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A Robust Study on Atangana-Baleanu Fractional Coupled Jaulent-Miodek System

Atangana–Baleanu Kesirli Bağlantılı Jaulent–Miodek Sistemi Üzerine Kapsamlı Bir Çalışma

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Abstract

This study explores the use of the Atangana–Baleanu fractional derivative within the context of the coupled Jaulent–Miodek system, with a focus on its theoretical foundations, computational aspects, and practical effectiveness. The Atangana–Baleanu operator, characterized by its Mittag–Leffler kernel, offers a flexible approach to modeling systems that exhibit strong memory effects and complex temporal behavior. The coupled Jaulent–Miodek system itself represents a well-known class of nonlinear wave equations and is widely used to describe interactions between different wave structures. However, classical formulations often fall short in capturing the influence of memory and long-term processes. To overcome these challenges, the integration of the Atangana–Baleanu derivative has been proposed as a way to better model such dynamics. In this work, both theoretical analysis and numerical simulations are carried out to show how effectively the Atangana–Baleanu framework handles the detailed structural features of the Jaulent–Miodek system. The findings further emphasize the potential of this fractional derivative in broadening the applications of fractional calculus in the study of complex physical phenomena.

Keywords: Jaulent-Miodek System, Atangana-Baleanu q-Elzaki homotopy analysis transform method, Atangana-Baleanu Elzaki transform.

Öz

Bu çalışma, kuramsal temelleri, hesaplamalı yönleri ve pratik etkinliği çerçevesinde Atangana–Baleanu kesirli türevinin, eşlenik Jaulent–Miodek sistemi bağlamındaki kullanımını incelemektedir. Mittag–Leffler çekirdeği ile karakterize edilen Atangana–Baleanu operatörü, güçlü bellek etkileri ve karmaşık zamansal davranış sergileyen sistemlerin modellenmesinde esnek bir yaklaşım sunmaktadır. Eşlenik Jaulent–Miodek sistemi ise, farklı dalga yapıları arasındaki etkileşimleri tanımlamak amacıyla yaygın olarak kullanılan, iyi bilinen bir doğrusal olmayan dalga denklemleri sınıfını temsil etmektedir. Ancak klasik formülasyonlar, bellek etkisi ve uzun vadeli süreçlerin etkisini yakalamada çoğu zaman yetersiz kalmaktadır. Bu zorlukların üstesinden gelmek amacıyla, Atangana–Baleanu türevinin entegrasyonu, söz konusu dinamikleri daha iyi modellemek için önerilmiştir. Bu çalışmada, Jaulent–Miodek sisteminin ayrıntılı yapısal özelliklerinin Atangana–Baleanu çerçevesiyle ne denli etkin bir biçimde ele alındığı hem kuramsal analizler hem de sayısal benzetimler aracılığıyla ortaya konmuştur. Elde edilen bulgular, bu kesirli türevin karmaşık fiziksel olayların incelenmesinde kesirli hesaplamaların uygulama alanlarını genişletme potansiyelini vurgulamaktadır.

Anahtar Kelimeler: Jaulent–Miodek Sistemi, Atangana–Baleanu q-Elzaki homotopi analiz dönüşüm yöntemi, Atangana–Baleanu Elzaki dönüşümü.

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1. Introduction

Many branches of science and engineering have shown a great deal of interest in fractional differential equations because they add derivatives of non-integer order to the classical framework. Areas such as control theory, signal processing, physics, and bioengineering increasingly rely on these models to better capture the complex behavior of dynamical systems. A key advantage of fractional models is their ability to naturally reflect memory effects and time-dependent characteristics, offering a deeper understanding of how a system's current state is influenced by its past. Because of these features, fractional calculus has been particularly successful in describing processes such as anomalous diffusion, viscoelastic behavior, and other forms of non-standard viscous flow (Oldham, 1974; Machado et al.,2011; Petras, 2011; Baleanu et al.,2012).

Fractional calculus is a generalization of classical calculus that involves derivatives and integrals of arbitrary (non-integer) order. This extended approach offers a powerful and adaptable tool for modeling complex system behaviors. Over the years, it has become a critical tool across various scientific and engineering disciplines, including dynamical systems, control theory, signal processing, and bioengineering. A major strength of fractional calculus is its ability to naturally account for memory effects and the influence of previous states, which enables a more accurate representation of systems that traditional models often fail to describe properly. This feature not only improves the precision of engineering applications but also advances the understanding of complex physical processes, supporting both theoretical progress and practical innovations across a wide range of areas (Kilbas et al.,2006; Diethelm, 2010).

Traditional definitions of fractional derivatives, such as those by Grünwald, Riemann-Liouville, and Caputo, are based on singular kernels (Güngör, 2025). Because of their fundamental importance in the early development of fractional calculus, these classical models have been extensively explored and discussed in the literature. In recent years, however, newer versions like the Atangana–Baleanu and Caputo–Fabrizio derivatives have been introduced, distinguished by their use of nonsingular kernels. These modern approaches overcome some of the limitations of the traditional methods, making them better suited for modeling physical processes and dynamical systems that require smoother memory behavior and improved computational efficiency (Shah, et al.,2022; Miller and Ross, 1993; Atangana, 2018; Engheta, 1998; Podlubny, 1999).

Various integral transform methods have been proposed in the literature to help solve fractional differential equations, offering alternatives to conventional analytical and numerical techniques. Among these methods, the Elzaki Transform (ET) has attracted particular interest for its ability to reduce complex fractional models into simpler algebraic forms (Elzaki, 2011). By applying ET, researchers have developed new strategies to address real-world problems, achieving highly accurate and reliable numerical solutions (Elzaki, 2012) Beyond improving computational efficiency, the Elzaki Transform has also contributed to a deeper understanding of fractional dynamic behavior, establishing itself as an important tool in the study of fractional-order systems across many scientific and engineering fields (Elzaki, 2011; Wang and Xia, 2013, Sahoo et al., 2020).

The coupled Jaulent–Miodek system has found extensive application across many scientific and engineering disciplines. It plays a key role in areas like thermodynamics, gas dynamics, and elasticity by helping to explain how materials and fluids react under stress and deformation. The system is equally important in diffusion theory, where it is used to model the transport of particles and heat through different media. In the field of shock wave studies, it provides a framework for understanding how abrupt changes in pressure or velocity propagate through various environments. Additionally, it contributes to research on turbulence by offering tools to analyze the highly complex and chaotic patterns observed in fluid flows (Jaulent and Miodek, 1976).

