Asymptotically $I$-Cesàro Equivalence of Sequences of Sets

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Abstract

In this paper, we defined concepts of asymptotically $I$-Cesàro equivalence and investigate the relationships between the concepts of asymptotically strongly $I$-Cesàro equivalence, asymptotically strongly $I$-lacunary equivalence, asymptotically $p$-strongly $I$-Cesàro equivalence and asymptotically $I$-statistical equivalence of sequences of sets.

1. Introduction

The concept of convergence of sequences of real numbers $\mathbb{R}$ has been transferred to statistical convergence by Fast [5] and independently by Schoenberg [16]. $I$-convergence was first studied by Kostyrko et al. [9] in order to generalize of statistical convergence which is based on the structure of the ideal $I$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [4] introduced new notions, namely $I$-statistical convergence and $I$-lacunary statistical convergence by using ideal. There are different convergence notions for sequence of sets. One of them handled in this paper is the concept of Wijsman convergence (see, [1], [2], [6], [7], [8], [9], [10], [11], [15], [19], [20]).

The concepts of statistical convergence and lacunary statistical convergence of sequences of sets were studied in [11, 18] in Wijsman sense. Also, new convergence notions, for sequences of sets, which is called Wijsman $I$-convergence, Wijsman $I$-statistical convergence and Wijsman $I$-Cesàro summability by using ideal were introduced in [7], [8], [20]. Marouf [10] presented definitions for asymptotically equivalent and asymptotic regular matrices. This concepts was investigated in [12, 13, 14]. The concept of asymptotically equivalence of sequences of real numbers which is defined by Marouf [10] has been extended by Ulusu and Nuray [19] to concepts of Wijsman asymptotically equivalence of set sequences. Moreover, natural inclusion theorems are presented. Kışı et al. [8] introduced the concepts of Wijsman $I$-asymptotically equivalence of sequences of sets.

2. Definitions and notations

Now, we recall the basic definitions and concepts (See [1, 2, 6, 7, 8, 9, 10, 11, 15, 19, 20]).

Let $(Y, \rho)$ be a metric space. For any point $y \in Y$ and any non-empty subset $U$ of $Y$, we define the distance from $y$ to $U$ by $d(y, U) = \inf_{u \in U} \rho(y, u)$.

Let $(Y, \rho)$ be a metric space and $U, U_i$ be any non-empty closed subsets of $Y$. The sequence $\{U_i\}$ is Wijsman convergent to $U$ if for each $y \in Y$,

$$\lim_{i \to \infty} d(y, U_i) = d(y, U).$$

Let $(Y, \rho)$ be a metric space and $U, U_i$ be any non-empty closed subsets of $Y$. The sequence $\{U_i\}$ is Wijsman statistical convergent to $U$ if $\{d(y, U_i)\}$ is statistically convergent to $d(y, U)$; i.e., for every $\varepsilon > 0$ and for each $y \in Y$,

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ i \leq n : |d(y, U_i) - d(y, U)| \geq \varepsilon \right\} \right| = 0.$$
A family of sets \( \mathcal{F} \subseteq 2^\mathbb{N} \) is called an ideal if and only if (i) \( \emptyset \in \mathcal{F} \), (ii) For each \( U, V \in \mathcal{F} \) we have \( U \cup V \in \mathcal{F} \), (iii) For each \( U \in \mathcal{F} \) and each \( V \subseteq U \) we have \( V \in \mathcal{F} \).

An ideal is called non-trivial ideal if \( \mathbb{N} \notin \mathcal{F} \) and non-trivial ideal is called admissible ideal if \( \{ n \} \in \mathcal{F} \) for each \( n \in \mathbb{N} \).

A family of sets \( \mathcal{I} \subseteq 2^\mathbb{N} \) is a filter if and only if (i) \( \emptyset \notin \mathcal{I} \), (ii) For each \( U, V \in \mathcal{I} \) we have \( U \cap V \in \mathcal{I} \), (iii) For each \( U \in \mathcal{I} \) and each \( V \supseteq U \) we have \( V \in \mathcal{I} \).

**Proposition 2.1.** \( \mathcal{I} \) is a non-trivial ideal in \( \mathbb{N} \) if and only if

\[
\mathcal{I}(\mathcal{F}) = \{ E \subseteq \mathbb{N} : (\exists U \in \mathcal{I})(E = N \setminus U) \}
\]

is a filter in \( \mathbb{N} \).

