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Approximately Subgroups in Proximal Relator Spaces

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Abstract

The focus of this article is on descriptive approximately subgroups and their homomorphisms in digital images endowed with descriptive proximity relations. Furthermore, descriptive approximately cosets, normal subgroups, homomorphisms of descriptive approximately groups and homomorphism theorems on descriptive approximately groups were introduced using non-abstract points that is pixels in digital images.

Keywords: Proximity spaces, Relator spaces, Descriptive approximations, Approximately subgroups.

Proksimal Relator Uzaylarında Yaklaşımlı Alt Gruplar

Özet

Bu makalede tanımsal proksimiti bağıntıları ile donatılmış dijital görüntülerde tanımsal yaklaşımlı alt grupların ve homomorfizmaların tanımlanması amaçlanmıştır. Ayrıca, soyut olmayan noktalar, yani dijital görüntülerdeki pikseller kullanılarak tanımlanan tanımsal yaklaşımlı gruplar üzerinde tanımsal yaklaşımlı kosetler, normal alt gruplar, tanımsal yaklaşımlı grupların homomorfizmaları ve homomorfizma teoremleri verilmiştir.

Anahtar Kelimeler: Proksimiti uzaylar, Relator uzaylar, Tanımsal yaklaşımlar, Yaklaşımlı alt gruplar.

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Introduction

Ordinary algebraic structures consists of a nonempty set of abstract points with one or more binary operations, which are required to satisfy certain axioms. For example, a groupoid is an algebraic structure $(A,\circ)$ consisting of a nonempty set A and a binary operation "$\circ$" defined on A [1]. In a groupoid, the binary operation "$\circ$" must be closed in A, i.e. for all a, b in A, the result of the operation $a \circ b$ is also in A. As for the proximal relator space, $(X, \mathcal{R})$, a nonempty set A is endowed with a proximal relator $\mathcal{R}$, which is a set of proximity relations. In the context of planar digital images, X is a set of picture points (pixels), describable with feature vectors in proximal relator spaces. Descriptively upper approximation of a nonempty set is obtained by using the set of points composed by the proximal relator space together with matching features of points. In the algebraic structures constructed on proximal relator spaces, the basic tool is consideration of descriptively upper approximations of the subsets of non-abstract points. In a groupoid A, the binary operation "$\circ$" may be closed in descriptively upper approximation of A, that is for all a, b in A, $a \circ b$ is in descriptively upper approximation of A.

There are two important differences between ordinary algebraic structures and descriptive approximately algebraic structures. The first one is working with pixels in digital images while the second one is considering of descriptively upper approximations of the subsets of non-abstract points for the closeness of binary operations. Using the theoretical background of this concept, one can be obtain functional algorithms for applied sciences such as image processing.

Essentially, the aim is to obtain descriptive approximately subgroups and their homomorphisms in proximal relator spaces. Furthermore, descriptive approximately cosets, normal subgroups, homomorphisms of descriptive approximately groups and homomorphism theorems on descriptive approximately groups are introduced.

Preliminaries

Let X be a nonempty set. Family of relations $\mathcal{R}$ on a nonempty set X is called a *relator*. The pair $(X, \mathcal{R})$ (or $X(\mathcal{R})$) is a relator space which is natural generalizations of uniform spaces [16]. If we consider a family of proximity relations on X, we have a proximal relator space $(X, \mathcal{R}_\delta)$ ($X(\mathcal{R}_\delta)$). As in [11], $\mathcal{R}_\delta$ contains proximity relations, namely, Efremovič proximity δ [2, 3], Lodato proximity [7], Wallman proximity, descriptive proximity δ_Φ in defining $\mathcal{R}_{\delta_\Phi}$ [9, 15].

In this article, we consider the Efremovič proximity δ [3] and the descriptive proximity δ_Φ in defining a descriptive proximal relator space (denoted by $(X, \mathcal{R}_{\delta_\Phi})$).

An Efremovič proximity δ is a relation on 2^X that satisfies

- $A \ \delta \ B \Rightarrow B \ \delta \ A$.
- $A \ \delta \ B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$.
- $A \cap B \neq \emptyset \Rightarrow A \ \delta \ B$.
- $A \ \delta \ (B \cup C) \Leftrightarrow A \ \delta \ B$ or $A \ \delta \ C$.
- $\{x\} \ \delta \ \{y\} \Leftrightarrow x = y$.
- EF axiom: $A \ \underline{\delta} \ B \Rightarrow \exists E \subseteq X$ such that $A \ \underline{\delta} \ E$ and $E^c \ \underline{\delta} \ B$.

Lodato proximity [7] swaps the EF axiom 2 for the following condition:

$$A \ \delta \ B \text{ and } \forall b \in B, \{b\} \ \delta \ C \Rightarrow A \ \delta \ C. \ (Lodato\ Axiom)$$

In a discrete space, a non-abstract point has a location and features that can be measured [6, 3]. Let X be a nonempty set of non-abstract points in a proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$ and let $\Phi = \{\phi_1, \ldots, \phi_n\}$ a set of probe functions that represent features of each $x \in X$.