The nonlinear time-fractional coupled Jaulent–Miodek (FCJM) equation is a noteworthy advancement of this classical system, incorporating fractional time derivatives to account for memory and hereditary effects that are not covered by conventional integer-order models. The FCJM equation provides a more versatile and generalized tool for researching wave interactions and other intricate dynamical behaviors where past states still affect the dynamics of the system today by integrating fractional calculus (Fokas et all., 1981; Shen et all, 2021; Anaç, 2022).

$$u_t + u_{xxx} + \frac{3}{2} v v_{xxx} + \frac{9}{2} v_x v_{xx} - 6uu_x - 6uvv_x - \frac{3}{2} u_x v^2 = 0. \tag{1}$$
$$v_t + v_{xxx} - 6u_x v - 6uv_x - \frac{15}{2} v_x v^2 = 0.$$

Nonlinear time-fractional Jaulent–Miodek (JM) system of equation is defined as follows (Shen et all, 2021).

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^3 u}{\partial x^3} + \frac{3}{2} v \frac{\partial^3 v}{\partial x^3} + \frac{9}{2} \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} - 6u \frac{\partial u}{\partial x} - 6uv \frac{\partial v}{\partial x} - \frac{3}{2} v^2 \frac{\partial u}{\partial x} = 0. \tag{2}$$
$$\frac{\partial^\alpha v}{\partial t^\alpha} + \frac{\partial^3 v}{\partial x^3} - 6v \frac{\partial u}{\partial x} - 6u \frac{\partial v}{\partial x} - \frac{15}{2} v^2 \frac{\partial v}{\partial x} = 0, \qquad 0 < \alpha \leq 1, 0 < t \leq 1.$$

In this study the Atangana-Baleano fractional coupled Jaulent–Miodekequation is examined as,

$$ {}^{ABC}D_t^{\alpha}u+\frac{\partial^3 u}{\partial x^3}+\frac{3}{2}v\frac{\partial^3 v}{\partial x^3}+\frac{9}{2}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}-6u\frac{\partial u}{\partial x}-6uv\frac{\partial v}{\partial x}-\frac{3}{2}v^2\frac{\partial u}{\partial x}=0. \tag{3}$$

$$ {}^{ABC}D_t^{\alpha}v+\frac{\partial^3 v}{\partial x^3}-6v\frac{\partial u}{\partial x}-6u\frac{\partial v}{\partial x}-\frac{15}{2}v^2\frac{\partial v}{\partial x}=0,\ \ 0<\alpha\leq 1.$$

with initial conditions (ICs) (Güngör, 2025)

$$u(x,0)=\frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda x}{2}\right)\right). \tag{4}$$

$$v(x,0)=\lambda sech\left(\frac{\lambda x}{2}\right).$$

Jaulent and Miodek used the inverse scattering transform, a crucial analytical method frequently used in soliton theory, to first formulate the coupled Jaulent–Miodek (JM) equations. They developed integrable nonlinear systems that could describe complicated wave interactions using energy-dependent Schrödinger potentials. This early contribution helped to construct integrable models in many areas of mathematical physics and served as a vital basis for further research on nonlinear wave propagation (Nisar et all, 2024)

In several scientific domains, the Jaulent–Miodek (JM) equation plays a significant role in the modeling and analysis of nonlinear systems. It offers a useful framework for researching the dynamics of complex wave events because of its many applications, especially in its coupled form. However, from a mathematical perspective, it is frequently quite difficult to derive correct analytical solutions to these equations, particularly when coupling effects are present. Consequently, numerical techniques have emerged as a widely used and successful strategy for examining and estimating solutions to these complex systems (Jaulent and Miodek, 1976).

This study's primary objective is to create and implement a novel method known as the Atangana–Baleanu q-Elzaki Homotopy Analysis Transform Method (ABq-EHATM), which aims to generate unique numerical solutions for the fractional coupled Jaulent–Miodek (JM) equation. The paper's structure is set out as follows: The research problem, the JM equation, and its extension to the time-fractional situation are all covered in Section 1. The main definitions of fractional derivatives are covered in Section 2, along with an explanation of how the Elzaki transform is applied when computing them. A thorough explanation of the suggested ABq-EHATM is provided in Section 3, along with details on its theoretical underpinnings and construction. The correctness and effectiveness of the procedure are demonstrated by the numerical findings that are highlighted in Section 4. Section 5 concludes by summarizing the key conclusions and going into the wider ramifications and possible uses of the suggested method (Baleanu et all, 2020).

2. Materials and Methods

In this section, we provide the fundamental definitions and theorems that form the basis for the subsequent analysis.

Definition 2.1 Let $\vartheta(t)$ be a continuously differentiable function. The Caputo derivative of order $\mu > 0$ is mathematically expressed as (Baleanu, et al., 2012);

$$ {}^{C}D_t^{\mu}[\vartheta(t)] = \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{\vartheta^{(n)}(s)}{(t-s)^{1+\mu-n}} ds, \tag{5} $$

$n-1 < \mu \leq n,\ n \in \mathbb{N},\ t > 0,\ \vartheta(t) \in C^{m-1}[0,t],\ s(t) = t^{n-\mu-1}$ is the singular kernel. ${}^{C}D_t^{\mu};\ t \in \mathbb{Z}, \mu \in \mathbb{R}^+$ gives n th integer and fractional order derivatives of $\vartheta(t)$ (Almeida, 2017; Kürkçü et al., 2019).