Throughout the paper, we let \((Y, \rho)\) be a separable metric space, \( \mathcal{I} \subseteq 2^\mathbb{N} \) be an admissible ideal and \( U, U_i \) be any non-empty closed subsets of \( Y \).

The sequence \( \{U_i\} \) is Wijsman \( \mathcal{I} \)-convergent to \( U \), if for every \( \varepsilon > 0 \) and for each \( y \in Y \), \( U(y, \varepsilon) = \{ i \in \mathbb{N} : |d(y, U_i) - d(y, U)| \geq \varepsilon \} \) belongs to \( \mathcal{I} \).

The sequence \( \{U_i\} \) is Wijsman \( \mathcal{I} \)-statistical convergent to \( U \), if for every \( \varepsilon > 0, \delta > 0 \) and for each \( y \in Y \),

\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ i \leq n : |d(y, U_i) - d(y, U)| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}
\]

and we write \( U_i \overset{S(\mathcal{I})}{\rightarrow} U \).

The sequence \( \{U_i\} \) is Wijsman \( \mathcal{I} \)-Cesàro summable to \( U \), if for every \( \varepsilon > 0 \) and for each \( y \in Y \),

\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y, U_i) - d(y, U)| \geq \varepsilon \right\} \in \mathcal{I}
\]

and we write \( U_i \overset{C(\mathcal{I})}{\rightarrow} U \).

The sequence \( \{U_i\} \) is Wijsman strongly \( \mathcal{I} \)-Cesàro summable to \( U \), if for every \( \varepsilon > 0 \) and for each \( y \in Y \),

\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y, U_i) - d(y, U)| \geq \varepsilon \right\} \in \mathcal{I}
\]

and we write \( U_i \overset{C_s(\mathcal{I})}{\rightarrow} U \).

The sequence \( \{U_i\} \) is Wijsman \( p \)-strongly \( \mathcal{I} \)-Cesàro summable to \( U \), if for every \( \varepsilon > 0 \), for each \( p \) positive real number and for each \( y \in Y \),

\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y, U_i) - d(y, U)|^p \geq \varepsilon \right\} \in \mathcal{I}
\]

and we write \( U_i \overset{C_{p}(\mathcal{I})}{\rightarrow} U \).

By a lacunary sequence we mean an increasing integer sequence \( \theta = \{k_r\} \) such that \( k_0 = 0 \) and \( h_r = k_r - k_{r-1} \to \infty \) as \( r \to \infty \). In this paper the intervals determined by \( \theta \) will be denoted by \( I_r = (k_{r-1}, k_r) \) and ratio \( \frac{k_r}{k_{r-1}} \) will be abbreviated by \( q_r \).

Let \( \theta \) be a lacunary sequence. The sequence \( \{U_i\} \) is Wijsman strongly \( \mathcal{I} \)-lacunary summable to \( U \), if for every \( \varepsilon > 0 \) and for each \( y \in Y \),

\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k_r \leq i \leq k_{r+1}} |d(y, U_i) - d(y, U)| \geq \varepsilon \right\} \in \mathcal{I}
\]

and we write \( U_i \overset{N\alpha(\mathcal{I})}{\rightarrow} U \).

Two nonnegative sequences \( a = (a_i) \) and \( b = (b_i) \) are said to be asymptotically equivalent if

\[
\lim_{i \to \infty} \frac{a_i}{b_i} = 1
\]

and denoted by \( a \sim b \).

We define \( d(y; U_i, V_i) \) as follows:

\[
d(y; U_i, V_i) = \begin{cases} 
\frac{d(y, U_i)}{d(y, V_i)}, & y \notin U_i \cup V_i \\
\mathcal{L}, & y \in U_i \cup V_i.
\end{cases}
\]

The sequences \( \{U_i\} \) and \( \{V_i\} \) are Wijsman asymptotically equivalent of multiple \( \mathcal{L} \), if for each \( y \in Y \),

\[
\lim_{i \to \infty} d(y; U_i, V_i) = \mathcal{L}.
\]
The sequences \( \{ U_i \} \) and \( \{ V_i \} \) are Wijsman asymptotically statistical equivalent of multiple \( \mathcal{L} \), if for every \( \varepsilon > 0 \) and for each \( y \in Y \),
\[
\lim_{n \to \infty} \frac{1}{n} \left| \{ i \leq n : |d(y; U_i, V_i) - \mathcal{L}| \geq \varepsilon \} \right| = 0.
\]
The sequences \( \{ U_i \} \) and \( \{ V_i \} \) are Wijsman asymptotically \( \mathcal{I} \)-equivalent of multiple \( \mathcal{L} \), if for every \( \varepsilon > 0 \) and each \( y \in Y \)
\[
\{ i \in \mathbb{N} : |d(y; U_i, V_i) - \mathcal{L}| \geq \varepsilon \} \in \mathcal{I}
\]
and we write \( U_i \overset{\mathcal{I}}{\sim} V_i \) and simply Wijsman asymptotically \( \mathcal{I} \)-equivalent if \( \mathcal{L} = 1 \).

