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A Different Geometric Model for Teaching Derivative and Integral Concepts

Alper Cihan KONYALIOĞLU, Senem KALAÇ
Department of mathematics Education, K.K. Education Faculty, Atatürk Universty,25240 Erzurum/Turkey E-mail: ackonyali@atauni.edu.tr

ABSTRACT

In this study, a different geometric model that is thought to be used in teaching derivatives and integrals will be introduced. This geometric model is based on the meanings of the words derivative and integral. The geometric models used for teaching consist of plane geometric regular shapes. With this study, whether this geometric model should be used in teaching derivatives and integrals, its deficiencies or incorrects will be opened to discussion by mathematics and mathematics educators.

Keywords: Derivative teaching, integral teaching, geometric model

INTRODUCTION

Mathematics starts with arithmetic and continues with algebra, analysis and calculus, and continues to progress by being reinforced with geometry, which allows the fixed points to gain different meanings at a more advanced level by giving them movement. Thanks to the spiral structure of mathematics, these meanings find a new area of use in both algebra and analysis. This situation roughly explains the systematic structure of mathematics in both the development and teaching process as follows: While quantities are abstracted with numbers with arithmetic, numbers are abstracted with letters with algebra. Later, analysis, where letters begin to show relations or functions, and finally calculus, which deals with the direction and movement capabilities of these relations at a certain point, is encountered.

Two of the most basic and important concepts in analysis and calculus are derivative and integral. Therefore, learning these concepts is important for mathematics. It is seen that various approaches are used in the literature for teaching these concepts. In particular, approaches other than the traditional approach have been used in the teaching process of these concepts. While Tall (1993) stated that the traditional approach may be insufficient in supporting students' conceptual understanding and may lead to rote-based learning, Heid (1988) stated that traditional teaching causes students to have difficulty in understanding the geometric meaning of derivative and integral. Konyalıoğlu et.al (2011a); and again Konyalıoğlu et al. (2011b) drew attention to the presence of procedural learning in these concepts . Freeman et al. (2014) showed that active learning methods significantly increase student success compared to traditional methods. In contrast, Hohenwarter & Fuchs (2004) and Hohenwarter & Preiner (2007) have shown that students understand concepts more deeply when the constructivist approach is combined with tools such as GeoGebra . Heid (2002) also stated that the use of technology in teaching derivatives and integrals strengthens conceptual understanding rather than computational skills. Kaput (1994) stated that multiple representations allow students to better understand the relationship between derivatives and integrals. In short, what is generally suggested in these approaches is the use of a geometric interpretation of the concepts of derivatives and integrals.

The abstract nature of the concepts of derivative and integral can complicate the learning process and negatively affect students' mathematical thinking skills(Tall, 1993; Artigue, 2000). However, using the geometric meanings of derivative and integral can help students understand the concepts more easily by making these abstract concepts concrete and this stands out as a critical strategy for students to develop their conceptual understanding (Arcavi, 2003; Duval, 2006). Visual learning techniques in particular are an effective tool for students to comprehend mathematical concepts in depth (Duval, 2006).

The derivative represents the rate of change of a function at a certain point or the slope of the tangent of its curve at that point; while the integral represents the total effect of a function over a certain interval or the area under the curve (Stewart, 2012). Teaching through the geometric meanings of the derivative and integral allows students to see the physical and real-life equivalents of these concepts. In this way, the learning process ceases to be only process-oriented and the conceptual learning process is supported (Orton, 1983). Supporting these concepts with visual representations allows students to grasp the geometric meaning along with symbolic and numerical operations (Tall, 1993; Yerushalmy, 1997).

The definition of the derivative can be explained by visualizing the slope of the tangent lines drawn to a point on the graph (Stewart, 2012). Such visualizations can be supported by GeoGebra, Desmos or similar dynamic mathematics software. Again, the concept of speed in the motion of an object can be shown graphically through the velocity function $v(t)$, which is the derivative of the position function $x(t)$. Students understand that speed means "the slope of the curve" by observing it on the graph (Tall, 1993). Yerushalmy (1997) states that visual tools in teaching derivatives, especially the rate of change, facilitate understanding as a dynamic process. Examining the change in slope step by step thanks to dynamic software also facilitates students' understanding of the concept of limit.

Integral refers to the total effect of a function over a certain interval or the area under the curve. In order to ensure that students understand this abstract concept, Riemann sums can be used to show step by step how the sum of the areas of rectangular particles is approached (Tall , 1993; Stewart , 2012). With dynamic mathematics software, students can visually watch how the lower and upper sums approach the integral value (Arcavi, 2003). Duval (2006) states that visual tools in teaching integrals allow students to establish the relationship between symbolic operations and the concept of area.