Definition 2.2 The fractional derivative of order μ of the function $\vartheta(t)$ is mathematically defined as follows (Baleanu et all, 2020; Anaç, 2022):

$$ D_t^{\mu}[\vartheta(t)] = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_{t_0}^t (s-t)^{n-\mu-1} [\vartheta_0(s) - \vartheta(s)]\, ds. \tag{6} $$

Definition 2.3. The Atangana-Baleanu (AB) fractional derivative of order $0 < \mu < 1$ for a function $\vartheta(t)$ is defined in terms of the Mittag-Leffler function as follows (Atangana and Baleanu, 2016; Owolabi, 2018);

$$ {}^{ABC}D_t^{\mu}[\vartheta(t)] = \frac{N(\mu)}{1-\mu} \int_0^t \vartheta'(\tau)\, E_\mu \left[-\frac{\mu(t-\tau)^\mu}{1-\mu} \right] d\tau, \qquad 0 < \mu \leq 1, \tag{7} $$

where $N(\mu)$ is the normalization function, which satisfies $N(\mu) = 1$ when $\mu = 0, \mu = 1$ and E_μ denotes the Mittag-Leffler function, defined as;

$$ E_\mu = \sum_{k=0}^{\infty} \frac{\tau^k}{\Gamma(\mu k + 1)}. $$

This fractional derivative is widely used in modeling complex systems and non-local phenomena due to its non-singular kernel and memory effects.

Definition 2.4. The Caputo-Fabrizio fractional derivative of order $0 < \mu$ for a differentiable function $\vartheta(t)$ is expressed as (Caputo, 1969; Caputo and Fabrizio, 2015);

$$ {}^{CF}D_{0,t}^{\mu}[\vartheta(t)] = \frac{M(\mu)}{1-\mu}\int_{0}^{t} \vartheta'(\tau)\, exp\left[-\frac{\mu(t-\tau)}{1-\mu}\right] d\tau, \qquad t > 0, \tag{8}$$

where $M(\mu)$ is the normalization function such that $M(0) = M(1) = 1$, and $g \in H^1(a,b), b > a$. Here, $M(\mu)$ represents a normalization function that ensures the operator satisfies fundamental properties of fractional calculus. The presence of the exponential kernel differentiates this derivative from classical fractional derivatives, as it provides a non-singular memory effect, making it suitable for real-world applications involving systems with smooth fading memory behavior (Alkan and Anaç, 2024).

Definition 2.5. (Haroon, et al., 2022): The Elzaki transform applied to the Atangana-Baleanu (AB) fractional derivative of order ${}^{AB}D_t^{\mu}[\vartheta(t)]$ under the Atangana-Baleanu-Caputo (ABC) operator is formulated as follows:

$$ {}^{ABC}E_{\alpha}\left[{}^{ABC}D_t^{\alpha}\vartheta(t)\right] = \frac{N(\alpha)}{\alpha\omega^{\alpha}+1-\alpha}\left[\frac{{}^{ABC}E_{\alpha}\,\vartheta(t)}{\omega} - \omega\vartheta(0)\right]. \tag{9}$$

3. Analysis of the ABq-EHATM

In this section, we present the method developed to solve nonlinear fractional partial differential equations (FPDEs) that involve the Atangana–Baleanu Caputo Fractional Derivative (ABCFD). The process starts with the formulation of the FPDE, after which the Atangana–Baleanu Elzaki Transform (ABET) is applied to simplify the problem into a more manageable form. Once the transform is applied, a homotopy analysis technique is used to iteratively refine and improve the solution. This combined strategy offers an effective and flexible framework for addressing fractional systems that exhibit both nonlinearities and nonlocal behaviors. Now, analyze the Atangana-Baleanu time FPDEs (ABTFPDEs):

$$ {}^{ABC}D_{\tau}^{\alpha}u(\xi,\tau) + \mathcal{K}u(\xi,\tau) + \mathcal{S}u(\xi,\tau) = \zeta(\xi,\tau), \qquad \tau \in (0,\infty), \qquad n-1 < \alpha \le n, \tag{10}$$

where $\mathcal{K}$ and $\mathcal{S}$ are linear and nonlinear operators, respectively, $\zeta(\xi,\tau)$ is a nonhomogeneous function, and ${}^{ABC}D_{\tau}^{\alpha}$ is ABCFD of order α .

By applying the Atangana-Baleanu Elzaki Transform (ABET) to this equation, along with the initial conditions, we arrive at the following equation:

$$\frac{N(\alpha)}{\alpha\omega^{\alpha}+1-\alpha}\left[\frac{{}^{ABC}E_{\alpha}\, u(\xi,\tau)}{\omega}-\omega u(\xi,0)\right]+{}^{ABC}E_{\alpha}\,[\mathcal{K}u(\xi,\tau)+\mathcal{S}u(\xi,\tau)-\zeta(\xi,\tau)]=0. \tag{11}$$

Rewriting Equation (11) leads to:

$${}^{ABC}E_{\alpha}\, u(\xi,\tau)-\omega^{2}u(\xi,0)+\frac{\alpha\omega^{\alpha+1}+\omega-\omega\alpha}{N(\alpha)} \tag{12}$$

$$\times\, {}^{ABC}E_{\alpha}\,[\mathcal{K}u(\xi,\tau)+\mathcal{S}u(\xi,\tau)-\xi(\xi,\tau)]=0.$$

Using the homotopy analysis method, the nonlinear operator for $\phi(x,t;q)$ can be expressed as:

$$N[\phi(\xi,\tau;q)]={}^{ABC}E[\phi(\xi,\tau;q)]-\omega^{2}\phi(\xi,\tau;q)(0^{+})+\frac{\alpha\omega^{\alpha+1}+w-w\alpha}{N(\alpha)} \tag{13}$$

$$\times\, {}^{ABC}E_{\alpha}\,[\mathcal{K}\phi(\xi,\tau;q)+\mathcal{S}\phi(\xi,\tau;q)-\zeta(\xi,\tau)],$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

A homotopy involving the parameter q is constructed as follows:

$$(1-nq)^{ABC}E_{\alpha}\,[\phi(\xi,\tau;q)-u_{0}(\xi,\tau)]=hqH^{*}(\xi,\tau)^{ABC}E_{\alpha}\,[\phi(\xi,\tau;q)], \tag{14}$$

where, $h\neq 0$ is an auxiliary parameter and ${}^{ABC}E_{\alpha}$ represents Atangana-Baleanu Comformable Time Fractional (ABCET). When $q=0$ and $q=\frac{1}{n}$, the results of Equation (14) can be determined as follows:

$$\psi(\xi,\tau;0)=u_{0}(\xi,\tau),\phi\left(\xi,\tau;\frac{1}{n}\right)=u(\xi,\tau). \tag{15}$$

