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SONLU GRUPLARDA SYLOW ÇİFTLERİ

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ABSTRACT

Consider a finite group X and let t denote a prime number. A Sylow $t-$subgroup of X is a $t-$subgroup S of X whose order is as large as is allowed by Lagrange's theorem, and $Syl_t(X)$ is the set all such subgroups. The essential theorem of group theory asserts that Sylow subgroups always exist and $|Syl_t(X)| \equiv 1$ mod t. In this note, we say that an ordered pair (t,n) is a Sylow pair if there exists group X with $|Syl_t(X)| = n$, where $n \geq 1$ is an integer. We prove that the ordered pair (7,15) is not a Sylow pair.

Keywords: Finite group, Sylow's theorem, Sylow pair

SYLOW PAIRS IN FINITE GROUPS

ÖZ

X bir sonlu grup ve t bir asal sayı olsun. X in bir Sylow t-alt grubu,Langrange teoremi ile büyük mertebeli X in bir alt grubu S ve $Syl_t(X)$ bu tür bütün alt grupların kümesini gösterir. Grup teorisinin temel teoremi, Sylow alt grupların her zaman mevcut olduğunu ve $|Syl_t(X)| \equiv 1$(modt) olduğunu belirtir. Bu makalede $n \geq 1$bir tamsayı ve $|Syl_t(X)| = n$ olacak şekilde bir X grubu varsa (t,n) sıralı çiftinin bir Sylow çifti olduğu gösterildi. Ayrıca (7,15) sıralı çiftinin de bir Sylow çifti olmadığı ispatlandı.

Anahtar Kelimeler: Sonlu grup, Sylow teoremi, Sylow çifti

INTRODUCTION

Fix a prime t. A finite group X is a t-group if $|X| = t^r$ for some $r \geq 0$. If $H \leq X$, then H is a $t-$subgroup of X if H is a $t-$group. A Sylow $t-$subgroup T of X is $t-$subgroup with $t \nmid$ t$|X:T|$, and $Syl_t(X)$ is the set of all such subgroups. In other words, if $|X| = t^b m$, where $b \geq 0$ and $t \nmid m \geq 1$, then $T \leq G$ is a Sylow $t-$subgroup of X provided that $|X| = t^b$. Thus, a Sylow $t-$subgroup of X is a $t-$subgroup whose order is as large as is allowed by Lagrange's theorem. The essential theorem of group theory asserts that Sylow subgroups always exist. This result is known as "Existence Sylow Theorem". Thus, this theorem is the converse of Lagrange's result for prime powers. Also, "Development Sylow Theorem" asserts that any $H \leq X$ with $|H| = t^c$ for some $c \geq 0$, is contained in a member of $Syl_t(X)$. As a corollary $|Syl_t(X)| \equiv 1$ mod t and $|Syl_t(X)| \mid |X|$.

Suppose t denotes a prime and $n(\geq 1) \in Z$. In this note, we call that the pair (t,n) is a Sylow pair if there exists group X with $|Syl_t(X)| = n$. As we said, if the ordered pair (t,n) is a Sylow pair, then n must be of the form $1+kt$ for some $k \geq 0$. As a Sylow $t-$subgroup of $Z_t \times Z_{t+1}$ is normal, the ordered pair $(t,1)$ is a Sylow pair. Also, we know that $|Syl_2(A_5)| = 5$, $|Syl_3(A_5)| = 10$, | and $|Syl_5(A_5)| = 6$. Hence, the pais (2,5),(3,10), and (5,6) are Sylow pairs.

Thus, the natural question is whether all ordered pairs $(t,1+kt)$ are Sylow pairs. In other words, if $k \geq 1$, can we find a finite group X with $|Syl_t(X)| = 1+kt$? In [1], the author proved that the answer is negative and he used a complicated proof to show that if t denotes a prime and $k \geq 0$ is an integer such that $1 < k < \frac{t+3}{2}$, then the ordered pair $(t,1+kt)$ is not a Sylow pair unless $1+kt = q^s$ for some prime q and some integer $s \geq 1$ or $k = \frac{t-3}{2}$ and $t > 3$ is a Fermat prime.

