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Primitive Pythagorean Triples Whose Difference of Legs Is a Perfect Square

Research Article

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Abstract

Let (x, y, z) be a primitive Pythagorean triple such that $x^2 + y^2 = z^2$ with $\gcd(x, y, z) = 1$. In this paper, we determine all primitive Pythagorean triples for which the difference of the legs is a perfect square. We show that infinitely many such Pythagorean triples exist. In fact, we provide the parametric formulas of all sides of such Pythagorean triples depending on two parameters. To get the results, we use some elementary methods along with generalized Pell equations and some classical Diophantine equations from number theory.

Keywords: Pythagorean triples, Diophantine equations, Pell equations.

1. INTRODUCTION

The study of Pythagorean triples – that is, integer solutions (x, y, z) to the Diophantine equation $x^2 + y^2 = z^2$, dates back several millennia and is deeply interwoven with the history of mathematics. While commonly attributed to Pythagoras (circa 582–481 BC), evidence suggests that such triples were known much earlier, as shown by the Babylonian clay tablet Plimpton 322 (circa 1800 BC) and the Hindu Sulbasutras (around 800–500 BC), where they were used in altar constructions. These sources indicate not only an empirical understanding of the Pythagorean relationship but also the possible existence of systematic methods for generating such triples.

A Pythagorean triple (x, y, z) is said to be *primitive* if gcd(x, y, z) = 1. Every primitive Pythagorean triple can be generated uniquely—up to sign and ordering—by the classical Euclidean parametrization:

$$x = u^2 - v^2, y = 2uv,\ z = u^2 + v^2\ , \gcd(u,v) = 1,\ 2 \mid uv\ ,\ \ u > v > 0 \quad (1)$$

that is, u and v are coprime, and not both odd. This formula provides a complete description of all primitive Pythagorean triples.

Many intriguing arithmetic properties of primitive Pythagorean triples have been established. For instance, in any such triple, one leg is divisible by 3, another by 4, and the product xyz is always divisible by 60. Primitive Pythagorean triples in which the sum or difference of the legs is prime were studied in a notable paper by Wegener (1975). The area of a primitive Pythagorean triangle is never a perfect square, a result traced back to Fermat. Furthermore, extensive classifications have been made based on fixed values of area, perimeter, or other algebraic constraints, see Agarwal (2020) for detailed information.

The present paper aims to investigate primitive Pythagorean triples (x, y, z) for which x – y or y – x is a perfect square. For each condition, when some solutions exist, we provide all solutions as parametric families depending on two parameters and show that infinitely many such triples occur. Our methods rely on elementary number theory, classical parametrizations, and certain Diophantine arguments, including generalized Pell equations.

2. MAIN RESULTS

Throughout this note, (x, y,z) denotes a primitive Pythagorean triple parametrized as in (1) We begin by citing a result from Conrad (Theorem 3.3) related to generalized Pell equations. Let $D > 0$ be an integer, not perfect square. Recall that the equation $x^2 - Dy^2 = n$ is called a *generalized Pell equation*.

Theorem 2.1 (Conrad, Theorem 3.3) Fix $u = r + s\sqrt{D}$ where $r^2 - Ds^2 = 1$ with r and s in $\mathbf{Z}^+$. For each $n \in \mathbf{Z} - \{0\}$, every integral solution of $x^2 - Dy^2 = n$ is $\left(x' + y'\sqrt{D}\right)u^k$ where $x'^2 - Dy'^2 = n$, $k \in \mathbf{Z}$, and

$|x'| \leq \frac{\sqrt{|n|}\left(\sqrt{u}+\frac{1}{\sqrt{u}}\right)}{2}$ and $|y'| \leq \frac{\sqrt{|n|}\left(\sqrt{u}+\frac{1}{\sqrt{u}}\right)}{2\sqrt{D}}$.