As a result, the solution $\phi(\xi,\tau;q)$gradually changes from $u_{0}(\xi,\tau)$to the intended solution $u(\xi,\tau)$ as q grows from 0 to $1/n$. It can be stated as follows by using the Taylor theorem with respect to q:

$$\phi(\xi,\tau;q)=u_{0}(\xi,\tau)+\sum_{i=1}^{\infty}u_{m}(\xi,\tau)q^{m}, \tag{16}$$

where,

$$u_m(\xi,\tau) = \frac{1}{m!}\frac{\partial^m \phi(\xi,\tau;q)}{\partial q^m}|_{q=0}. \quad (17)$$

Equation (16) converges when $q = \frac{1}{n}$ under suitable choices for $u_0(\xi,\tau)$, n and h. Consequently, the nonlinear equation's numerical solution is provided by:

$$u(x,t) = u_0(\xi,\tau) + \sum_{m=1}^{\infty} u_m(\xi,\tau)\left(\frac{1}{n}\right)^m. \quad (18)$$

Differentiating the zeroth-order deformation Equation (16) m −times with respect to q and then dividing by $m!$, yields the following expression for $q = 0$:

$$^{ABC}E_\alpha\, [u_m(\xi,\tau) - k_m u_{m-1}(\xi,\tau)] = hH^*(\xi,\tau)\mathcal{R}_m(\vec{u}_{m-1}), \quad (19)$$

where the vectors are defined as follows:

$$\vec{u}_m = \{u_0(\xi,\tau), u_1(\xi,\tau), \dots, u_m(\xi,\tau)\}. \quad (20)$$

When the inverse ABCET (IABCET) is applied to Equation (20), the resulting expression is:

$$u_m(\xi,\tau) = k_m u_{m-1}(\xi,\tau) + h\left({}^{ABC}E_\alpha\right)^{-1}[H^*(\xi,\tau)\mathcal{R}_m(\vec{u}_{m-1})], \quad (21)$$

where

$$\mathcal{R}_m(\vec{w}_{m-1}) = {}^{ABC}E_\alpha\, [w_{m-1}(\xi,\tau)] - \left(1 - \frac{k_m}{n}\right)\omega^2 w_0(\xi,\tau) + \frac{\alpha\omega^{\alpha+1} + w - w\alpha}{N(\alpha)}\, {}^{ABC}E_\alpha\, [K w_{m-1}(\xi,\tau) + {H^*}_{m-1}(\xi,\tau) - \zeta(\xi,\tau)]. \quad (22)$$

and

$$k_m = \begin{cases} 0, & m \le 1, \\ n, & m > 1. \end{cases} \quad (23)$$

where, ${H^*}_m$ represents the homotopy polynomials and is defined as follows:

$${H^*}_m = \frac{1}{m!}\frac{\partial^m \phi(\xi,\tau;q)}{\partial q^m}|_{q=0}.$$

and

$$\phi(\xi,\tau;q) = \phi_0 + q\phi_1 + q^2\phi_2 + \cdots. \quad (24)$$

Using Equations (23) and (24), the resulting expression is:

$$w_m(\xi,\tau) = (k_m + h)w_{m-1}(\xi,\tau) - \left(1 - \frac{k_m}{n}\right)\omega^2 u_0(\xi,\tau)$$

$$+h\left({}^{ABC}E_\alpha\right)^{-1}\left[\left(\frac{\alpha\omega^{\alpha+1} + w - w\alpha}{N(\alpha)}\,{}^{ABC}E_\alpha\left[Ku_{m-1}(\xi,\tau) + H^*{}_{m-1}(\xi,\tau) - \zeta(\xi,\tau)\right]\right)\right]. \tag{25}$$

Through the ABq-EHATM approach, the following result is obtained:

$$u(\xi,\tau) = u_0(\xi,\tau) + \sum_{c=1}^{\infty} u_c(\xi,\tau)\left(\frac{1}{n}\right)^c. \tag{26}$$

In summary, the methodology combines fractional calculus (via ABCFD), integral transform techniques (ABET), and a homotopy-based iterative framework. This hybrid approach enables the solution of complex FPDEs that involve nonlocal operators and nonlinear terms, making it a versatile and powerful tool in the analysis of fractional dynamic systems.

4. Applications

In this section, we demonstrate the application of the ABq-EHATM method to the Atangana-Baleanu time-fractional coupled Jaulent–Miodek equation (ABTFJME).

Example: Let us analyze the nonlinear ABCTFJME (Wang, 2013)

$$\begin{aligned} &{}^{ABC}D_t^\alpha u + \frac{\partial^3 u}{\partial x^3} + \frac{3}{2}v\frac{\partial^3 v}{\partial x^3} + \frac{9}{2}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} - 6u\frac{\partial u}{\partial x} - 6uv\frac{\partial v}{\partial x} - \frac{3}{2}v^2\frac{\partial u}{\partial x} = 0. \\ &{}^{ABC}D_t^\alpha v + \frac{\partial^3 v}{\partial x^3} - 6v\frac{\partial u}{\partial x} - 6u\frac{\partial v}{\partial x} - \frac{15}{2}v^2\frac{\partial v}{\partial x} = 0, \quad 0 < \alpha \le 1, 0 < t \le 1. \end{aligned} \tag{27}$$

with ICs

$$\begin{aligned} &u(x,0) = \frac{1}{8}\lambda^2\left(1 - 4sech^2\left(\frac{\lambda x}{2}\right)\right). \\ &v(x,0) = \lambda sech\left(\frac{\lambda x}{2}\right). \end{aligned} \tag{28}$$

Applying the Atangana-Baleanu Elzaki Transform (ABCET) to Equations (27) and (28), and incorporating the initial conditions, the resulting expression is given by:

$$\frac{N(\alpha)}{\alpha w^{\alpha}+1-\alpha}\left(\frac{{}^{ABC}\mathbb{E}_{\alpha}[u(x,t)]}{w}-wu(x,0)\right)$$
$$+{}^{AB}\mathbb{E}_{\alpha}\left[\frac{\partial^3 u}{\partial x^3}+\frac{3}{2}v\frac{\partial^3 v}{\partial x^3}+\frac{9}{2}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}-6u\frac{\partial u}{\partial x}-6uv\frac{\partial v}{\partial x}-\frac{3}{2}v^2\frac{\partial u}{\partial x}\right]=0 \tag{29}$$

$$\frac{N(\alpha)}{\alpha w^{\alpha}+1-\alpha}\left(\frac{{}^{ABC}\mathbb{E}_{\alpha}[v(x,t)]}{w}-wv(x,0)\right)$$
$$+{}^{AB}\mathbb{E}_{\alpha}\left[\frac{\partial^3 v}{\partial x^3}-6v\frac{\partial u}{\partial x}-6u\frac{\partial v}{\partial x}-\frac{15}{2}v^2\frac{\partial v}{\partial x}\right]=0 \tag{30}$$