The sequences \( \{ U_i \} \) and \( \{ V_i \} \) are Wijsman asymptotically \( \mathcal{I} \)-statistical equivalent of multiple \( \mathcal{L} \), if for every \( \varepsilon > 0, \delta > 0 \) and for each \( y \in Y \),
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ i \leq n : |d(y; U_i, V_i) - \mathcal{L}| \geq \varepsilon \} \right| \geq \delta \right\} \in \mathcal{I}
\]
and we write \( U_i \overset{\mathcal{S}(\mathcal{I}^{\delta})}{\sim} V_i \) and simply Wijsman asymptotically \( \mathcal{I} \)-statistical equivalent if \( \mathcal{L} = 1 \).

3. Main results

In this section, we defined notions of asymptotically \( \mathcal{I} \)-Cesàro equivalence of sequences of sets. Also, we investigate the relationships between the concepts of asymptotically strongly \( \mathcal{I} \)-Cesàro equivalence, asymptotically strongly \( \mathcal{I} \)-lacunary equivalence, asymptotically \( p \)-strongly \( \mathcal{I} \)-Cesàro equivalence and asymptotically \( \mathcal{I} \)-statistical equivalence of sequences of sets.

**Definition 3.1.** The sequences \( \{ U_i \} \) and \( \{ V_i \} \) are asymptotically \( \mathcal{I} \)-Cesàro equivalence of multiple \( \mathcal{L} \), if for every \( \varepsilon > 0 \) and for each \( y \in Y \),
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y; U_i, V_i) - \mathcal{L}| \geq \varepsilon \right\} \in \mathcal{I}
\]
and we write \( U_i \overset{\mathcal{C}(\mathcal{I}^{\mathcal{L}})}{\sim} V_i \) and simply asymptotically \( \mathcal{I} \)-Cesàro equivalent if \( \mathcal{L} = 1 \).

**Definition 3.2.** The sequences \( \{ U_i \} \) and \( \{ V_i \} \) are asymptotically strongly \( \mathcal{I} \)-Cesàro equivalence of multiple \( \mathcal{L} \), if for every \( \varepsilon > 0 \) and for each \( y \in Y \),
\[
\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y; U_i, V_i) - \mathcal{L}| \geq \varepsilon \right\} \in \mathcal{I}
\]
and we write \( U_i \overset{\mathcal{C}(\mathcal{I}^{\mathcal{L}})}{\sim} V_i \) and simply asymptotically strongly \( \mathcal{I} \)-Cesàro equivalent if \( \mathcal{L} = 1 \).

**Theorem 3.3.** Let \( \theta \) be a lacunary sequence. If \( \lim \inf q_r > 1 \) then,
\[
U_i \overset{\mathcal{C}(\mathcal{I}^{\mathcal{L}})}{\sim} V_i \Rightarrow U_i \overset{\mathcal{S}(\mathcal{I}^{\delta})}{\sim} V_i.
\]