A *probe function* $\Phi: X \to \mathbb{R}$ represents a feature of a sample point in a picture. Let $\Phi(x) = (\phi_1(x), \ldots, \phi_n(x))$ $(n \in \mathbb{N})$ be an object description denote a feature vector of x, which provides a description of each $x \in X$. After the choosing a set of probe functions, one obtain a descriptive proximity relation δ_Φ.

Definition 1 [Set Description; 8] *Let X be a nonempty set of non-abstract points, Φ an object description and A a subset of X. Then the set description of A is defined as*

$$Q(A) = \{\Phi(a) | a \in A\}.$$

Definition 2 [Descriptive Set Intersection; 8, 10] *Let X be a nonempty set of non-abstract points, A and B any two subsets of X. Then the descriptive (set) intersection of A and B is defined as*

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B | \Phi(x) \in Q(A) \ and \ \Phi(x) \in Q(B)\}.$$

Definition 3 [9] *Let X be a nonempty set of non-abstract points, A and B any two subsets* *of X. If $Q(A) \cap Q(B) \neq \emptyset$, then A is called descriptively near B and denoted by $A\delta_{\Phi}B$. If $Q(A) \cap Q(B) = \emptyset$ then $A\underline{\delta}_{\Phi}B$ reads A is descriptively far from B.*

Definition 4 [Descriptive Nearness Collections, 9] *Let X be a nonempty set of non-abstract points and A any subset of X. Then the descriptive proximal collection $\xi_{\Phi}(A)$ is defined by*

$$\xi_{\Phi}(A) = \{B \in \mathcal{P}(X) | A\delta_{\Phi}B\}.$$

Let $(X, \mathcal{R}_{\delta_{\Phi}})$ be descriptive proximal relator space and $A \subset X$, where A contains non-abstract objects [14]. Let $(A,\cdot)$ and $(Q(A),\circ)$ be groupoids. Let consider the object description Φ by means of a function

$$\Phi: A \subset X \to Q(A) \subset \mathbb{R},\ a \mapsto \Phi(a),\ a \in A.$$

The object description Φ of A into $Q(A)$ is an *object description homomorphism* if $\Phi(a \cdot b) = \Phi(a) \circ \Phi(b)$ for all $a, b \in A$.

Also a descriptive closure of a point $a \in A$ is defined by

$$cl_{\Phi}(a) = \{x \in X | \Phi(a) = \Phi(x)\}.$$

Descriptively lower approximation of $A \subset X$ begins by determining which $cl_{\Phi}(a)$ are subsets of *set* A. This discovery process leads to define descriptively lower approximation of $A \subseteq X$, denoted by $\Phi_* A$.

Definition 5 [Descriptively Lower Approximation of a Set; 5] *Let $(X, \mathcal{R}_{\delta_{\Phi}})$ be descriptive proximal relator space and $A \subset X$. A descriptively lower approximation of A is defined as*

$$\Phi_* A = \{a \in A | cl_{\Phi}(a) \subseteq A\}.$$

Definition 6 [Descriptively Upper Approximation of a Set; 5] *Let $(X, \mathcal{R}_{\delta_{\Phi}})$ be descriptive proximal relator space and $A \subset X$. A descriptively upper approximation of A is defined as*

$$\Phi^* A = \{x \in X | x\delta_{\Phi}A\}.$$

Lemma 1 [5] *Let $(X, \mathcal{R}_{\delta_{\Phi}})$ be descriptive proximal relator space and $A, B \subset X$, then*

(1) $Q(A \cap B) = Q(A) \cap Q(B)$,

(2) $Q(A \cup B) = Q(A) \cup Q(B)$.

Definition 7 [5] *Let $(X, \mathcal{R}_{\delta_{\Phi}})$ be descriptive proximal relator space and let "$\cdot$" a binary operation on X. $G \subset X$ is called a descriptive approximately groupoid in descriptive proximal* *relator space if* $x \cdot y \in \Phi^* G$ *for all* $x, y \in G$.

Definition 8 [4] *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be descriptive proximal relator space and let "$\cdot$" a binary operation on* X. $G \subset X$ *is called a descriptive approximately group in descriptive proximal relator space if the followings are true:*

$(\mathcal{A}G_1)$ *For all* $x, y \in G$, $x \cdot y \in \Phi^* G$,

$(\mathcal{A}G_2)$ *For all* $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *property holds in* $\Phi^* G$,

$(\mathcal{A}G_3)$ *There exists* $e \in \Phi^* G$ *such that* $x \cdot e = e \cdot x = x$ *for all* $x \in G$ *(*e *is called the approximately identity element of* G*),*

$(\mathcal{A}G_4)$ *There exists* $y \in G$ *such that* $x \cdot y = y \cdot x = e$ *for all* $x \in G$ *(*y *is called the inverse of* x *in* G *and denoted as* x^{-1}*).*

A subset S of the set of X is called a *descriptive approximately semigroup* in descriptive proximal relator space if

$(\mathcal{A}S_1)$ $x \cdot y \in \Phi^* S$ for all $x, y \in S$ and

$(\mathcal{A}S_2)$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* S$ for all $x, y, z \in S$ properties are satisfied.

If descriptive approximately semigroup have an approximately identity element $e \in \Phi^* S$ such that $x \cdot e = e \cdot x = x$ for all $x \in S$, then S is called a *descriptive approximately monoid* in descriptive proximal relator space.

