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Half inverse problem for singular Sturm-Liouville operator with discontinuity conditions inside an interval.

Nilüfer TOPSAKAL [1,*], Yalçin GÜLDÜ[1] and Rauf AMİROV[1]

[1] *Department of Mathematics, Faculty of Science, Cumhuriyet University, SIVAS 58140, TURKEY*

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Abstract. In this study, half inverse problem for singular Sturm-Liouville operator is considered. It is shown by Hochstadt and Lieberman's method that if the potential function $q(x)$ prescribed on interval $\left(\frac{\pi}{2},\pi\right)$, then a single spectrum sufficies to determine $q(x)$ on the whole interval $(0,\pi)$.

1. Introduction

Consider the following Sturm-Liouville operator L defined by

$$Ly := -y'' + \left[\frac{C}{x^a} + q(x)\right] y = \lambda y,\ \lambda = k^2 \tag{1.1}$$

on the interval $0 < x < \pi$ with the boundary conditions

$$U(y) := y(0) = 0, V(y) := \ y(\pi) = 0 \tag{1.2}$$

and with the jump conditions

$$\begin{cases} y\left(\frac{\pi}{2}+0\right) = \alpha y\left(\frac{\pi}{2}-0\right) \\ y'\left(\frac{\pi}{2}+0\right) = \alpha^{-1} y'\left(\frac{\pi}{2}-0\right) \end{cases} \tag{1.3}$$

where λ is a spectral parameter, $C \in \mathbb{R}, \alpha \neq 1, \alpha > 0, a \in [1, 3/2)$, $q(x)$ is a real valued bounded function and $q(x) \in L_2(0,\pi)$.

Inverse problem for Sturm-Liouville operators consists of reconstruction of the operator by its spectral data. Half inverse problem for Sturm-Liouville operators is to determine a differential operator by one spectrum and half of its potential (see [1]-[6]). Hochstadt and Lieberman [1] firstly considered the half inverse problem for the Sturm-Liouville problem and showed that if potential function $q(x)$ is prescribed on $\left(\frac{\pi}{2},\pi\right)$, then $q(x)$ on the whole interval $(0,\pi)$ can be uniquely determined by one spectrum. After that Hald [2] proved that if the potential is known over half of the interval and if one boundary condition is given, then the potential and the other boundary condition are uniquely determined by the eigenvalues. In [3], [4], Malamud and Gesztezy, Simon obtained some new uniqueness results in inverse spectral

*Corresponding author. *Email address:* `ntopsakal@cumhuriyet.edu.tr`

analysis with partial information for scalar and matrix Sturm-Liouville equations, respectively. In 2001, Sakhnovich [5] proved the existence of the solution for the half-inverse problem of Sturm-Liouville problems and gave a method of reconstructing this solution under some conditions. In [6], Hryniv and Mykytyuk studied the half-inverse spectral problems for Sturm-Liouville operators with singular potentials. In [7], Koyunbakan and Panakhov considered the half inverse problem for diffusion operators on the finite interval.

In this study we discuss the half-inverse problem for singular Surm-Liouville operator by using Hochstadt and Lieberman's method [1].

2. The integral representation for the solution

We define $y_1(x) = y(x)$, $y_2(x) = (\Gamma y)(x) = y'(x) - u(x)\, y(x)$, where $u(x) = C \ln x$ for $a = 1$, $u(x) = C\dfrac{x^{1-a}}{1-a}$, for $1 < a < 3/2$. Let us write the expression in left hand side of equation (1.1) as follows

$$Ly = -\left[(\Gamma y)(x)\right]' - u(x)(\Gamma y)(x) - u^2(x)\, y + q(x)\, y. \tag{2.1}$$

Then equation (1.1) reduces to the system,

$$\begin{cases} y_1' - y_2 = u(x)\, y_1 \\ y_2' + k^2 y_1 = -u(x)\, y_2 - u^2(x)\, y_1 + q(x)\, y_1 \end{cases} \tag{2.2}$$

with the boundary conditions

$$y_1(0) = 0, y_1(\pi) = 0 \tag{2.3}$$

and with the jump conditions

$$\begin{cases} y\left(\dfrac{\pi}{2} + 0\right) = \alpha y\left(\dfrac{\pi}{2} - 0\right) \\ y'\left(\dfrac{\pi}{2} + 0\right) = \alpha^{-1} y'\left(\dfrac{\pi}{2} - 0\right). \end{cases} \tag{2.4}$$