Rewriting Equations (29) and (30), the following expressions are obtained:

$${}^{AB}\mathbb{E}_{\alpha}[u(x,t)]-w^2u(x,0)+\frac{\alpha w^{\alpha}+1-\alpha}{N(\alpha)}$$
$$\times\,{}^{AB}\mathbb{E}_{\alpha}\left[\frac{\partial^3 u}{\partial x^3}+\frac{3}{2}v\frac{\partial^3 v}{\partial x^3}+\frac{9}{2}\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}-6u\frac{\partial u}{\partial x}-6uv\frac{\partial v}{\partial x}-\frac{3}{2}v^2\frac{\partial u}{\partial x}\right]=0 \tag{31}$$

$${}^{AB}\mathbb{E}_{\alpha}[v(x,t)]-w^2v(x,0)+\frac{\alpha w^{\alpha}+1-\alpha}{N(\alpha)}$$
$$\times\,{}^{AB}\mathbb{E}_{\alpha}\left[\frac{\partial^3 v}{\partial x^3}-6v\frac{\partial u}{\partial x}-6u\frac{\partial v}{\partial x}-\frac{15}{2}v^2\frac{\partial v}{\partial x}\right]=0 \tag{32}$$

Using the homotopy analysis method, the nonlinear operator associated with $\varphi(x,t;q),\psi(x,t;q)$ is expressed as:

$$N^1[\varphi(x,y,\tau;q),\psi(x,y,\tau;q)]={}^{AB}\mathbb{E}_{\alpha}[\varphi(x,y,\tau;q)]-w^2\varphi(x,t;q)\,(0^+) \tag{33}$$
$$+\frac{\alpha w^{\alpha}+1-\alpha}{N(\alpha)}\left\{{}^{AB}\mathbb{E}_{\alpha}\left[\frac{\partial^3\varphi(x,y,\tau;q)}{\partial x^3}+\frac{3}{2}\psi(x,y,\tau;q)\frac{\partial^3\psi(x,y,\tau;q)}{\partial x^3}\right.\right.$$
$$+\frac{9}{2}\frac{\partial\psi(x,y,\tau;q)}{\partial x}\frac{\partial^2\psi(x,y,\tau;q)}{\partial x^2}-6\varphi(x,y,\tau;q)\frac{\partial\varphi(x,y,\tau;q)}{\partial x}$$
$$\left.\left.-6\varphi(x,y,\tau;q)\psi(x,y,\tau;q)\frac{\partial\psi(x,y,\tau;q)}{\partial x}-\frac{3}{2}\psi(x,y,\tau;q)^2\frac{\partial\varphi(x,y,\tau;q)}{\partial x}\right]\right\}.$$

$$N^2[\varphi(x,y,\tau;q),\psi(x,y,\tau;q)]={}^{AB}\mathbb{E}_{\alpha}[\psi(x,y,\tau;q)] \tag{34}$$
$$-w^2\psi(x,t;q)\,(0^+)+\frac{\alpha w^{\alpha}+1-\alpha}{N(\alpha)}\left\{{}^{AB}\mathbb{E}_{\alpha}\left[\frac{\partial^3\psi(\zeta,\mu,\tau;q)}{\partial x^3}\right.\right.$$

$$-6\psi(x,y,\tau;q)\ \frac{\partial\varphi(x,y,\tau;q)}{\partial x} - 6\varphi(x,y,\tau;q)\frac{\partial\psi(x,y,\tau;q)}{\partial x}$$
$$-\frac{15}{2}\psi(x,y,\tau;q)^2\frac{\partial\psi(x,y,\tau;q)}{\partial x}\Bigg]\Bigg\},$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

Applying the proposed algorithm, the $m-th$ order deformation equation can be expressed as:

$$^{AB}\mathbb{E}_\alpha[u_m\,(x,t) - \hbar_m\, u_{m-1}(x,t)] = h\mathcal{R}_{1,m}\,(\vec{u}_{m-1},\vec{v}_{m-1}). \tag{35}$$

$$^{AB}\mathbb{E}_\alpha[v_m\,(x,t) - \hbar_m\, v_{m-1}(x,t)] = h\mathcal{R}_{2,m}\,(\vec{u}_{m-1},\vec{v}_{m-1}). \tag{36}$$

where,

$$\mathcal{R}_{1,m}\,(\vec{u}_{m-1},\vec{v}_{m-1}) = {}^{AB}\mathbb{E}_\alpha[u_{m-1}(x,\mathrm{y},t)] - \left(1-\frac{\hbar_m}{n}\right)w^2u_0(x,y,t)$$
$$+\frac{\alpha w^\alpha+1-\alpha}{N(\alpha)}{}^{AB}\mathbb{E}_\alpha\left[\sum_{j=0}^{i}\frac{\partial^3 u_{i-j}(x,\mathrm{y},t)}{\partial x^3}+\frac{3}{2}\sum_{j=0}^{i}v_j(x,\mathrm{y},t)\frac{\partial^3 v_{i-j}(x,\mathrm{y},t)}{\partial x^3}\right. \tag{37}$$
$$+\frac{9}{2}\sum_{j=0}^{i}\frac{\partial v_{i-j}(x,\mathrm{y},t)}{\partial x}\frac{\partial^2 v_{i-j}(x,\mathrm{y},t)}{\partial x^2} - 6\sum_{j=0}^{i}u_j(x,\mathrm{y},t)\frac{\partial u_{i-j}(x,\mathrm{y},t)}{\partial x}$$
$$-6\sum_{j=0}^{i}\left(\sum_{r=0}^{m}u_r(x,\mathrm{y},t)\,v_{m-r}(x,\mathrm{y},t)\right)\frac{\partial v_{i-j}(x,\mathrm{y},t)}{\partial x}$$
$$\left.-\frac{3}{2}\sum_{j=0}^{i}\left(\sum_{r=0}^{m}v_r(x,\mathrm{y},t)\,v_{m-r}(x,\mathrm{y},t)\right)\frac{\partial u_{i-j}(x,\mathrm{y},t)}{\partial x}\right].$$