If $x \cdot y = y \cdot x$, for all $x, y \in S$ property holds in $\Phi^* G$, then G is *commutative descriptive approximately groupoid, semigroup, monoid or group* in descriptive proximal relator space.

Suppose that G is a descriptive approximately groupoid with the binary operation "$\cdot$" in $(X, \mathcal{R}_{\delta_\Phi})$, $g \in G$ and $A, B \subseteq G$. We define the subsets $g \cdot A, A \cdot g, A \cdot B \subseteq \Phi^* G \subseteq X$ as follows:

$$g \cdot A = gA = \{ga : a \in A\},$$
$$A \cdot g = Ag = \{ag : a \in A\},$$
$$A \cdot B = AB = \{ab : a \in A, b \in B\}.$$

Lemma 2 [5] *Let* (X, δ_Φ) *be descriptive proximity space and* $A, B \subset X$. *If* $\Phi : X \to \mathbb{R}$ *is an object descriptive homomorphism, then*

$$\mathcal{Q}(A)\mathcal{Q}(B) = \mathcal{Q}(AB).$$

Example 1 [5] *Let X be a digital image endowed with descriptive proximity relation δ_Φ and consists of* 25 *pixels as in Figure 1.*

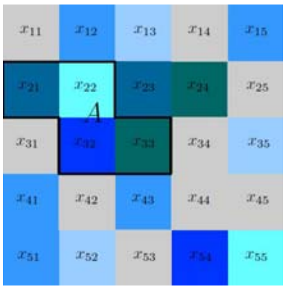

Figure 1. Digital image

A pixel x_{ij} is an element at position (i, j) (row and column) in digital image X. Let ϕ be a probe function that represent RGB colour of each pixel are given in Table 1.

Table 1. RGB colour of each pixel

	Red	*Green*	*Blue*
x_{11}	204	204	204
x_{12}	51	153	255
x_{13}	204	255	255
x_{14}	204	204	204
x_{15}	51	153	255
x_{21}	0	102	153
x_{22}	102	255	255
x_{23}	0	102	153
x_{24}	0	102	102
x_{25}	204	204	204
x_{31}	204	204	204
x_{32}	0	51	255
x_{33}	0	102	102
x_{34}	204	204	204
x_{35}	204	255	255
x_{41}	51	153	255
x_{42}	204	204	204
x_{43}	51	153	255
x_{44}	204	204	204
x_{45}	204	204	204
x_{51}	51	153	255
x_{52}	204	255	255
x_{53}	204	204	204
x_{54}	0	51	255
x_{55}	102	255	255

Let

$$\cdot : X \times X \to X$$

$$(x_{ij}, x_{kl}) \mapsto x_{ij} \cdot x_{kl} = x_{pr} \text{ on X, where } p = \min\{i, k\} \text{ and } r = \min\{j, l\}.$$

Let $A = \{x_{21}, x_{22}, x_{32}, x_{33}\}$ be a subimage (subset) of X.

In [5], from Example 3. 11., the descriptively upper approximation of A is

$$\Phi^* A = \{x_{21}, x_{22}, x_{23}, x_{24}, x_{32}, x_{33}, x_{54}, x_{55}\}$$

as shown in Figure 1.

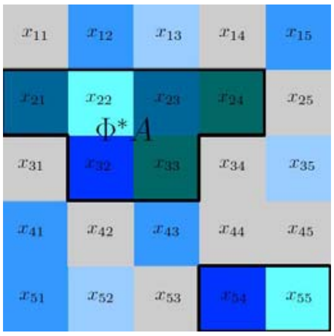

Figure 2. Descriptively upper approximation of A.

Since

($\mathcal{A}S_1$) For all $x_{ij}, x_{kl} \in A$, $x_{ij} \cdot x_{kl} \in \Phi^* A$,

($\mathcal{A}S_2$) For all $x_{ij}, x_{kl}, x_{mn} \in A$, $(x_{ij} \cdot x_{kl}) \cdot x_{mn} = x_{ij} \cdot (x_{kl} \cdot x_{mn})$ property holds in $\Phi^* A$

are satisfied, the subimage A of the image X is indeed a descriptive approximately semigroup in descriptive proximity space (X, δ_Φ) with binary operation "$\cdot$". Also, since $x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}$ for all $x_{ij}, x_{kl} \in A$ property holds in $\Phi^* A$, A is a commutative descriptive approximately semigroup.

Descriptive Approximately Subgroups and Cosets

Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space, $G \subset X$ a descriptive approximately group and H a descriptive approximately subgroup of G. The right compatible relation "ρ_r" defined as

$$x \rho_r y :\Leftrightarrow x \cdot y^{-1} \in H \cup \{e\}.$$

Since G is a descriptive approximately group, $x^{-1} \in G$ for all $x \in G$, $x \cdot x^{-1} = e$, that is $x \rho_r x$. Further, if $x \rho_r y$, then $x \cdot y^{-1} \in H \cup \{e\}$, that is $x \cdot y^{-1} \in H$ or $x \cdot y^{-1} \in \{e\}$ for all $x, y \in G$. If $x \cdot y^{-1} \in H$, then, since H is a descriptive approximately subgroup of G, we have $(x \cdot y^{-1})^{-1} = y \cdot x^{-1} \in H$. Hence $y \rho_r x$. If $x \cdot y^{-1} \in \{e\}$, then $x \cdot y^{-1} = e$. That means $y \cdot x^{-1} = (x \cdot y^{-1})^{-1} = e^{-1} = e$, and thus $y \rho_r x$. Therefore "ρ_r" is compatible relation over the descriptive approximately group G [4].