Research shows that visualization significantly increases student success in teaching derivatives and integrals. For example, according to Yerushalmy (1997), students in classes where dynamic visualization tools are used understand the relationship between derivatives and integrals better. Graphing software makes it easier for students to understand the relationship between derivatives and slopes or the concept of integrals and areas (Keller & Hirsch, 1998). With GeoGebra , students can observe slope changes on a graph by taking the derivative of a function or visually explore integral and area calculations (Hohenwarter & Preiner, 2007). Again, Desmos offers students an interactive experience to explore the concepts of derivatives and integrals step by step (Abramovich & Norton, 2006). Again, WolframAlpha supports the learning process by providing step-by-step solutions in derivative and integral calculations. Mathematica is effective for visual analysis and modeling in mathematics teaching (Heid, 2002). It also supports the teaching of derivatives and integrals with online videos and interactive exercises (Khan, 2012).

In the literature that is tried to be summarized in general terms for teaching the concepts of derivative and integral, it is seen that the geometric meanings of the concepts of derivative and integral, known in the classical literature, as slope and area under the curve, respectively, are emphasized. Here, the geometric meanings of the definite integral are generally considered as area and volume, and it has been determined that the concept of the curve bundle, which is the geometric meaning of the indefinite integral, is not generally used.

In this study, a different geometric model that can be used in teaching derivatives and integrals will be introduced. Below, it will be discussed how the derivatives and integrals geometric meanings of this model, which will be tried to be expressed with examples, can be used. Whether this model can be used in teaching derivatives and integrals in mathematics education, and its missing or incorrect aspects will be opened to discussion by mathematics and mathematics educators. In fact, this geometric model, known by most mathematicians and mathematics educators, is based on the literal meanings of the concepts of derivative and integral. In this study, roughly, derivative refers to the most basic parts that form a geometric shape and integral refers to the resulting shape. In other words, while derivative means the source of change, the part, integral means the addition of parts to form the whole.

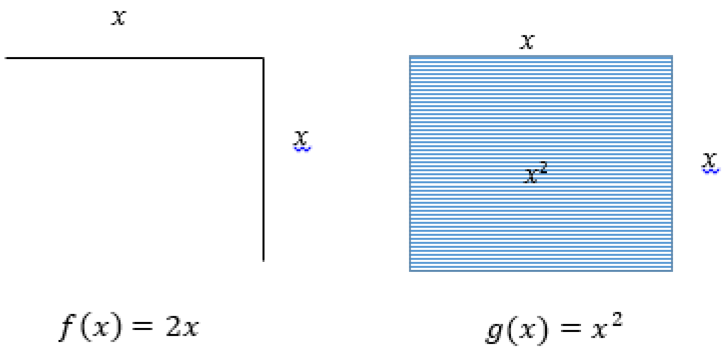
METHOD

Qualitative data collection methods were used in the study. Eight mathematics education and mathematics field experts working at different universities were consulted about whether the given geometric model could be used. Expert opinions were obtained through one-to-one interviews. Before the interviews, the experts were asked to examine the geometric model. During the interviews, the experts were asked the questions; Is the model correct, can it be used? The answers were written and the data were content analysed. Direct quotations were made from the expert opinions and interpreted. Expert opinions are indicated as E1, E2, E3, E4, E5, E6, E7 and E8.

Geometric Model

The geometric model to be discussed in the study is limited to plane geometric (Euclidean Geometry) shapes.

Let the derivative function $f(x) = 2x$. We can write this function $f(x) = x + x$ in the form. In this case, $f(x)$ the function $g(x) = x^2$ is the elementary function that generates the integral function. Conversely $g(x)$, the derivative of $f(x)$ gives .



As can be seen, these are two x- line segments. Derivative; It meant the source of change, the derivator, the segment. In this case, let these two line segments be the basic shapes, and let the horizontal and vertical infinite ones of them come side by side. In this case;