$$\mathcal{R}_{2,m}\,(\vec{u}_{m-1},\vec{v}_{m-1}) = {}^{AB}\mathbb{E}_\alpha[v_{m-1}(x,\mathrm{y},t)] - \left(1-\frac{\hbar_m}{n}\right)w^2v_0(x,y,t)$$
$$+\frac{\alpha w^\alpha+1-\alpha}{N(\alpha)}{}^{AB}\mathbb{E}_\alpha\left[\sum_{j=0}^{i}v_j(x,\mathrm{y},t)\frac{\partial^3 v_{i-j}(x,\mathrm{y},t)}{\partial x^3} - 6\sum_{j=0}^{i}v_j(x,\mathrm{y},t)\frac{\partial u_{i-j}(x,\mathrm{y},t)}{\partial x}\right. \tag{38}$$
$$\left.-6\sum_{j=0}^{i}u(x,\mathrm{y},t)\frac{\partial v_{i-j}(x,\mathrm{y},t)}{\partial x} - \frac{15}{2}\sum_{j=0}^{i}\left(\sum_{r=0}^{m}v_r(x,\mathrm{y},t)\,v_{m-r}(x,\mathrm{y},t)\right)\frac{\partial v_{i-j}(x,\mathrm{y},t)}{\partial x}\right].$$

By implementing the inverse Atangana-Baleanu Elzaki Transform (IABCET) to Equations (35) and (36), the resulting expressions are:

$$u_m\,(x,\mathrm{y},t) = \hbar_m\, u_{m-1}(x,y,t) + h\left({}^{AB}\mathbb{E}_\alpha\right)^{-1}\left[\mathcal{R}_{1,m}\,(\vec{u}_{m-1})\right]. \tag{39}$$

$$v_m\,(x,y,t) = \hbar_m\, v_{m-1}(x,y,t) + h\left({}^{AB}\mathbb{E}_\alpha\right)^{-1}\left[\mathcal{R}_{2,m}\,(\vec{v}_{m-1})\right]. \tag{40}$$

With ICs, it is obtained by

$$u_0(x,y,t) = \frac{1}{8}\lambda^2\left(1-4sech^2\left(\frac{\lambda x}{2}\right)\right). \tag{41}$$
$$v_0(x,y,t) = \lambda sech\left(\frac{\lambda x}{2}\right).$$

Substituting $m=1, m=2$ into Equations (39) and (40), the resulting Equations (42) through (45) are obtained as follows:

$$u_1(x,y,t) = -\frac{h\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) sinh\left(\frac{\lambda x}{2}\right)\lambda^5}{4N(\alpha)cosh^3\left(\frac{\lambda x}{2}\right)}. \tag{42}$$

$$v_1(x,y,t) = \frac{h\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) sinh\left(\frac{\lambda x}{2}\right)\lambda^4}{4N(\alpha)cosh^2\left(\frac{\lambda x}{2}\right)}. \tag{43}$$

$$u_2(x,t) = -(n+h)\frac{h\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) sinh\left(\frac{\lambda x}{2}\right)\lambda^5}{4N(\alpha)cosh^3\left(\frac{\lambda x}{2}\right)} -h^2\frac{\lambda^8\left(cosh^2\left(\frac{\lambda x}{2}\right)-\frac{3}{2}\right)\left(1-\alpha+2\alpha(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)}{8N^2(\alpha)cosh^4\left(\frac{\lambda x}{2}\right)}. \tag{44}$$

$$v_2(x,t) = (n+h)\frac{h\left(1-\alpha+\frac{\alpha t^{\alpha}}{\Gamma(\alpha+1)}\right) sinh\left(\frac{\lambda x}{2}\right)\lambda^4}{4N(\alpha)cosh^2\left(\frac{\lambda x}{2}\right)} +h^2\frac{\lambda^7\left(cosh^2\left(\frac{\lambda x}{2}\right)-2\right)\left(1-\alpha+2\alpha(1-\alpha)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\alpha^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)}{16N^2(\alpha)cosh^3\left(\frac{\lambda x}{2}\right)}. \tag{45}$$

As a result, the ABq-EHATM solution of Equation (28) can be expressed as:

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{46}$$

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)\left(\frac{1}{n}\right)^m. \tag{47}$$

Figure 1 illustrates the 2D graphs of the solutions $\mathrm{u}(x,t)$ and $\mathrm{v}(x,t)$ obtained using the ABq-EHATM method for different values of α. These graphs are presented to visualize the behavior of the method under varying parameters and to analyze its impact on the solutions.