Definition 9 [4] *A compatible class defined by relation "* ρ_r *" is called descriptive approximately right coset. A descriptive approximately right coset that contains element* g *is denoted by* $\bar{g}_r$*, that is*

$$\bar{g}_r = \{h \cdot g | h \in H, g \in G, h \cdot g \in G\} \cup \{g\}.$$

Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space, $G \subset X$ a descriptive approximately group and H a descriptive approximately subgroup of G. The left compatible relation "ρ_l" defined as

$$x \rho_l y :\Leftrightarrow x^{-1} \cdot y \in H \cup \{e\}.$$

Theorem 1 [4] *"ρ_l" is a compatible relation over the descriptive approximately group G.*

Definition 10 [4] *A compatible class defined by relation "ρ_l" is called descriptive approximately left coset. The descriptive approximately left coset that contains the element g is denoted by $\bar{g}_l$, that is*

$$\bar{g}_l = \{g \cdot h | h \in H, g \in G, g \cdot h \in G\} \cup \{g\}.$$

Clearly $\bar{g}_r = H \cdot g$ and $\bar{g}_l = g \cdot H$. Since the binary operation of a descriptive approximately group is not commutative, the relations "ρ_r" and "ρ_l" are different from each other. Consequently, the descriptive approximately left cosets and descriptive approximately right cosets are different from each other.

Theorem 2 [4] *The descriptive approximately left cosets and descriptive approximately right cosets are equal in number.*

Definition 11 [4] *The number of both descriptive approximately left cosets and descriptive approximately right cosets is called the index of subgroup H in G.*

Main Results

Definition 12 *A descriptive approximately subgroup N of descriptive approximately group G is called a descriptive approximately normal subgroup, if $g \cdot N = N \cdot g$ for all $g \in G$.*

Theorem 3 *Let $G \subset X$ be descriptive approximately group and $N \subset G$ descriptive approximately subgroup of G. Then*

(1) *N is a descriptive approximately normal subgroup of G if and only if $g \cdot N \cdot g^{-1} = N$ for all $g \in G$.*

(2) *N is a descriptive approximately normal subgroup of G if and only if $g \cdot n \cdot g^{-1} \in N$*

for all $g \in G$ *and* $n \in N$.

Proof. (1) Let N be a descriptive approximately normal subgroup of G. From definition, we get $g \cdot N = N \cdot g$ for all $g \in G$. Since G is a descriptive approximately group,

$$(g \cdot N) \cdot g^{-1} = (N \cdot g) \cdot g^{-1}$$
$$\Rightarrow g \cdot N \cdot g^{-1} = N \cdot (g \cdot g^{-1})$$
$$\Rightarrow g \cdot N \cdot g^{-1} = N.$$

Conversely, let N be a descriptive approximately subgroup of G and $g \cdot N \cdot g^{-1} = N$ for all $g \in G$. Then $(g \cdot N \cdot g^{-1}) \cdot g = N \cdot g$, that is $g \cdot N = N \cdot g$.

Consequently N is a descriptive approximately normal subgroup of G.

(2) Let N be a descriptive approximately normal subgroup of G. We have $g \cdot N \cdot g^{-1} = N$ for all $g \in G$. Hence $g \cdot n \cdot g^{-1} \in N$ for any $n \in N$.

Conversely, let N be a descriptive approximately subgroup of G and $g \cdot n \cdot g^{-1} \in N$ for all $g \in G$ and $n \in N$. We get $g \cdot N \cdot g^{-1} \subset N$. Since $g^{-1} \in G$, we obtain $g \cdot (g^{-1} \cdot N \cdot g) \cdot g^{-1} \subset g \cdot N \cdot g^{-1}$, that is $N \subset g \cdot N \cdot g^{-1}$. Since $g \cdot N \cdot g^{-1} \subset N$ and $N \subset g \cdot N \cdot g^{-1}$, we get $g \cdot N \cdot g^{-1} = N$. As a result, N is a descriptive approximately normal subgroup.

Homomorphisms of Descriptive Approximately Groups

Let $(X, \mathcal{R}_{\delta_\Phi})$ and $(Y, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator spaces, and let "$\cdot$", "$\circ$" be binary operations on X and Y, respectively.

Definition 13 *Let* $G \subset X$, $H \subset Y$ *be descriptive approximately groups and* χ *a mapping from* Φ^*G *onto* Φ^*H *such that* Φ^*G, Φ^*H *be groupoids. If* $\chi(x \cdot y) = \chi(x) \cdot \chi(y)$ *for all* $x, y \in G$, *then* χ *is called a descriptive approximately group epimorphism and also,* G *is called descriptive approximately homomorphic to* H, *denoted by* $G \simeq H$.