**Proof.** If \( \lim \inf q_r > 1 \), then there exists \( \delta > 0 \) such that \( q_r \geq 1 + \delta \) for all \( r \geq 1 \). Since \( h_r = k_r - k_{r-1} \), we have
\[
\frac{k_r}{h_r} \leq \frac{1 + \delta}{\delta} \quad \text{and} \quad \frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}.
\]
Let \( \varepsilon > 0 \) and for each \( y \in Y \), we define the set
\[
S = \left\{ k_r \in \mathbb{N} : \frac{1}{k_r} \sum_{i=1}^{k_r} |d(y; U_i, V_i) - \mathcal{L}| < \varepsilon \right\}.
\]
We can easily say that \( S \in \mathcal{F}(\mathcal{I}) \), which is a filter of the ideal \( \mathcal{I} \), so we have
\[
\frac{1}{n} \sum_{i=1}^{k_r} |d(y; U_i, V_i) - \mathcal{L}| = \frac{1}{n} \frac{k_r}{n} \sum_{i=1}^{k_r} |d(y; U_i, V_i) - \mathcal{L}| - \frac{1}{n} \frac{k_{r-1}}{n} \sum_{i=1}^{k_{r-1}} |d(y; U_i, V_i) - \mathcal{L}|
\]
\[
= \frac{k_r}{n} \cdot \frac{1}{k_r} \sum_{i=1}^{k_r} |d(y; U_i, V_i) - \mathcal{L}|
\]
\[
- \frac{k_{r-1}}{n} \cdot \frac{1}{k_{r-1}} \sum_{i=1}^{k_{r-1}} |d(y; U_i, V_i) - \mathcal{L}|
\]
\[
\leq \left( 1 + \frac{\delta}{\delta} \right) \varepsilon - \frac{1}{\delta} \varepsilon.',
\]
for each \( y \in Y \) and for each \( k_r \in S \). Choose \( \eta = \left( \frac{1+\delta}{\delta} \right) \varepsilon + \frac{1}{\delta} \varepsilon' \). Therefore, for each \( y \in Y \)

\[
\left\{ r \in \mathbb{N} : \frac{1}{k_r} \sum_{j \in k_r} |d(y; U_i, V_i) - \mathcal{L}| < \eta \right\} \in \mathcal{F} (\mathcal{L}).
\]

Therefore, \( U_r^{\mathcal{N}^b_r} |_{\mathcal{F} r} \sim V_i \).

\[ \square \]

**Theorem 3.4.** Let \( \theta \) be a lacunary sequence. If \( \limsup_r q_r < \infty \) then,

\[ U_r^{\mathcal{N}^b_r} |_{\mathcal{F} r} \sim V_i \Rightarrow U_r^{C^1 |_{\mathcal{F} r}} |_{\mathcal{F} r} V_i, \]

\[ \text{Proof.} \] If \( \limsup_r q_r < \infty \), then there exists \( K > 0 \) such that \( q_r < K \) for all \( r \geq 1 \). Let \( U_r^{\mathcal{N}^b_r} |_{\mathcal{F} r} \sim V_i \) and for each \( y \in Y \), we define the sets \( T \) and \( R \)

\[ T = \left\{ r \in \mathbb{N} : \frac{1}{k_r} \sum_{j \in k_r} |d(y; U_i, V_i) - \mathcal{L}| < \epsilon_1 \right\} \]

and

\[ R = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y; U_i, V_i) - \mathcal{L}| < \epsilon_2 \right\}. \]

Let

\[ a_j = \frac{1}{k_r} \sum_{j \in k_r} |d(y; U_i, V_i) - \mathcal{L}| < \epsilon_1 \]

for each \( y \in Y \) and for all \( j \in T \). It is obvious that \( T \in \mathcal{F} (\mathcal{L}) \). Choose \( n \) is any integer with \( k_{r-1} < n < k_r \), where \( r \in T \). Then, for each \( y \in Y \) we have

\[
\frac{1}{k_r} \sum_{i \in k_r} |d(y; U_i, V_i) - \mathcal{L}| \leq \frac{1}{k_{r-1}} \sum_{i \in k_{r-1}} |d(y; U_i, V_i) - \mathcal{L}|
\]

\[ = \frac{1}{k_{r-1}} \left( \sum_{i \in k_{r-1}} |d(y; U_i, V_i) - \mathcal{L}| + \sum_{i \in k_r} |d(y; U_i, V_i) - \mathcal{L}| + \ldots + \sum_{i \in k_r} |d(y; U_i, V_i) - \mathcal{L}| \right) \]

\[ = \frac{1}{k_{r-1}} \left( \sum_{i \in k_{r-1}} |d(y; U_i, V_i) - \mathcal{L}| \right) + \frac{k_r-k_{r-1}}{k_{r-1}} \left( \sum_{i \in k_r} |d(y; U_i, V_i) - \mathcal{L}| \right) + \ldots + \frac{k_r-k_{r-1}}{k_{r-1}} \left( \sum_{i \in k_r} |d(y; U_i, V_i) - \mathcal{L}| \right) \]

\[ = \frac{1}{k_{r-1}} a_1 + \frac{k_r-k_{r-1}}{k_{r-1}} a_2 + \ldots + \frac{k_r-k_{r-1}}{k_{r-1}} a_r \]

\[ \leq \left( \sup_{j \in T} a_j \right) \frac{k_r-k_{r-1}}{k_{r-1}} \leq \epsilon_1 \cdot K. \]

Choose \( \epsilon_2 = \frac{\epsilon}{K} \) and in view of the fact that

\[ \bigcup \{ n : k_{r-1} < n < k_r, r \in T \} \subset R, \]

where \( T \in \mathcal{F} (\mathcal{L}) \), it follows from our assumption on \( \theta \) that the set \( R \) also belongs to \( \mathcal{F} (\mathcal{L}) \) and therefore, \( U_r^{C^1 |_{\mathcal{F} r}} |_{\mathcal{F} r} V_i \).

\[ \square \]

We have the following Theorem by Theorem 3.3 and Theorem 3.4.