In this paper, we show that the ordered pair (7,15) is not a Sylow pair. Equivalently, we prove that the equation $|Syl_7(X)| = 15$ has no solution in finite groups.

Theorem 1. The ordered pair (7,15) is a not a Sylow pair. Equivalently, for any f inite group G, $|Syl_7(G)| \neq 15$.

MATERIAL AND METHODS

Because of the importance of the Sylow theorems, we bring them here.

Theorem 2. ***(Sylow's Thorems)*** Consider a prime t and let X denote a finite group. Then

(1) ***(Existence)*** $Syl_t(X) \neq \phi$.

(2) ***(Conjugacy)*** If $P,Q \in Syl_t(X)$, then $Q = g^{-1}Pg$ for some $g \in X$.

(3) ***(Development)*** For any $t-$subgroup $L \leq X$, $P \in Syl_t(X)$ containing L exists.

(4) ***(Number)*** $|Syl_t(X)| = |X : N_X(P)| = 1+kt$ for some $k \geq 0$, where $P \in Syl_t(X)$.

Proof. This is [[3], Theorem 1.7, 1.12, 1.14, 1.15, 1.17].

We need the following lemmas to prove the main result.

Lemma 3. Let $m \geq 1$ be an odd integer and X denote a finite group with $|X| = 2m$. Then $H \leq X$ of index 2 exists.

Proof. Take $x \in X$ of order 2. Let X act on itself by right multiplication and note that since $x \neq 1$, the element x has no fixed points in this action. Since $x^2 = 1$, the cycle structure of x consists of only $2-$cycles and $1-$cycles, but since there are no fixed points, we have only $2-$cycles. It follows that this permutation is a multiplication of m disjoint transpositions and hence is odd. Hence, $x \in X \cap A_{2m}$, and so $X \cap A_{2m}$ has index 2 in X.

Lemma 4. Consider a finite group X and a prime t. Assume that $O_t(X)$ is the biggest normal $t-$subgroup of X. If $P \in Syl_t(X)$ is abelian, then $Syl_t(X)$ contains two members T and S such that $T \cap S = O_t(X)$.

Proof. This is [[2], Theorem 5.28].

Lemma 5. If t is prime, X finite group, and $n = |Syl_t(H)|$, then $H \leq S_n$ with $n = |Syl_t(H)|$ exists.

Proof. First, we show that if $K \lhd X$ and $P \in Syl_t(X)$, then $N_X(PK) = N_X(P)K$. To prove this, since $P \in Syl_t(PK)$ and $PK \lhd L := N_X(PK)$, by using Frattini Argument in L, we get

$$L = N_X(PK) = N_L(P)PK = N_L(P)K = KN_X(P)$$

because $K \lhd X$ and we know that $N_L(P) \subseteq N_X(P) \subseteq N_X(PK)$. Conjugacy Sylow theorem (part 2 of Theorem 2) implies that conjugation is an action of X on $Syl_t(X)$. Let $K = \{g \in X \mid P^g = P \;\; for\, all \;\; P \in Syl_t(X)\}$ be the kernel of this action. Clearly, $K \lhd X$ and $K \subseteq N_X(P)$, where $P \in Syl_t(X)$. We obtain that

$$|X/K : N_{X/K}(PK/K)|$$
$$= |X/K : N_X(PK)/K| = |X : N_X(PK)|$$
$$= |X : N_X(P)K| = |X : N_X(P)| = |Syl_t(X)| = n.$$

As X/K is a subgroup of S_n, we have the desired conclusion by taking $H = X/K \leq S_n$.

In Lemma 6, a finite group X acts faithfully on $Syl_t(X)$ by conjugation and $|Syl_t(X)| = 1+kt$, where $k < t$ is an integer. Then we show that $t^2 \nmid |X|$.

Lemma 6. Let X begroup, t a prime, and $k \geq 1$ an integer. Assume that $|Syl_t(X)| = 1+kt$ and $\{g \in X \mid \forall P \in Syl_t(X) : g^{-1}Pg = P\} = \{1\}$. If $k < t$, then $t^2 \nmid |X|$.