Similarly, it may be mentioned descriptive approximately semigroup or monoid homomorphisms are maps between descriptive approximately semigroups or monoids that preserves the operations in these algebraic structures.

In this section χ is a descriptive approximately homomorphism such that $\chi\colon \Phi^*G \to \Phi^*H$.

Theorem 4 *Let* G *and* H *be descriptive approximately homomorphic groups. If "*$\cdot$*" is commutative, then "*$\circ$*" is also commutative.*

Proof. Consider χ such that $\chi(x \cdot y) = \chi(x) \circ \chi(y)$ for all $x, y \in \Phi^*G$. For every $\chi(x)$, $\chi(y) \in \Phi^*H$ since χ is surjection, there exist $x, y \in \Phi^*G$ such that $x \mapsto \chi(x)$, $y \mapsto \chi(y)$. Thus $\chi(x \cdot y) = \chi(x) \circ \chi(y)$ and $\chi(y \cdot x) = \chi(y) \circ \chi(x)$. As a result, by $x \cdot y = y \cdot x$, we get $\chi(x) \circ \chi(y) = \chi(y) \circ \chi(x)$.

Theorem 5 *Let $G \subset X$, $H \subset Y$ be descriptive approximately groups that are descriptive approximately homomorphic, Φ^*G is a groupoid and $\Phi^*(\chi(G)) = \Phi^*H$. Then $\chi(G)$ is also a descriptive approximately group.*

Proof. $(\mathcal{A}G_1)$ For all $x', y' \in \chi(G)$, consider $x, y \in G$ such that $x \mapsto x'$, $y \mapsto y'$. Then $\chi(x \cdot y) = \chi(x) \circ \chi(y) \in \Phi^*H = \Phi^*\chi(G)$, that is $x' \circ y' \in \Phi^*\chi(G)$.

$(\mathcal{A}G_2)$ Since $e \in \Phi^*G$, $\chi(e) \in \Phi^*H$ and $\varphi(x) \in \chi(G)$, $\chi(e) \circ \chi(x) = \chi(e \cdot x) = \chi(x)$.

$(\mathcal{A}G_3)$ Since G is a descriptive approximately group, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in G$. Thus

$$\chi\big(x \cdot (y \cdot z)\big) = \chi(x) \circ \chi(y \cdot z) = \chi(x) \circ \big(\chi(y) \circ \chi(z)\big),$$
$$\chi\big((x \cdot y) \cdot z\big) = \chi(x \cdot y) \circ \chi(z) = \big(\chi(x) \circ \chi(y)\big) \circ \chi(z),$$
$$\big(\chi(x) \circ \chi(y)\big) \circ \chi(z) = \chi(x) \circ \big(\chi(y) \circ \chi(z)\big).$$

$(\mathcal{A}G_4)$ There exists a $x \in G$ such that $x \mapsto x'$ for all $x' \in \chi(G)$. Since G is a descriptive approximately group, $x^{-1} \in G$. Then $\chi(x^{-1}) \in \chi(G)$ and $\chi(x) \circ \chi(x^{-1}) = \chi(x^{-1}) \circ \chi(x) = \chi(e)$. Thus, $(x')^{-1} = \chi(x^{-1})$.

Consequently, $\chi(G)$ is a descriptive approximately group.

Theorem 6 *Let $G \subset X$, $H \subset Y$ be descriptive approximately groups that are descriptive approximately homomorphic. Let e and e' be the descriptive approximately identity elements of G and H, respectively. Then $\chi(e) = e'$ and $\chi(x^{-1}) = \chi(x)^{-1}$ for all $x \in \Phi^*G$.*

Definition 14 *Let $G \subset X$, $H \subset Y$ be descriptive approximately homomorphic groups. The kernel of χ is defined by $Ker\chi = \{x \in G | \chi(x) = e'\}$, where e' is the descriptive approximately identity element of H and denote by K.*

Theorem 7 *Let $G \subset X$, $H \subset Y$ be descriptive approximately homomorphic groups and K be kernel of a descriptive approximately homomorphism. If Φ^*K is a groupoid, then K is a descriptive approximately normal subgroup of G.*

Proof. Let χ be a descriptive approximately epimorphism from Φ^*G to Φ^*H, and $K = Ker\chi$. Then $\chi(x) = e'$, $\chi(y) = e'$ for all $x, y \in K$. Hence $\chi(x \cdot y) = \chi(x) \circ \chi(y) = e' \circ e' = e'$, that is $x \cdot y \in K$. Since $\chi(x^{-1}) = \chi(x)^{-1} = e'^{-1} = e'$, we have $x^{-1} \in K$. Consequently, K is a descriptive approximately normal subgroup of G.