$$\int 2xdx = x^2 + c$$

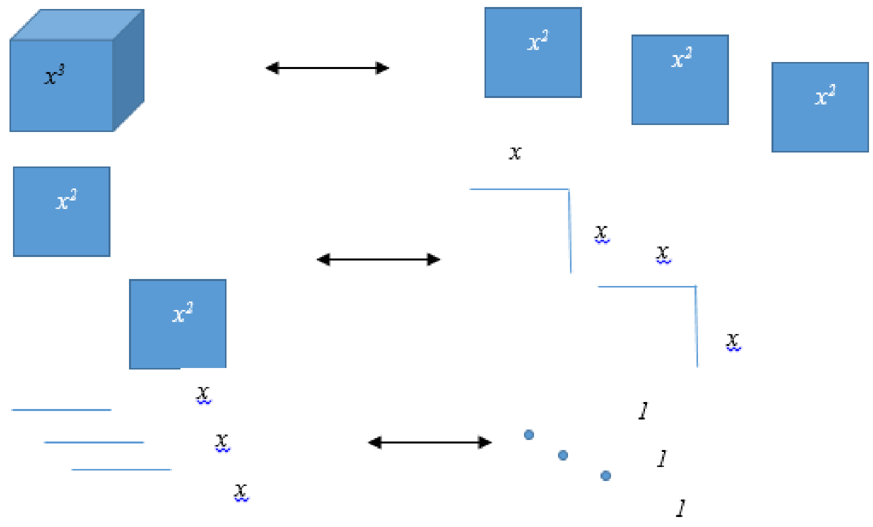
$$(x^2)^/ = 2x$$

since the line segments are certain or taken in certain lengths, c=0 can be accepted. Or, in this geometric model, the constant c represents a point that has no area. As can be seen, there is a part and whole relationship between the derivative and the integral.

In fact, when taking a derivative, there is a process of revealing the purest state or unit that will create the visual of the expression.

Now another example;

$$f(x) = x^3 + 2x^2 + 3x \Rightarrow f^/(x) = 3x^2 + 4x + 3$$



Now let's consider polynomial type and non -transcendental functions. In this case, Maclaurin series comes into play. Maclaurin series expansion allows any function to be written as a polynomial function. After this is done, plane geometric shapes are used for several terms in the expansion as above. The first few terms in the Maclaurin series expansions of some functions are below.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

From these series $sinx^/ = cosx$, $cosx^/ = -sinx$ or its reverse integrals can be easily seen.

RESULTS

Content analysis of expert responses to the question "Is the model correct?

Participant statements based on the first question are given below with direct quotations.

E1: Dimension reduction was made in the first way. However, I think the model is not correct because there is no mathematical proof.

E2: The model needs to be based on the definition of limit epistemologically. I think it is not correct.

E3: The model is not correct because it is against the rules of generalisation.

E4: The subject is approached from a different perspective, the model can be verified if the definitions are deepened and proofs can be made.

E5: With this logic, a visual connection can be established between the derivatives of functions in R, R^2, R^3 and the indefinite integral. However, how will it be in the case of x^n and n>3. From now on, it will be even more difficult to make this connection. At the basic education level, it can be explained with this connection that the derivative is not only slope. In the same way, the idea of definite integral and area can be connected and compared with the visual. The transition to the Maclaurin series was very fast. I don't understand why this series was used.

E6: The mathematical representation of x^2 is made as a geometric square. The mathematical representation is a function graph. From this point of view, I think that the model is correct only for one question and can be used only for that question.

E7: I can say that the model is not correct because I think it is against generalisation. I don't understand the transition to Maclaurin series, it needs to be explained in detail.

E8: It would be better to start with the Riemann integral and move on to $f(x)=2x$ and $g(x)=x^2$ with the fact that the sum of lines is the area.

When the expert opinions about the model were examined, the experts generally stated that the model was not correct. When the objections were examined, the experts stated that the model was not correct because of the generalisation problem. E5 and E6 stated that the model can be used for a specific question and generalisation cannot be made. Moreover, E5 and E7 stated that they did not understand the transition to the Maclaurin series and that it was too fast. As a result of the content analysis, it was concluded that the experts generally did not find the model correct and stated that it could be used for a specific question, but no generalisation could be made.

Content analysis of expert responses to the question "Can the model be used?

Participant statements based on the second question are given below with direct quotations.

E1: I think it cannot be used because the model is not correct

E2: It cannot be used because the visual model relationship established is wrong.

E3: This model can be used for a very specific question or for initial comprehension.

E4: The model seems incomplete and wrong as it is. It can be used if it can be improved.

E5: The model cannot be used as it is.

E6: It can be used at the beginning for comprehension. But it cannot be used afterwards.

E7: It cannot be used.

E8: It can be used as a model if it is improved.

When the expert opinions about the use of the model were examined, the experts stated that the model was generally not suitable for use. Two experts stated that the model can be used provided that it is improved, one expert stated that it can be used for a very specific question and generalisation cannot be made, and one expert stated that it can be used to comprehend the relationship between integral and derivative in the initial stage, but it is not appropriate to use it in the continuation of the subject.

CONCLUSION

In this study, a geometric model other than the geometric meanings of these concepts, which are generally used in teaching derivatives and integrals, has been introduced. Discussing this model in terms of its use in mathematical content and teaching-learning processes by both mathematicians and mathematics educators will contribute to the teaching process of these concepts. If this model is suitable, it can be used, if not, it will take its place in the literature as a situational knowledge that should not be done.

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