Proof. Since the action of X on $Syl_t(X)$ is faithful and $|Syl_t(X)| = 1+kt > 1$, we see that $X \leq S_{1+kt}$. We know that disjoint cycles in the symmetric groups commute, and hence

$P = <(1,...,t),(t+1,...,2t),...,((k-1)t+1,...,kt)>$ is elementary abelian with $|P| = t^k$ of S_{1+kt}. Also, we observe that the largest power of t in $|S_{1+kt}| = (1+\mathrm{kt})!$ is k. Thus, $H \in Syl_t(S_{1+kt})$. As $X \leq S_{1+kt}$, any $P \in Syl_t(X)$ is elementary abelian.

By using Lemma 4, we choose $P, Q \in Syl_t(X)$ with $P \cap Q = O_t(X)$. But the action on $Syl_t(X)$ is faithful. This forces that $O_t(X) = P \cap Q = 1$.

We claim that $P \cap Q = P \cap N_X(Q)$. To do this, we see that $P \cap N_X(Q)$ is a $t-$subgroup of $N_X(Q)$, and so $P \cap N_X(Q) \subseteq \mathrm{Q}$. This implies that $P \cap Q = P \cap N_X(Q)$.

Now,we have $|Orbit_P(Q)| = |P : P \cap N_X(Q)| = |P : P \cap Q| = |P|$. But we know that $Orbit_P(Q) \subseteq Syl_t(X)$ and
$|\mathrm{P}| = |Orbit_P(Q)| \leq |Syl_t(X)| = 1 + kt$
$$\leq 1 + (t-1)t < t^2,$$
where we used the hypothesis that $k \leq t-1$. We obtain that $|P| < t^2$ and $|P| = t$.

MAIN RESULTS

Now, we prove Theorem 1.

Theorem 7. The ordered pair (7,15) is a not a Sylow pair. Equivalently, for any finite group G, $|Syl_7(G)| \neq 15$.

Proof. We suppose the finite group G satisfies $|Syl_7(G)| = 15$ and $|G|$ is minimal. Hence, G has fifteen Sylow $7-$subgroups and $|G|$ is minimal with this property. This implies that G acts faithfully on $Syl_7(G)$, and so $G \leq S_{15}$. Also, if $H \leq G$ with $|G : H| = 2$, then clearly we have $Syl_7(G) = Syl_7(H)$ which is impossible as $|G|$ is minimal with $|Syl_7(G)| = 15$. This forces that $|G : H| \geq 3$ for every $H < G$ and G does not contain any odd permutation in S_{15}. Therefore, $G \leq A_{15}$.

Let $P \in Syl_7(G)$. We show that $C_G(P) = P$. To do this, as the action of group G on $Syl_7(G)$ is faithful and $|Syl_7(G)| = 1 + 2(7)$, Lemma 6 yields that $|P| = 7$ and $P \subseteq C_G(P)$. We assume that $P = <x>$. If $Q \in Syl_7(G)$ with $Q^x = Q$, we observe that $P \subseteq N_G(Q)$ and so $P = Q$ as $Q \in Syl_7(N_G(Q))$ and $Q \trianglelefteq N_G(Q)$. This implies that the only fixed point of the permutation $x \in A_{15}$ in the action of G on $Syl_7(G)$ is P. But the order of x in G is 7. We deduce that $x \in G \leq A_{15}$ is of the form $(1,2,...,7)(8,9,...,14) \in A_{15}$. We get that $|C_{15}(x)| = 49$. Since $P \subseteq C_G(x) = G \cap C_{A_{15}}(x) \subseteq C_{A_{15}}(x)$ and $P \subseteq C_G(x)$, we have $|C_G(x)| = 7$ because $49 \nmid |G|$, Thus, $|P| = |C_G(P)| = 7$ and so $C_G(P) = P$.