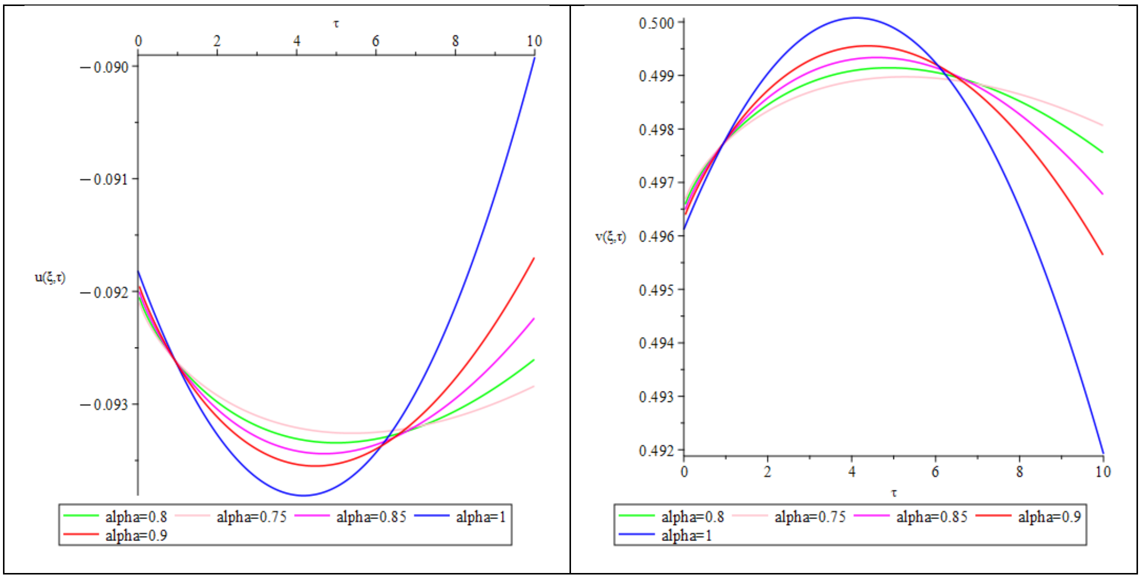
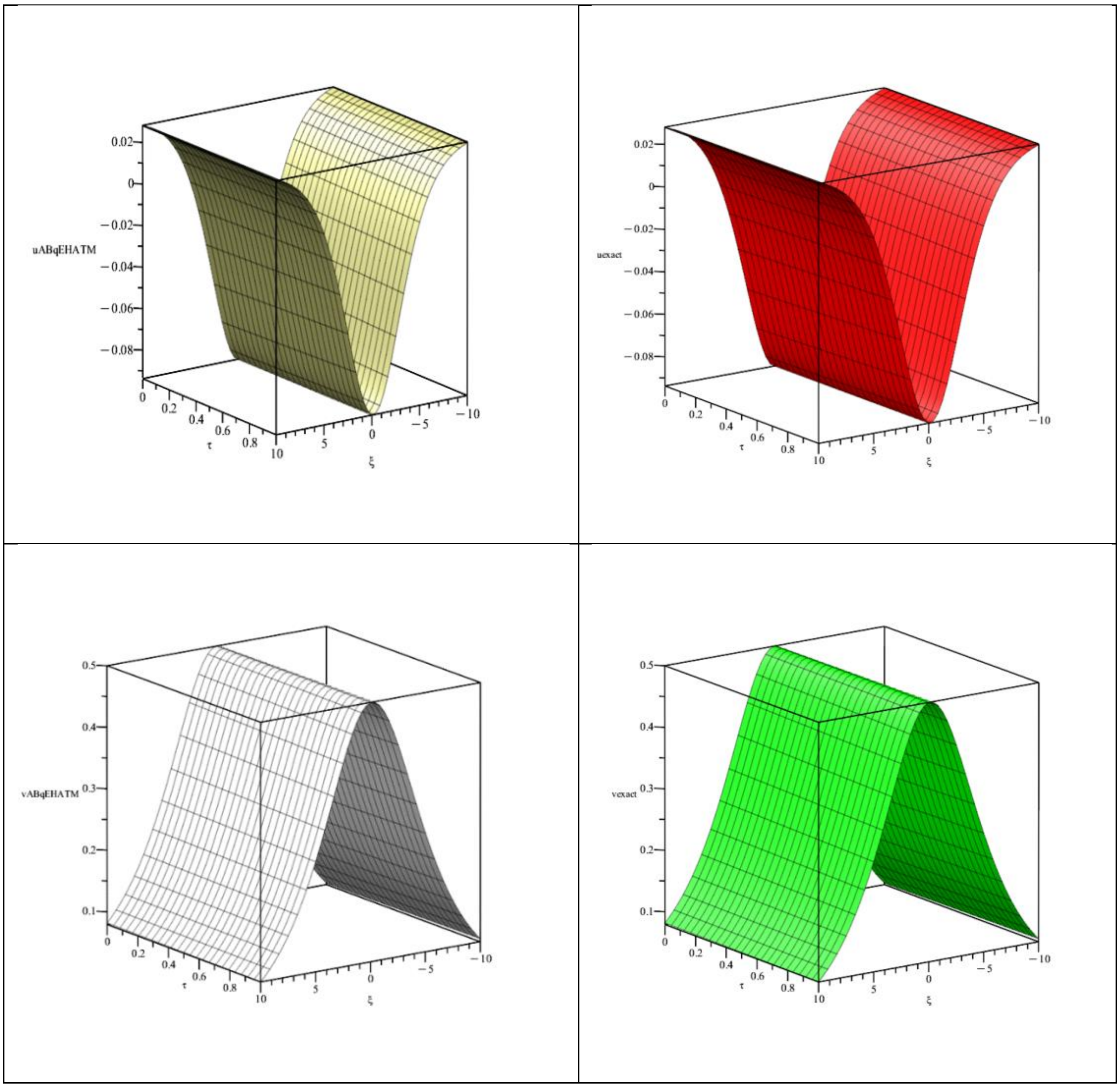


Figure 1. Analysis of the ABq-EHATM solutions for $\mathrm{u}(x,t)$ and $\mathrm{v}(x,t)$ in relation to the exact solution, and (b) evaluation of the ABq-EHATM solutions for $\mathrm{u}(x,t)$ and $\mathrm{v}(x,t)$ against the exact solution at $t = 0.5, R = 100, \mathrm{h} = -1, \mathrm{n} = 1$ for different values of α.

Figure 2 displays the 3D representations of ABq-EHATM, the exact solution, and the absolute error for $\mathrm{u}(x,t)$ and $\mathrm{v}(x,t)$, providing a visual comparison of their accuracy and behavior.