Theorem 2.2 : Let (x, y, z) be a primitive Pythagorean triple with the parameters given in equation (1).

a) The equation $x - y = d^2$ holds for some nonnegative integer d if and only if there exist relatively prime positive integers a and b such that

$$u \coloneqq a^2 + 2ab + 2b^2, \quad v \coloneqq 2ab \quad \text{and} \quad d \coloneqq |a^2 - 2b^2|.$$

b) The equation $y - x = d^2$ holds for some nonnegative integer d if and only if there exist relatively prime positive integers a and b such that

$$u = \begin{cases} 2(a+b)b\ , & if\ \ a^2 < b^2 + 2ab \\ 2(a-b)a\ , & if\ b^2 + 2ab < a^2 \end{cases}, \quad v = a^2 + b^2 \ \text{ and } \ d = |(a+b)^2 - 2b^2|\,.$$

Proof : Let (x, y, z) be a primitive Pythagorean triple given in the form equation (1). Assume first that $x - y = d^2$. Then we have that

$$x - y = u^2 - v^2 - 2uv = (u - v)^2 - 2v^2 = d^2.$$

This equation is of the form

$$d^2 + 2V^2 = U^2$$

where $U = u - v, V = v$. In this equation, $\gcd(U, V) = 1$ since $\gcd(u, v) = 1$. This is a classical Diophantine equation and all its solutions are given by

$$U \coloneqq u - v = a^2 + 2b^2,$$
$$V \coloneqq v = 2ab,$$

for some relatively prime positive integers a and b such that $d := |a^2 - 2b^2|$, see P3.5 in (Ribenboim, pp. 39). . Thus, we get parametric solutions

$$u := a^2 + 2ab + 2b^2,$$
$$v := 2ab,$$

where $d := |a^2 - 2b^2|$.
Assume that $y - x = d^2$ for some positive integer d. Then we have that

$$y - x = 2uv - (u^2 - v^2) = d^2,$$
$$(u - v)^2 - 2v^2 = -d^2.$$

This equation is of the form

$$U^2 + d^2 = 2V^2$$

where $U = u - v,\ V = v$. In this equation, $\gcd(U, V) = 1$ since $\gcd(u, v) = 1$. This is another classical Diophantine equation, and all its solutions are given by

$$U := u - v = |a^2 - 2ab - b^2| = |(a - b)^2 - 2b^2|,$$
$$d := |a^2 + 2ab - b^2| = |(a + b)^2 - 2b^2|,$$
$$V := v = a^2 + b^2.$$

For proof of this, we cite Corollary 6.3.14. (2) in (Cohen, pp. 353). So,

$$u = \begin{cases} 2(a + b)b, & if\ a^2 < b^2 + 2ab \\ 2(a - b)a, & if\ b^2 + 2ab < a^2 \end{cases}$$
$$v = a^2 + b^2$$

provided that $d = |(a + b)^2 - 2b^2|$. It is easy to check that in either case, the given parametric forms of u, v and d leads to a primitive Pythagorean triple. This completes the proof.

Even though Theorem 2.2 provides a complete theoretical characterization of primitive Pythagorean triples such that the difference of whose legs is a perfect square, in practice, to find such Pythagorean triples for any given $d > 0$ requires more calculation if it really exists, since all parametric solutions of $|x - y| = d$ depends on the satisfying parameters,

$$a^2 - 2b^2 = \pm d \quad \text{or} \quad (a + b)^2 - 2b^2 = \pm d. \tag{2}$$

Both equations are generalized Pell equations. Thus, our first corollary reveals the connection between the aforementioned primitive Pythagorean triples and generalized Pell equations.

Corollary 2.3 : There exist primitive Pythagorean triples whose legs differ by the perfect square of non-zero integer d if and only if the generalized Pell equation $X^2 - 2Y^2 = \pm d$ has a non-trivial relatively prime solution, that is a solution with $\gcd(X, Y) = 1, XY \neq 0$.

By Theorem 2.1, we know that infinitely many solutions can be derived from a seed solution, if it exists. Thus, we state the following.

Corollary 2.4: Let $d > 0$ be a positive integer. Then, either there exist infinitely many primitive Pythagorean triples with the difference of legs equal to d^2 or there do not exist such primitive Pythagorean triples with this property.
To clarify the method for obtaining primitive Pythagorean triples with the mentioned property for any given integer d, we work on some examples with some small values of d . First, we state the following known particular case $d = 1$ just for completeness.

Corollary 2.5: There exist infinitely many primitive Pythagorean triples such that the difference of their legs is 1.

Corollary 2.6: There do not exist any primitive Pythagorean triples such that the difference of whose legs is 3^2 or 5^2.

Proof: Let us consider the generalized Pell equation

$$X^2 - 2Y^2 = \pm d$$

where $\gcd(X,\ Y) = 1,\ d = 3, 5$. By taking modulo 3 (modulo 5) we see that it has no solutions in integers. So, there do not exist any parameters a and b satisfying (2).