We can write the system (2.2) in matrix form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -k^2 - u^2 + q & -u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.5}$$

or $y' = Ay$ such that $A = \begin{pmatrix} u(x) & 1 \\ -k^2 - u^2(x) + q(x) & -u(x) \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Since $x = 0$ is a regular singular end point at equation (2.5), by Theorem 2 in [9] (see Remark 1-2, p.56), there exists only one solution of the system (2.2) which satisfies the initial condititons $y_1(0) = 1$, $y_2(0) = ik$.

Definition 2.1. *The first component of the solution of system (2.2) which satisfies the initial condititons $y_1(\xi) = \upsilon_1, y_2(\xi) = (\Gamma y)(\xi) = \upsilon_2$ is called the solution of equation (1.1) which satisfies the initial conditions $y_1(0) = 1$, $y_2(0) = ik$.*

The following theorem was proved in [10].

Theorem 2.1. *For each solution of system (2.2) satisfying the initial conditions* $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ ik \end{pmatrix}$ *and the jump conditions (2.4) the following expression is valid:*

$$\begin{cases} y_1 = e^{ikx} + \int\limits_{-x}^{x} K_{11}(x,t)\, e^{ikt} dt \\ y_2 = ike^{ikx} + b(x)\, e^{ikx} + \int\limits_{-x}^{x} K_{21}(x,t)\, e^{ikt} dt + ik \int\limits_{-x}^{x} K_{22}(x,t)\, e^{ikt} dt \end{cases} \text{, for } x < \frac{\pi}{2}$$

$$\begin{cases} y_1 = \alpha^+ e^{ikx} + \alpha^- e^{ik(\pi - x)} + \int\limits_{-x}^{x} K_{11}(x,t)\, e^{ikt} dt \\ y_2 = ik\left(\alpha^+ e^{ikx} - \alpha^- e^{ik(\pi - x)}\right) \\ \quad + b(x)\left[\alpha^+ e^{ikx} + \alpha^- e^{ik(\pi - x)}\right] \\ \quad + \int\limits_{-x}^{x} K_{21}(x,t)\, e^{ikt} dt + ik \int\limits_{-x}^{x} K_{22}(x,t)\, e^{ikt} dt \end{cases} \text{, for } x > \frac{\pi}{2}$$

where

$$b(x) = -\frac{1}{2}\int\limits_0^x \left[u^2(s) - q(s)\right] e^{-\frac{1}{2}\int\limits_s^x u(t)dt} ds,$$

$$K_{11}(x,x) = \frac{\alpha^+}{2} u(x),$$

$$K_{21}(x,x) = b'(x) - \frac{1}{2}\int\limits_0^x \left[u^2(s) - q(s)\right] K_{11}(s,s)\, ds - \frac{1}{2}\int\limits_0^x u(s)\, K_{21}(s,s)\, ds,$$

$$K_{22}(x,x) = -\frac{\alpha^+}{2}\left[u(x) + 2b(x)\right],$$

$$K_{11}(x, \pi - x + 0) - K_{11}(x, \pi - x - 0) = \frac{\alpha^-}{2} u(x), \quad \alpha^{\pm} = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right),$$

$$\frac{\partial K_{ij}(x,.)}{\partial x}, \frac{\partial K_{ij}(x,.)}{\partial t} \in L_2(0,\pi), i,j = 1,2.$$

3. Properties of the spectrum

In this section, properties of the spectrum of problem L will be studied. Let us denote problem L as L_0 in the case of $C = 0$ and $q(x) \equiv 0$.