**Theorem 3.5.** Let \( \theta \) be a lacunary sequence. If \( 1 < \liminf_r q_r < \limsup_r q_r < \infty \) then,

\[ U_r^{C^1 |_{\mathcal{F} r}} |_{\mathcal{F} r} V_i \Leftrightarrow U_r^{\mathcal{N}^b_r} |_{\mathcal{F} r} V_i. \]

**Definition 3.6.** The sequences \( \{ U_i \} \) and \( \{ V_i \} \) are asymptotically \( p \)-strongly \( \mathcal{F} \)-Cesàro equivalence of multiple \( \mathcal{L} \) if for every \( \varepsilon > 0 \), for each \( p \) positive real number and for each \( y \in Y \),

\[ \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y; U_i, V_i) - \mathcal{L}|^p \geq \varepsilon \right\} \in \mathcal{F} \]

and we write \( U_r^{C^1 |_{\mathcal{F} r}} |_{\mathcal{F} r} V_i \) simply asymptotically \( p \)-strongly \( \mathcal{F} \)-Cesàro equivalent if \( \mathcal{L} = 1 \).
Theorem 3.7. If the sequences \( \{U_i\} \) and \( \{V_i\} \) are asymptotically \( p \)-strongly \( \mathcal{F} \)-Cesàro equivalence of multiple \( \mathcal{L} \) then, \( \{U_i\} \) and \( \{V_i\} \) are asymptotically \( \mathcal{F} \)-statistical equivalence of multiple \( \mathcal{L} \).

Proof. Let \( U_i \overset{C^{(p)}_\mathcal{F}}{\sim} V_i \) and \( \varepsilon > 0 \) given. Then, for each \( y \in Y \) we have

\[
\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}|^p \geq \varepsilon \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}|^p
\]

and so

\[
\frac{1}{\varepsilon^p} \cdot \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}|^p \geq \frac{1}{n} \cdot \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}| \geq \varepsilon.
\]

Hence, for each \( y \in Y \) and for a given \( \delta > 0 \),

\[
\left\{ \frac{1}{n} \cdot \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}| \geq \varepsilon \right\} \subseteq \left\{ \frac{1}{n} \cdot \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}|^p \geq \varepsilon - \delta \right\} \in \mathcal{F}.
\]

Therefore, \( U_i \overset{\mathcal{S}^{(p)}_\mathcal{F}}{\sim} V_i \).

Theorem 3.8. Let \( d(y;U_i) = \mathcal{O}(d(y;V_i)) \). If \( \{U_i\} \) and \( \{V_i\} \) are asymptotically \( \mathcal{F} \)-statistical equivalence of multiple \( \mathcal{L} \) then, \( \{U_i\} \) and \( \{V_i\} \) are asymptotically \( \mathcal{F} \)-Cesàro equivalence of multiple \( \mathcal{L} \).

Proof. Suppose that \( d(y;U_i) = \mathcal{O}(d(y;V_i)) \) and \( U_i \overset{\mathcal{S}^{(p)}_\mathcal{F}}{\sim} V_i \). Then, there is a \( K > 0 \) such that \( d(y;U_i,V_i) \leq K \) for all \( i \) and for each \( y \in Y \). Given \( \varepsilon > 0 \) and for each \( y \in Y \), we have

\[
\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}|^p \leq \frac{1}{n} \cdot \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}| \leq K^p \cdot \frac{\varepsilon}{K^p}.
\]

Then, for any \( \delta > 0 \),

\[
\left\{ \frac{1}{n} \cdot \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}| \geq \delta \right\} \subseteq \left\{ \frac{1}{n} \cdot \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathcal{L}|^p \geq \delta \right\} \in \mathcal{F}.
\]

Therefore, \( U_i \overset{C^{(p)}_\mathcal{F}}{\sim} V_i \).

References