Theorem 8 *Let $G \subset X$, $H \subset Y$ be descriptive approximately homomorphic groups. Let G', N be descriptive approximately subgroup and descriptive approximately normal subgroup of G, respectively and Φ^*G' a groupoid. Then we have the following:*

(1) *If $\chi(\Phi^*G') = \Phi^*\chi(G')$, then $\chi(G')$ is a descriptive approximately subgroup of H.*

(2) *If $\chi(G') = H$ and $\chi(\Phi^*N) = \Phi^*\chi(N)$, then $\chi(N)$ is a descriptive approximately normal subgroup of H.*

Proof. (1) Let χ be an onto mapping from Φ^*G to Φ^*H such that $\chi(x \cdot y) = \chi(x) \circ \chi(y)$ for all $x, y \in \Phi^*G$. From the definition of χ, there exists $x, y \in G'$ such that $x \mapsto \chi(x)$ and $\chi(x) \circ \chi(y) = \chi(x \cdot y) \in \chi(\Phi^*G')$ for all $\chi(x)$, $\chi(y) \in \chi(G')$. Since $\chi(\Phi^*G') = \Phi^*\chi(G')$, we get $\chi(x) \circ \chi(y) \in \Phi^*\chi(G')$.

Also, from the definition of χ, there exists $x \in G'$ such that $x \mapsto \chi(x)$ for all $\chi(x) \in \chi(G')$. Since G' is a descriptive approximately subgroup of G, we obtain $x^{-1} \in G'$.

Therefore $\chi(x)^{-1} = \chi(x^{-1}) \in \chi(G')$. As a result $\chi(G')$ is a descriptive approximately subgroup of H.

(2) From (1), if $\chi(\Phi^*N) = \Phi^*(\chi(N))$, we observe that $\chi(N)$ is a descriptive approximately subgroup of H. Since $\chi(G') = H$, we obtain $\chi(x) \in \chi(G')$ for all $\chi(x) \in H$.

Therefore $x \in G'$, $x^{-1} \in G'$ and $\chi(x^{-1}) \in \chi(G') = H$. Since N is a descriptive approximately normal subgroup of G', we get $x \cdot n \cdot x^{-1} \in N$. Thus $\chi(x.n.x^{-1}) = \chi(x) \circ \chi(n) \circ \chi(x^{-1}) \in \chi(N)$ for all $\chi(x) \in$ H, $\chi(n) \in \chi(N)$. Consequently, $\chi(N)$ is a descriptive approximately normal subgroup of H.

Theorem 9 *Let $G \subset X$, $H \subset Y$ be descriptive approximately homomorphic groups. Let H', N' be a descriptive approximately subgroup and a descriptive approximately normal subgroup of H, respectively and Φ^*G' a groupoid, where G' is the inverse image of H'. Then we have the following:*

(1) *If $\chi(\Phi^*G') = \Phi^*H'$, then G' is a descriptive approximately subgroup of G.*

(2) *If $\chi(G) = H$ and $\chi(\Phi^*N) = \Phi^*N'$, then N, which is the inverse image of N', is a*

descriptive approximately normal subgroup of G.

Proof. (1) Since G' is the inverse image of H', that is, $\chi(x)$, $\chi(y) \in H'$ for all $x, y \in G'$. Since H' is a descriptive approximately subgroup of H, we get $\chi(x \cdot y) = \chi(x) \circ \chi(y) \in \Phi^* H' = \chi(\Phi^* G')$. Hence $x.y \in \Phi^* G'$. Since H' is a descriptive approximately subgroup of H, we have $\chi(x)^{-1} = \chi(x^{-1}) \in H'$. Therefore $x^{-1} \in G'$.

(2) From (1), we can easily shown that N is a descriptive approximately subgroup of H if $\chi(\Phi^* N) = \Phi^*(\chi(N))$. We obtain $\chi(x) \in \chi(G) = H$, $\chi(x)^{-1} = \chi(x^{-1}) \in \chi(G) = H$, $\chi(n) \in N'$ for all $x \in G$, $n \in N$. Since N' is a descriptive approximately normal subgroup of H, we get $\chi(x) \circ \chi(n) \circ \chi(x^{-1}) = \chi(x \cdot n \cdot x^{-1}) \in N'$. Thus $x \cdot n \cdot x^{-1} \in N$. As a result, inverse image of N' is a descriptive approximately normal subgroup of G.

Descriptive Approximately Group of Descriptive Approximately Cosets

Definition 15 *Let $G \subset X$ be a descriptive approximately group and H a descriptive approximately subgroup of G. For $x, y \in G$, let xH and yH be two descriptive approximately left cosets that determined the elements x and y, respectively. The product of two descriptive approximately left cosets that are determined by $x \cdot y \in \Phi^* G$ can be defined as*

$$(x \cdot y)H = \{(x \cdot y) \cdot h | h \in H, x \cdot y \in \Phi^* G, (x \cdot y) \cdot h \in G\} \cup \{x \cdot y\},$$

and denoted by

$$xH \circ yH = (x \cdot y)H.$$

Let $G \subset X$ be a descriptive approximately group and H a descriptive approximately subgroup of G. Then

$$G/_{\rho_l} = \{xH | x \in G\}$$

is a set of all descriptive approximately left cosets of G by H. In this case, if we consider $\Phi^* G$ instead of descriptive approximately group G

$$(\Phi^* G)/_{\rho_l} = \{xH | x \in \Phi^* G\}.$$

Then $xH = \{x \cdot h | h \in H, x \in \Phi^* G, x \cdot h \in G\} \cup \{x\}$.