When $C = 0$ and $q(x) \equiv 0$ it is easily shown that solution $\varphi_0(x,k)$ satisfying the initial conditions $\varphi_0(0,k) = 0, (\Gamma\varphi_0)(0,k) = k$ and the jump conditions (2.4) can be written as

$$\varphi_0(x,k) = \frac{y_0(x,k) - \overline{y_0(x,k)}}{2i} = \begin{cases} \sin kx & \text{, for } x < \frac{\pi}{2} \\ \alpha^+ \sin kx + \alpha^- \sin k(\pi - x) & \text{, for } x > \frac{\pi}{2} \end{cases} \tag{3.1}$$

Clearly the characteristic function of problem L_0 is

$$\Delta_0(k) = \alpha^+ \sin k\pi = 0 \tag{3.2}$$

We denote the characteristic function, eigenvalues and normalizing numbers of problem L, by $\Delta(k)$, $\{k_n\}$ and $\{\alpha_n\}$, respectivly. Let $\varphi(x,k)$, $C(x,k)$ and $\psi(x,k)$ are solutions of equation (1.1) under the following initial conditions:

$$\varphi(0,k) = 0, (\Gamma\varphi)(0,k) = 1, \psi(\pi,k) = 0, (\Gamma\psi)(\pi,k) = 1, C(0,k) = 1, (\Gamma C)(0,k) = 0.$$

Obviously,

$$\Delta(k) = \langle \psi(x,k), \varphi(x,k) \rangle, \tag{3.3}$$

where

$$\langle y(x), z(x) \rangle := y(x)(\Gamma z)(x) - (\Gamma y)(x) z(x).$$

According to the Liouville formula, $\langle \psi(x,k), \varphi(x,k) \rangle$ is not depended on x. Clearly, for each x, functions $\langle \psi(x,k), \varphi(x,k) \rangle$ are entire in k and

$$\Delta(k) = V(\varphi) = U(\psi) = \varphi(\pi,k) = \psi(0,k). \tag{3.4}$$

From Teorem 1 we have

$$\varphi(x,k) = \varphi_0(x,k) + \int_0^\pi \widetilde{K}_{11}(\pi,t)\sin kt dt.$$

Therefore we obtain

$$\Delta(k) = \Delta_0(k) + \int_0^\pi \widetilde{K}_{11}(\pi,t)\sin kt dt \tag{3.5}$$

where $\widetilde{K}_{11}(x,t) = K_{11}(x,t) - K_{11}(x,-t)$.

Lemma 3.1. *[12]* ***i)*** *Eigenvalues of problem L are real and simple.*

ii)*Eigenvalues of problem L have the following asymptotic behaviour*

$$k_n = k_n^0 + \frac{d_n}{k_n^0} + \frac{\delta_n}{k_n^0} \tag{3.6}$$

where $\delta_n \in \ell_2$ and

$$d_n = \frac{\alpha^+ \cos\left(k_n^0 + \varepsilon_n\right)\pi - \alpha^-}{2\dot{\Delta}_0\left(k_n^0\right)} u(\pi)$$

is a bounded sequence.

4. Half inverse problem

In this section, we state the main result. Together with L, we consider the problem $\widetilde{L}$ of the same form but the different potential function $\widetilde{q}(x)$ and coefficients $\widetilde{C}$, $\widetilde{\alpha}$.

The following Lemmas are important to prove our result.

Lemma 4.1. *[2] If $k_n = \widetilde{k}_n$, $n = 1, 2, ...$ then $\alpha = \widetilde{\alpha}$.*

Lemma 4.2. *If $k_n = \widetilde{k}_n$, $n = 1, 2, ...$ then $C = \widetilde{C}$.*

Proof. Using asymptotic behavior of k_n and [2], then we have

$$k_n^0 + \frac{d_n}{k_n^0} + \frac{\delta_n}{k_n^0} = \widetilde{k}_n^0 + \frac{\widetilde{d}_n}{\widetilde{k}_n^0} + \frac{\widetilde{\delta}_n}{\widetilde{k}_n^0},$$

$$\widetilde{d}_n - d_n = \delta_n - \widetilde{\delta}_n,$$

$$\lim_{n\to\infty} \left(\widetilde{d}_n - d_n\right) = 0.$$

It means that $C = \widetilde{C}$. □

Theorem 4.1. *If $k_n = \widetilde{k}_n$ for all $n \in \mathbb{N}$ and $q(x) = \widetilde{q}(x), x \in \left(\frac{\pi}{2}, \pi\right)$, then $q(x) = \widetilde{q}(x)$ almost everywhere in $[0, \pi]$.*