By using the bounds given in Theorem 2.1, we can detect the equation $X^2 - 2Y^2 = \pm d$ has a solution. This is the idea behind of the following corollary.

Corollary 2.7 : Let $1 \le d \le 100$ be an odd integer. Then a primitive Pythagorean triple exists for which the difference of whose legs is the square of d if and only if $d \in \{1, 7, 17, 23, 31, 41, 47, 49, 71, 73, 79, 89, 97\}$.

Now, we focus on the case $d = 7$ in detail to illustrate how we can find *allprimitive Pythagorean triples* with $|x - y| = 49$.

Example 2.8: The case $d = 7$. There exist infinitely many primitive Pythagorean triples such that the difference of their legs is 7^2.

First, assume that $x - y = 7^2$. By Theorem 2.2.(a), we need to find all parameters a and b satisfying $|a^2 - 2b^2| = 7$. Thus, we handle this equation in two subcases $a^2 - 2b^2 = 7$ and $a^2 - 2b^2 = -7$. Note that the fundamental solution of the Pell equation $X^2 - 2Y^2 = 1$ is $u = 3 + 2\sqrt{2} \cong 5.8$. Therefore, by Theorem 2.1, we check for integer solutions of the generalized Pell equation

$$x'^2 - 2y'^2 = \pm 7$$

in the range $|x'| \le 3.75$ and $|y'| \le 2.6$. We find the solutions $(\pm 3, \pm 1)$ and $(\pm 1, \pm 2)$. Thus, by applying Theorem 2.1 we get that

$$a_k + b_k\sqrt{2} = \pm\left(3 \pm \sqrt{2}\right)(3 \pm 2\sqrt{2})^k, \quad k \ge 0,\ d = 7, \qquad (3)$$

$$a_k + b_k\sqrt{2} = \pm\left(1 \pm 2\sqrt{2}\right)(3 \pm 2\sqrt{2})^k, \quad k \ge 0,\ d = -7. \qquad (4)$$

Each value of $a \coloneqq a_k$ and $b \coloneqq b_k$ gives a new set of parameters $u$and v, and therefore a new triple (x, y, z). The results are shown in Table 1 and Table 2.

Table 1. $x - y = 7^2$ with $a^2 - 2b^2 = 7$

k	a_k	b_k		u	v	x	y	z
0	3	1		17	6	253	204	325
1	13	9		565	234	264469	264420	373981
1	5	3		73	30	4429	4380	6229
2	75	53		19193	7950	305168749	305168700	431573749
2	27	19		2477	1026	5082853	5082804	7188205
3	437	309		651997	270066	352164443653	352164443604	498035732365
3	157	111		84145	34854	5865579709	5865579660	8295182341
4	2547	1801		22148705	9174294	406397462778589	406397462778540	574732803575461
4	915	647		2858453	1184010	6768873873109	6768873873060	9572633233309
5	14845	10497		752403973	311655930	468982319882019829	468982319882019780	663241157290349629
5	5333	3771		97103257	40221486	7811274583959853	7811274583959804	11046810456056245

Table 2. $x - y = 7^2$ with $a^2 - 2b^2 = -7$

k	a_k	b_k	u	v	x	y	z
0	1	2	13	4	153	104	185
1	11	8	425	176	149649	149600	211601
1	5	4	97	40	7809	7760	11009
2	65	46	14437	5980	172666569	172666520	244187369
2	31	22	3293	1364	8983353	8983304	12704345
3	379	268	490433	203144	199257042753	199257042704	281792012225
3	181	128	111865	46336	10366753329	10366753280	14660803121
4	2209	1562	16660285	6900916	229942454642169	229942454642120	325187737920281
4	1055	746	3800117	1574060	11963224330089	11963224330040	16918554097289
5	12875	9104	565959257	234428000	265353393399992049	265353393399992000	375266367767992049
5	6149	4348	129092113	53471704	13805550510141153	13805550510141104	19523996767468385