Definition 16 *Let $(X, \mathcal{R}_{\delta_\Phi})$ be descriptive proximal relator space, $G \subset X$ a descriptive approximately group and H a descriptive approximately subgroup of G. Let $G/_{\rho_l}$ be a set of all* *descriptive approximately left cosets of G by H, $\xi_{\Phi}(A)$ a descriptive proximal collection and $A \in \mathcal{P}(X)$. Then*

$$\Phi^*\left(G/_{\rho_l}\right) = \bigcup_{\xi_\Phi(A) \underset{\Phi}{\cap} G/_{\rho_l} \neq \emptyset} \xi_\Phi(A)$$

is called upper approximation of $G/_{\rho_l}$.

Theorem 10 *Let G be a descriptive approximately group, H a descriptive approximately subgroup of G and $G/_{\rho_l}$ a set of all descriptive approximately left cosets of G by H. If $(\Phi^*G)/_{\rho_l} \subseteq \Phi^*\left(G/_{\rho_l}\right)$, then $G/_{\rho_l}$ is a descriptive approximately group under the operation given by $xH \circ yH = (x \cdot y)H$ for all $x, y \in G$.*

Proof. $(\mathcal{A}G_1)$ Since G is a descriptive approximately group, we have that $x \cdot y \in \Phi^*G$ and $xH \circ yH = (x \cdot y)H \in (\Phi^*G)/_{\rho_l}$ for all xH, $yH \in G/_{\rho_l}$. From the hypothesis, $xH \circ yH = (x \cdot y)H \in \Phi^*\left(G/_{\rho_l}\right)$ for all xH, $yH \in G/_{\rho_l}$.

$(\mathcal{A}G_2)$ Since G is a descriptive approximately group, the associative property holds in Φ^*G. Hence, for all xH, yH, $zH \in G/_{\rho_l}$

$$(xH \circ yH) \circ zH = (x \cdot y)H \circ zH = \big((x \cdot y) \cdot z\big)H = \big(x \cdot (y \cdot z)\big)H$$
$$= xH \circ (y \cdot z)H = xH \circ (yH \circ zH)$$

holds in $(\Phi^*G)/_{\rho_l}$. Therefore, by the hypothesis, the associative property holds in $\Phi^*\left(G/_{\rho_l}\right)$.

$(\mathcal{A}G_3)$ Since G is a descriptive approximately group, there exists $e \in \Phi^*G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$. Hence

$$xH \circ eH = (x \cdot e)H = xH$$

and

$$eH \circ xH = (e \cdot x)H = xH$$

for all $xH \in G/_{\rho_l}$. Thus $eH \in (\Phi^*G)/_{\rho_l} \subseteq \Phi^*\left(G/_{\rho_l}\right)$ is the descriptive approximately identity element of $G/_{\rho_l}$.

$(\mathcal{A}G_4)$ Since G is a descriptive approximately group, there exists $x' \in G$ such that $x \cdot x' = x' \cdot x = e$ for all $x \in G$. In this case, we have that

$$xH \circ x'H = (x \cdot x')H = eH$$

and

$$x'H \circ xH = (x' \cdot x)H = eH$$

for all $xH \in G/_{\rho_l}$. Hence $x'H$ is the inverse of xH in $G/_{\rho_l}$. As a result, $G/_{\rho_l}$ is a descriptive approximately group.

Definition 17 *Let G be a descriptive approximately group and H a descriptive approximately subgroup of G. The descriptive approximately group $G/_{\rho_l}$ is called a descriptive approximately group of all descriptive approximately left cosets of G by H and denoted by $G/_{\rho}H$.*

Theorem 11 *Let $G \subset X$ be a descriptive approximately group and $H \subset G$ a descriptive approximately subgroup of G. Then the mapping $\Pi : \Phi^*G \to \Phi^*(G/_{\rho}H)$ defined by $\Pi(x) = xH$ for all $x \in \Phi^*G$ is a descriptive approximately homomorphism.*

Proof. From the definition of Π, Π is well defined from Φ^*G into $\Phi^*(G/_{\rho}H)$. By using the Definition 15,

$$\Pi(x \cdot y) = (x \cdot y)H = xH \circ yH = \Pi(x) \circ \Pi(y)$$

for all $x, y \in G$. Thus Π is a descriptive approximately homomorphism from Definition 13.

Definition 18 *The descriptive approximately homomorphism Π is called a descriptive approximately canonical homomorphism from Φ^*G into $\Phi^*(G/_{\rho}H)$.*

Definition 19 *Let $G \subset X$, $H \subset Y$ be descriptive approximately groups and G' a nonempty subset of G. Let*

$$\chi : \Phi^*G \to \Phi^*H$$

be a mapping and

$$\chi_{G'} = \chi\,\big|_{G'} : G' \to \Phi^*H$$

a restricted mapping. If $\chi(x \cdot y) = \chi_{G'}(x \cdot y) = \chi_{G'}(x) \cdot \chi_{G'}(y) = \chi(x) \cdot \chi(y)$ for all $x, y \in G'$, then χ is called a restricted descriptive approximately homomorphism and also, G is called restricted descriptive approximately homomorphic to H, denoted by $G \simeq_{rda} H$.