Now, we show that $P < N_G(P)$. If $N_G(P) = P$, then
$$|G| = |G : N_G(P)| \times |P| = 15 \times 7.$$
For $x \in G$, let $o(x)$ be the order of x. We see that the group G contains $90 = 15(7-1)$ elements of order 7. As $|G| = 105$, we get that $L \in Syl_5(G)$ is normal, and $|\{x \in G \,|\, o(x) = 5\}| = 4$. If $||Syl_3(G)| = 7$, then $|\{x \in G \,|\, o(x) = 3\}| = 14$, which is impossible since $|G| = 105$. We conclude that $B \in Syl_3(G)$ should be normal and $|\{x \in G \,|\, o(x) = 3\}| = 2$. But this is also a contradiction as $|G| = 105$. This forces that $P = N_G(P)$ is impossible and so $P < N_G(P)$.

Since $|P| = 7$, we get $N_G(P)/C_G(P) = N_G(P)/P \leq Aut(P)$ has order $2, 3$, or 6 because $|Aut(P)| = 6$. We know that $|G : H| \geq 3$ for all $H < G$, and so $|G|$ is not of the form $2m$, where m is odd. This implies that the only possible case is $|N_G(P)/P| = 3$ and so $|G| = 3^2 \times 5 \times 7$. Using Sylow's theorem yields that $|Syl_3(G)| = 1$ or 7. If $T \in Syl_3(G)$ is unique, then P acts on T by conjugation, where $P \in Syl_7(G)$. As $|P| = 7$, we have $|C_P(g)| = 1$ or 7 for all $g \in T$. As $|T| = 9$, we should have an element $g_1(\neq 1) \in T$ with $|C_P(g_1)| = 7$. This forces that $P = C_P(g_1)$ and $g_1 \in C_G(P) = P$, which is not possible. We deduce that $|Syl_3(G)| = 7$.

Suppose that $Syl_3(G) = U_1, ..., U_7\}$. As $|U| = 9$, we see that U is abelian. Applying Lemma

4 forces that there are U_i and U_j in $Syl_3(G)$ such that $O_3(G)=U_i \bigcap U_j$. Without loss of generality, let $O_3(G)=U \bigcap U_2$. Also, U acts on $Syl_3(G)$ by conjugation. If there exists $i>1$ with $Stab_U(U_i)=U$, then $U \subseteq N_G(U_i)$ and so $U=U_i$ as U_i is the only Sylow $3-$subgroup of $N_G(U_i)$. This is a contradiction. We obtain that $Stab_U(U_i)=U \bigcap N_G(U_i) \quad =U \bigcap U_i < U$. Also, if there exists $j>1$ with $Stab_U(U_j)=1$, then we have an orbit of length 9 in the action of U on $Syl_3(G)$, which is impossible as $|Syl_3(G)|=7$. Therefore, for all $i>1$, we have:

$$|U:Stab_U(U_i)|=|U:U \bigcap N_G(U_i)|=|U:U \bigcap U_i|=3.$$

In particular, we see that $|O_3(G)|=|U \bigcap U_2|=3$. As

$$3=\min\{p \mid p \textit{ is prime and } p \mid |G|\},$$

we get that $O_3(G) \subseteq Z(G)$. This forces that $O_3(G) \subseteq C_G(P)=P$ and 3 divides $|P|=7$, which is a contradiction.

DISCUSSION AND CONCLUSION

In group theory, Sylow theorems provides a prominent tool for proving results about finite groups. The third Sylow theorem indicates that $|Syl_P(G)|=1+kp$ for some $k \geq 0$. In this note, we show that the converse of this result is not correct. The converse of this result says that if $k \geq 0$ and p is prime, then can we construct a group G with $|Syl_P(G)|=1+kp$? We could prove that the answer is negative for $p=7$ and $k=2$.

REFERENCES

[1] Marshall Hall. Jr. (1967). On the number of Sylow subgroups in a finite group. Journal of Algebra (7), 363-371.

[2] I. M. Isaacs. (1994). Algebra, A Graduate Course. American Mathematical Society.

[3] I. M. Isaacs. (2008). Finite Group Theory. American Mathematical Society.

[4] G. Navarro. (2003). Number of Sylow subgroups in p-solvable groups. Proc. American.