} = \left\|\mathcal{F}^{-1}\left[\Lambda(\xi)\hat{T}(\xi)\right]\right\|_{L^p(\Omega)} \geq \inf_{\xi\in\mathbb{R}^d}|\Lambda(\xi)| \cdot \|T\|_{L^p}.$$

We analyze $|\Lambda(\xi)|$ in two asymptotic regimes.

(i) For $|\xi| \to \infty$.

$$\Lambda(\xi) = -\kappa|\xi|^{2\beta} + \tau^{1-\alpha}|\xi|^{\alpha}e^{i\alpha\frac{\pi}{2}\mathrm{sgn}(\xi)} + iv(x,t)\cdot\xi + \lambda.$$

The leading term is the fractional Laplacian $(-\kappa|\xi|^{2\beta})$ since $2\beta - \alpha \leq 0$. Thus

$$|\Lambda(\xi)| \geq \kappa|\xi|^{2\beta} - C_1|\xi|^{\alpha} - C_2|\xi| - \lambda,$$

where $C_1, C_2 > 0$ depend on $\tau^{1-\alpha}$ and v. For sufficiently large $|\xi|$, the negative term dominates, guaranteeing $|\Lambda(\xi)| \geq c_1 > 0$.

(ii) For $|\xi| \to 0$:

$$\Lambda(\xi) = \lambda + \tau^{1-\alpha}|\xi|^{\alpha} e^{i\alpha\frac{\pi}{2}\mathrm{sgn}(\xi)} + iv(x,t)\cdot\xi - \kappa|\xi|^{2\beta}.$$

The dominant contribution is $\lambda > 0$, thus

$$|\Lambda(\xi)| \geq \lambda - C_3|\xi|^{\alpha} - C_4|\xi| - C_5|\xi|^{2\beta} \geq c_2 > 0.$$

Combining both regimes:

$$\inf_{\xi\in R^d}|\Lambda(\xi)| \geq c$$

where $c := \min\{c_1, c_2\} > 0$. Therefore,

$$\|(\mathcal{L} + \lambda I)T\|_{L^p} \geq c\|T\|_{L^p}.$$

Finally, for small $|\xi|$, the term $\tau^{1-\alpha}|\xi|^{\alpha}$ dominates over the advection and diffusion terms, yielding

$$\|(\mathcal{L} + \lambda I)T\|_{L^p} \geq c\tau^{1-\alpha}\|T\|_{L^p}.$$

Remark 4.3 Theorems 4.1 and 4.2 ensure that the operator $\mathcal{L}$ generates an analytic semigroup in $L^p(\Omega)$. Therefore, the abstract Cauchy problem associated with Eq. (3.1)

$$\begin{cases} \mathcal{L}T(x,t) = Q(x,t),\ x\in\Omega,\ t>0, \\ T(x,0) = T_0(x),\ x\in\Omega, \\ T(x,t) = g(x,t),\ x\in\partial\Omega,\ t>0, \end{cases}$$

admits a unique mild solution under suitable assumptions on the data. This result directly follows from the classical theory of semigroups of linear operators and the framework of fractional evolution equations (see [11, 13]).

5. Conclusion

In this paper, we investigated a fractional advection–diffusion model that incorporates both temporal memory effects through the Caputo fractional derivative and spatial nonlocal behavior through the fractional Laplacian. A rigorous spectral analysis of the associated operator was carried out, and we established a uniform coercivity estimate under suitable assumptions. Moreover, we proved that the operator is sectorial, which, in turn, guarantees the existence and uniqueness of mild solutions via semigroup theory. These results provide a solid theoretical foundation for the continuous problem and extend previous works that mainly focused on numerical methods or special cases.

The main advantage of our approach lies in its unified analytical framework: it simultaneously captures memory and nonlocal effects and ensures stability through sectoriality and coercivity estimates. However, the analysis also has certain limitations. The results rely on parameter restrictions such as $2\beta - \alpha \leq 0$, and the work is confined to the continuous setting without addressing discretization or numerical implementation. Future research may extend the present framework to more general parameter regimes, develop numerical methods consistent with the theoretical stability, and explore applications of such fractional transport models to real-world phenomena.

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