Theorem 12 *Let $G \subset X$, $H \subset Y$ be descriptive approximately groups and χ a descriptive approximately homomorphism from Φ^*G into Φ^*H. Let $(\Phi^*Ker\chi, \cdot)$ be a groupoid and $(\Phi^*G)/_{\rho_l}$ be a set of all left cosets of Φ^*G by $Ker\chi$. If $(\Phi^*G)/_{\rho_l} \subseteq \Phi^*(G/_{\rho}Ker\chi)$ and $\Phi^*\chi(G) = \chi(\Phi^*G)$, then*

$$G/_{\rho}Ker\chi \simeq_{rda} \chi(G).$$

Proof. We have that $Ker\chi$ is a descriptive approximately subgroup of G. Since $(\Phi^{*}Ker\chi, \cdot)$ is a groupoid, from Theorem 7 $Ker\chi$ is a descriptive approximately subgroup of G. Because $(\Phi^{*}G)/_{\rho_l} \subseteq \Phi^{*}\big(G/_{\rho}Ker\chi\big)$, $G/_{\rho}Ker\chi$ is a descriptive approximately group of all left cosets of G by $Ker\chi$ from Theorem 10. Since $\Phi^{*}\chi(G) = \chi(\Phi^{*}G)$, $\chi(G)$ is a descriptive approximately subgroup of H by Theorem 8. Define

$$\theta: \Phi^{*}\big(G/_{\rho}Ker\chi\big) \to \Phi^{*}\chi(G)$$

$$A \mapsto \theta(A) = \begin{cases} \theta_{G/_{\rho}Ker\chi}(A), A \in (\Phi^{*}G)/_{\rho_l} \\ e_{\chi(G)}, A \notin (\Phi^{*}G)/_{\rho_l} \end{cases}$$

and

$$\theta_{G/_{\rho}Ker\chi} = \theta\Big|_{G/_{\rho}Ker\chi} : G/_{\rho}Ker\chi \to \Phi^{*}\chi(G)$$

$$xKer\chi \mapsto \theta\Big|_{G/_{\rho}Ker\chi}(xKer\chi) = \chi(x)$$

for all $xKer\chi \in G/_{\rho}Ker\chi$.

Since

$$xKer\chi = \{x \cdot k | k \in Ker\chi, x \cdot k \in G\} \cup \{x\},$$

$$yKer\chi = \{y \cdot k' | k' \in Ker\chi, y \cdot k' \in G\} \cup \{y\}$$

and the mapping χ is a descriptive approximately homomorphism,

$$\begin{aligned}
& xKer\chi = yKer\chi \\
& \Rightarrow x \in yKer\chi \\
& \Rightarrow x \in \{y \cdot k' | k' \in Ker\chi, y \cdot k' \in G\} \text{ } or \text{ } x \in \{y\} \\
& \Rightarrow x = y \cdot k' \ (k' \in Ker\chi, y \cdot k' \in G) \text{ } or \text{ } x = y \\
& \Rightarrow y^{-1} \cdot x = (y^{-1} \cdot y) \cdot k' \ (k' \in Ker\chi) \text{ } or \text{ } \chi(x) = \chi(y) \\
& \Rightarrow y^{-1} \cdot x = k' \ (k' \in Ker\chi) \\
& \Rightarrow y^{-1} \cdot x \in Ker\chi \\
& \Rightarrow \chi(y^{-1} \cdot x) = e_{\chi(G)} \\
& \Rightarrow \chi(y^{-1}) \cdot \chi(x) = e_{\chi(G)} \\
& \Rightarrow \chi(y)^{-1} \cdot \chi(x) = e_{\chi(G)}
\end{aligned}$$

$$\Rightarrow \chi(x) = \chi(y).$$

Therefore $\theta \big|_{G/_\rho Ker\chi}$ is well defined.

For $A, B \in \Phi^*\left(G/_\rho Ker\chi\right)$, we suppose that $A = B$. Since the mapping $\theta_{G/_\rho Ker\chi}$ is well defined,

$$\theta(A) = \begin{cases} \theta_{G/_\rho Ker\chi}(A), A \in (\Phi^*G)/_{\rho_l} \\ e_{\chi(G)}, A \notin (\Phi^*G)/_{\rho_l} \end{cases}$$

$$= \begin{cases} \theta_{G/_\rho Ker\chi}(B), B \in (\Phi^*G)/_{\rho_l} \\ e_{\chi(G)}, B \notin (\Phi^*G)/_{\rho_l} \end{cases}$$

$$= \theta(B).$$

Hence θ is well defined.

For all $xKer\chi, yKer\chi \in G/_\rho Ker\chi$,

$$\begin{aligned} \theta(xKer\chi \circ yKer\chi) &= \theta\big((x \cdot y)Ker\chi\big) \\ &= \theta_{G/_\rho Ker\chi}\big((x \cdot y)Ker\chi\big) \\ &= \chi(x \cdot y) \\ &= \chi(x) \cdot \chi(y) \\ &= \theta_{G/_\rho Ker\chi}(xKer\chi) \cdot \theta_{G/_\rho Ker\chi}(yKer\chi). \end{aligned}$$

Therefore θ is a restricted descriptive approximately homomorphism by Definition 19. Consequently, $G/_\rho Ker\chi \simeq_{rda} \chi(G)$.

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