Secondly, assume that $y - x = 7^2$. By Theorem 2.2.(b), this time we need to find all parameters a and b satisfying $|(a+b)^2 - 2b^2| = 7$. Again, we need to consider two subcases $(a+b)^2 - 2b^2 = 7$ and $(a+b)^2 - 2b^2 = -7$ separately. By a similar calculation as in the previous case, we find the parameters $a := a_k$ and $b := b_k$. Note that, this time we have to use

$$(a_k + b_k) + b_k\sqrt{2} = \pm\left(3 \pm \sqrt{2}\right)(3 \pm 2\sqrt{2})^k, \quad k \geq 0, \ d = 7, \tag{5}$$

$$(a_k + b_k) + b_k\sqrt{2} = \pm\left(1 \pm 2\sqrt{2}\right)(3 \pm 2\sqrt{2})^k, \quad k \geq 0, \ d = -7. \tag{6}$$

Therefore, by substitution, we get the corresponding u and v given in Theorem 2.2.(b). Thus, we obtain primitive Pythagorean triples satisfying $y - x = 7^2$. We give the results in Table 3 and Table 4.

Table 3. $y - x = 7^2$ with $(a+b)^2 - 2b^2 = 7$

k	a_k	b_k	u	v	x	y	z
0	2	1	6	5	11	60	61
0	4	-1	40	17	1311	1360	1889
1	4	9	234	97	45347	45396	64165
1	2	3	30	13	731	780	1069
2	22	53	7950	3293	52358651	52358700	74046349
2	8	19	1026	425	872051	872100	1233301
3	128	309	270066	111865	60421866131	60421866180	85449422581
3	46	111	34854	14437	1006374347	1006374396	1423228285
4	746	1801	9174294	3800117	69726781184747	69726781184796	98608559612125
4	268	647	1184010	490433	1161355152611	1161355152660	1642404207589
5	4348	10497	311655930	129092113	80464645065360131	80464645065360180	113794192342969669
5	1562	3771	40221486	16660285	1340202839766971	1340202839767020	1895333032329421

Table 4. $y - x = 7^2$ with $(a+b)^2 - 2b^2 = -7$

k	a_k	b_k	u	v	x	y	z
0	-1	2	6	5	11	60	61
0	3	-2	30	13	731	780	1069
1	3	8	176	73	25647	25696	36305
1	-1	-4	40	17	1311	1360	1889
2	19	46	5980	2477	29624871	29624920	41895929
2	-9	-22	1364	565	1541271	1541320	2179721
3	111	268	203144	84145	34187103711	34187103760	48347865761
3	-53	-128	46336	19193	1778653647	1778653696	2515396145
4	647	1562	6900916	2858453	39451888085847	39451888085896	55793395192265
4	-309	-746	1574060	651997	2052564795591	2052564795640	2902764971609
5	3771	9104	234428000	97103257	45527444663991951	45527444663992000	64385529704008049
5	-1801	-4348	53471704	22148705	2368657995486591	2368657995486640	3349788261840641

Remark: Even though one can use a brute force search to find some primitive Pythagorean triples where the difference of the legs is $7^2 = 49$., our approach is entirely different. In fact, the equations (3), (4), (5) and (6) provide *all* necessary parameters that lead to *all* Pythagorean triples possessing the studied property. This is why we give the results by using four separate tables instead of combining all of them into one table. In the tables above, we omitted repeated solutions by taking into account the effect of $\pm$ just for saving space.

3. CONCLUSION

In this paper, we have fully characterized primitive Pythagorean triples for which the difference of the right sides is a perfect square. Our main result establishes that such triples exist if and only if the corresponding generalized Pell equation has a non-trivial solution, leading to infinite families of solutions in certain cases. We have also provided explicit parametrizations and comprehensive tables, which serve as a valuable reference for further investigations. While the focus of this paper has been on the difference of the right sides, future investigations may explore analogous properties, such as sums or other algebraic combinations.

Furthermore, it is an intriguing challenge to find the exact parametrization of primitive Pythagorean triples (x, y, z)such that

$$|x - y| = d^4 .$$

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REFERENCES

1. Agarwal, R. P. (2020). Pythagorean triples before and after Pythagoras. Computation, 8(3), 62. https://doi.org/10.3390/computation8030062
2. Cohen, H., (2007). Number theory: Volume I: Tools and Diophantine equations. Springer.
3. Conrad, K. (n.d.). Pell’s equation II. Retrieved from https://kconrad.math.uconn.edu/blurbs/gradnumthy/pelleqn2.pdf
4. Ribenboim, P. (1994). Catalans conjecture: Are 8 and 9 the only consecutive powers? Academic Press.
5. Wegener, D. P. (1975). Primitive Pythagorean triples with sum or difference of legs equal to a prime, Fibonacci Q. 13, 263-277.