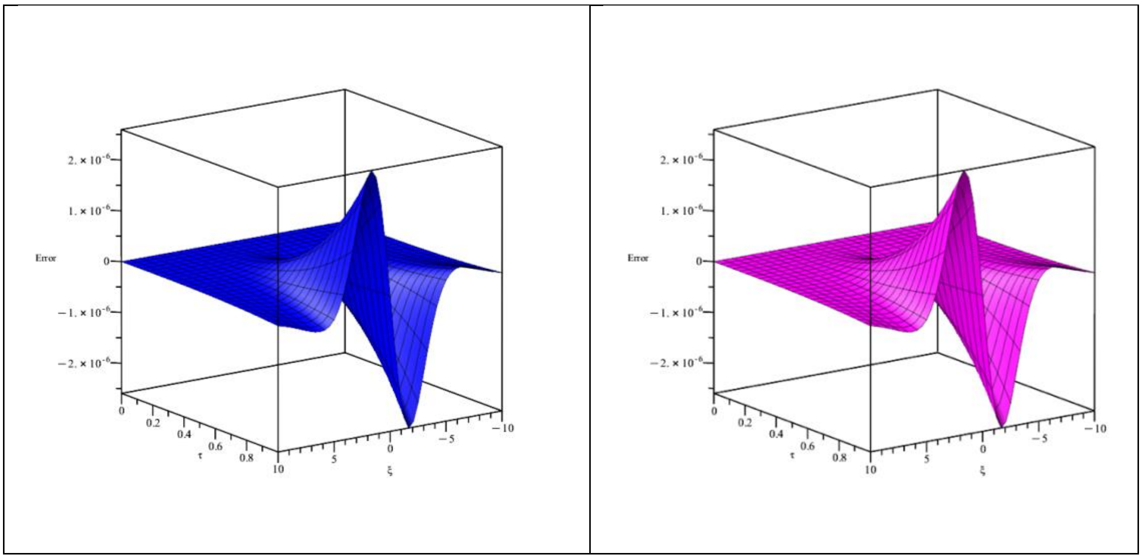


Figure 2. (a) Behavior of the ABq-EHATM solution $\mathrm{u}(x,t)$, (b) Behavior of the exact solution $\mathrm{u}(x,t)$, (c) Behavior of the ABq-EHATM solution $\mathrm{v}(x,t)$, (d) Behavior of the exact solution $\mathrm{v}(x,t)$, (e) Distribution of the absolute error $\left|u_{exact}-u_{ABq-EHATM}\right|$ and (f) Distribution of the absolute error $\left|u_{exact}-u_{ABq-EHATM}\right|$ at $\lambda=0.5$, $\alpha=1$, for Example 4.1.

Table 1 displays the numerical values of $u(x,y,t)$ solutions with ABq-EHATM at various fractional values α.

Table 1. Numerical solution of $u(x,t)$ by ABq-EHATM at various fractional values with different x,t and α at $\lambda=0.5, n=1, h=-1.$.

x	t	$\alpha=0.75$	$\alpha=0.80$	$\alpha=0.85$	$\alpha=0.90$	$\alpha=1$
0.1	0.001	1.43×10^{-7}	7.81×10^{-8}	4.39×10^{-8}	2.28×10^{-8}	2.48×10^{-17}
0.2	0.001	2.59×10^{-7}	1.53×10^{-8}	8.93×10^{-8}	4.65×10^{-8}	4.07×10^{-17}
0.3	0.001	3.64×10^{-7}	2.28×10^{-8}	1.35×10^{-8}	6.06×10^{-8}	7.47×10^{-17}
0.4	0.001	5.09×10^{-7}	3.09×10^{-8}	1.65×10^{-8}	9.27×10^{-8}	9.75×10^{-17}
0.5	0.001	6.17×10^{-7}	3.75×10^{-8}	2.19×10^{-8}	1.09×10^{-8}	1.32×10^{-16}

Table 2 displays the numerical values of $v(x,t)$ solutions with ABq-EHATM at various fractional values α.

Table 2. Numerical solution of $w(x,t)$ by ABq-EHATM at various fractional values with different with different x,t and α at $\lambda=0.5, n=1, h=-1$.

x	t	$\alpha=0.75$	$\alpha=0.80$	$\alpha=0.85$	$\alpha=0.90$	$\alpha=1$
0.1	0.001	2.47×10^{-6}	1.51×10^{-7}	8.95×10^{-7}	4.65×10^{-7}	3.08×10^{-17}
0.2	0.001	4.95×10^{-6}	3.08×10^{-7}	1.75×10^{-7}	9.32×10^{-7}	6.29×10^{-17}
0.3	0.001	7.45×10^{-6}	4.54×10^{-7}	2.65×10^{-6}	1.35×10^{-6}	9.34×10^{-17}
0.4	0.001	1.14×10^{-7}	5.98×10^{-7}	3.45×10^{-6}	1.91×10^{-6}	1.15×10^{-16}
0.5	0.001	1.19×10^{-7}	7.55×10^{-6}	4.37×10^{-6}	2.27×10^{-6}	1.45×10^{-15}

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5. Results and Discussion

Figure 1 illustrates the two-dimensional graphs of the $u(x,t)$, $v(x,t)$ obtained by ABq-EHATM for several values of α, namely $\alpha = 0.75,\ \alpha = 0.8,\ \alpha = 0.85,\ \alpha = 0.9,$ and $\alpha = 1$ in the context of ABCTFCJME. Figure 2 displays two-dimensional graphs, denoted as $u(x,t)$, obtained through the use of ABq-EHATM for several values of α, specifically $\alpha = 0.75,\ \alpha = 0.8,\ \alpha = 0.85,$ $\alpha = 0.9,$ and $\alpha = 1$ The $u(x,t),\ v(x,t)$ solution produced by ABq-EHATM for various values of $\alpha = 0.75,\ \alpha = 0.8,\ \alpha = 0.85,\ \alpha = 0.9,$ and $\alpha = 1$ is presented in Tables 1-2.

6. Conclusion

The ABq-EHATM demonstrates considerable promise as an effective approach for addressing the ABCTFCJME. The method's ability to break down complex equations into simpler parts, together with its accuracy and compatibility with other numerical techniques, makes it a valuable tool in mathematics. Furthermore, the ABq-EHATM offers some advantages over traditional numerical approaches, facilitating easier implementation and achieving greater accuracy. Because of its significant potential, this method is anticipated to become more and more significant in the solution of nonlinear differential equations in the years to come. As a result, scientists are encouraged to investigate its possible uses in greater detail. Future research may focus on extending the ABq-EHATM to other nonlinear fractional models describing real-world phenomena such as diffusion, viscoelasticity, and heat transfer. The method’s flexibility also makes it applicable to multi-dimensional and coupled systems, enhancing its use in practical problems. These studies would further demonstrate the versatility and effectiveness of ABq-EHATM in various scientific and engineering contexts.

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Authors’ Contributions

The author solely conceived the study, performed the analysis, and wrote the manuscript.

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The author declares that there is no conflict of interest regarding the publication of this paper.

Statement of Research and Publication Ethics

The author declares that this study complies with the principles of research and publication ethics.

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