Proof. Since $\varphi(x, k)$ and $\widetilde{\varphi}(x, k)$ are solutions of equation (1.1), we can write

$$-\widetilde{\varphi}''(x, k) + [\widetilde{u}'(x) + \widetilde{q}(x)]\, \widetilde{\varphi}(x, k) = k\widetilde{\varphi}(x, k)$$
$$-\varphi''(x, k) + [u'(x) + q(x)]\, \varphi(x, k) = k\varphi(x, k).$$

Multiplying the first equation by $\varphi(x, k)$, the second equation by $\widetilde{\varphi}(x, k)$ and substracting and integrating from 0 to π, we obtain

$$\frac{d}{dx} \langle \widetilde{\varphi}(x, k), \varphi(x, k)\rangle = (q(x) - \widetilde{q}(x))\, \widetilde{\varphi}(x, k)\, \varphi(x, k),$$

$$\langle \widetilde{\varphi}(x, k), \varphi(x, k)\rangle \left[\Big|_0^{\frac{\pi}{2}-0} + \Big|_{\frac{\pi}{2}+0}^{\pi}\right] = \int_0^{\pi} [q(x) - \widetilde{q}(x)]\, \widetilde{\varphi}(x, k)\, \varphi(x, k)\, dx.$$

Applying the assumption $q(x) = \widetilde{q}(x), x \in \left(\frac{\pi}{2}, \pi\right)$ in our hypothesis we get

$$\langle \widetilde{\varphi}(x, k), \varphi(x, k)\rangle \left[\Big|_0^{\frac{\pi}{2}-0} + \Big|_{\frac{\pi}{2}+0}^{\pi}\right] = \int_0^{\pi} [q(x) - \widetilde{q}(x)]\, \widetilde{\varphi}(x, k)\, \varphi(x, k)\, dx.$$

Define

$$F(k) := \int_0^{\pi/2} [q(x) - \widetilde{q}(x)]\, \widetilde{\varphi}(x, k)\, \varphi(x, k)\, dx \tag{4.1}$$

where

$$\widetilde{\varphi}(x,k)\varphi(x,k) = -\cos 2kx + \int_0^x K_1(x,t)\cos 2kt dt + \int_0^x K_2(x,t)\sin 2kt dt \quad (4.2)$$

and $K_{ij}(x,t), i = 1,2$ depend only on x, t.

Then from the boundary condition (1.3), we have

$$F(k_n) = 0, \text{ for all } n.$$

Let

$$\chi(k) := \frac{F(k)}{\Delta(k)} \quad (4.3)$$

which is an entire function. Since $F(k) = O(\exp \tau\pi)$ and $|\Delta(k)| \geq C_\delta |k| \exp \tau\pi$ for $k \in G_\delta := \{k : |k - k_n| > \delta\}$ where $\tau = |Imk|$, $C_\delta > 0$ [12],then $\chi(k)$ is a constant by the Liouville theorem. Therefore, using Theorem 1, (3.1), (4.2) and the Riemann-Lebesque lemma we get

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{R}}} \chi(k) = 0. \quad (4.4)$$

It means that $\chi(k) = 0$ on the whole $k-$plane.

It follows from (4.1) and (4.2) that

$$\int_0^{\pi/2} Q(x)\left\{\cos 2kx - \int_0^x K_1(x,t)\cos 2kt dt - \int_0^x K_2(x,t)\sin 2kt dt\right\} dx = 0 \quad (4.5)$$

for all k where $Q(x) := q(x) - \widetilde{q}(x)$. We can write equation (4.5) as

$$\int_0^{\pi/2} \cos 2ks \left[Q(s) + \int_0^{\pi/2} Q(x) K_1(x,t) dx\right] dt + \int_0^{\pi/2} \sin 2kt \int_0^{\pi/2} Q(x) K_2(x,t) dx dt = 0.$$

Therefore, we obtain from the completeness of the functions $(\cos 2kt, \sin 2kt)^T \in L_2(0,\pi) \oplus L_2(0,\pi)$ that

$$Q(s) + \int_0^{\pi/2} Q(x) K_1(x,t) dx = 0, \quad 0 < s < \frac{\pi}{2}. \quad (4.6)$$

Since the equation (4.6) is homogenous Volterra integral equation we get

$$Q(x) = 0, \text{ i.e. } q(x) = \widetilde{q}(x)$$

almost everywhere for $x \in (0,\pi)$. □

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