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$\mathbb{L}^{3}$' de Spacelike Odak Eğrisinin Harmonik Eğrilikleri Üzerine Bir Çalışma

Ezgi YAĞCI[1], Esen İYİGÜN[2]

[1] B.U.Ü., Fen Edebiyat Fakültesi, Matematik Bölümü, Bursa, TURKEY,ORCID ID 0009-0008-7903-388X
[2] B.U.Ü., Fen Edebiyat Fakültesi, Matematik Bölümü, Bursa, TURKEY,ORCID ID 0000-0001-6821-0248

Corresponding Author: Ezgi YAĞCI, 502411011@ogr.uludag.edu.tr

Özet

Bu makalede, [6]'da verilen, indeksi 1 olan Lorentz 3-uzay $\mathbb{L}^3$'deki spacelike odak eğrisinin Frenet denklemlerini inceliyoruz. Eğrinin sırasıyla teğet, esas normal ve binormal Frenet vektörleri olan T, N ve W'yi ve Frenet denklemlerini kullanarak, spacelike odak eğrisi için bazı önemli tanımlar, teoremler ve sonuçlar veriyoruz. Ek olarak, indeksi 1 olan üç boyutlu Lorentz uzayındaki spacelike odak eğrisini, harmonik eğrilikler açısından inceledik. Son olarak, bir örnekle $\mathbb{L}^3$' deki spacelike odak eğrisini ve Lorentz dairesel helisini elde ettik.

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Öne Çıkanlar

Harmonic Curvatures of a Spacelike Focal Curve

A Study on Harmonic Curvatures of a Spacelike Focal Curve in $\mathbb{L}^3$

Abstract

In this paper,we study the Frenet equations given in [6] for a spacelike focal curve in Lorentz 3-space with index 1. Using the Frenet equations, we derive important definitions, theorems and results for a spacelike focal curve, where T, N and W are the tangent, principal normal and binormal Frenet vectors of the curve, respectively. Additionally, we examine a spacelike focal curve in the three-dimensional Lorentz space with index 1 in terms of harmonic curvatures. Finally, we present an example of a spacelike focal curve and a Lorentzian circular helix of a spacelike curve in $\mathbb{L}^3$.

Keywords

Lorentzian space, Focal curve, Harmonic curvatures

Highlights

Harmonic Curvatures of a Spacelike Focal Curve

1. Introduction

In a Semi-Euclidean manifold, there are three types of curve: spacelike, timelike, and null (or lightlike) curves, according to their causal character. The geometry of focal curves has been studied by several authors (see references [5], [6], and [7]). In particular, Yıldırım [5] studied the surfaces of canals and tubes around a timelike focal curve in Minkowski 3-space. Meanwhile, Yıldırım [6] examined canal surfaces surrounding a spacelike focal curve in a Lorentzian 3-space. Özdemir [7] studied the focal curvatures of non-lightlike curves in Minkowski (m+1)-space. In this paper, we derive theorems and results for a spacelike focal curve in three-dimensional Lorentz space of index 1, expressed in terms of harmonic curvatures.

2. Material and method

2.1. Material

Semi-Euclidean 3−space $\mathbb{L}^3$ with index 1 is the Euclidean 3−space $\mathbb{E}_1^3$ equipped with an indefinite flat metric g given by g $= dx_1^2 - dx_2^2 + dx_3^2$, where $X = (x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{L}^3$. A vector $X = (x_1, x_2, x_3)$ in $\mathbb{L}^3$ is a called a spacelike, a timelike or a null (lightlike), if respectively holds g$(X, X) > 0$, g $(X, X) < 0$ or g $(X, X) = 0$ and $X \neq 0 = (0, 0, 0)$. The norm of a vector X is given by
$\|X\| = \sqrt{|g(X, X)|}$. Two vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ in $\mathbb{L}^3$ are said to be orthogonal, if g$(X, Y) = 0$.
An arbitrary curve $\alpha = \alpha$(s) in $\mathbb{L}^3$ can locally be spacelike, timelike or null, if respectively all of its velocities α'(s) are spacelike, timelike or null. Spacelike or timelike curve α (s) is said to be parametrized by arclength functions s, if g(α'(s), α'(s)) = ± 1 .
We also recall that the pseudosphere S_1^2 and the pseudohyperbolic space H_0^2 are the hyperquadrics in $\mathbb{L}^3$, defined respectively by:

$S_1^2(c,\ r)= \{\alpha\ \epsilon \mathbb{L}^3\colon \mathrm{g}(\alpha - c, \alpha - c) = r^2\},$
$H_0^2(c, - r)= \{\alpha\ \epsilon\ \mathbb{L}^3\colon \mathrm{g}(\alpha - c, \alpha - c) = -r^2\},$
where center c and radius r ϵR+ .

Let $X = (x_1, x_2, x_3)$ and Y = (y_1, y_2, y_3) be two vectors in $\mathbb{L}^3$. The vector product in $\mathbb{L}^3$ is defined using the determinant

$$X \wedge Y = \begin{vmatrix} e_1 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - y_2x_3,\ x_1y_3 - y_1x_3,\ x_1y_2 - y_1x_2)$$

where e_1, e_2 and e_3 are coordinate direction vectors [3].

We know that its velocity vector is $T(s) = \alpha'\ (s)$. Let be the unit vector $N(s) = T'\ \frac{T'(s)}{\| T'(s)\|}$.
Finally, define the vector $W(s)=N(s)\wedge T(s)$. The family $\{T(s), N(s), W(s)\}$ is an orthonormal triad. These three vectors are known as the tangent, the principal normal and the binormal vectors, respectively. This family $\{T(s), N(s), W(s)\}$ is called the Frenet frame [6].
The moving Frenet frame along the curve $\alpha = \alpha$ (s) is denoted by $\{T, N, W\}$. T, N and W then represenet the tangent, principal normal and binormal vectors of the curve α(s), respectively. Depending on the causal character of the curve α, we have the following Frenet formulas:

$$\left\{\begin{matrix} T' = k_1 N, \\ N' = k_1 T + k_2 W, \\ W' = k_2 N \end{matrix}\right\} \tag{1}$$

and

$$\left\{\begin{matrix} g(T,T) = g(W,W) = 1, \\ g(N,N) = -1, \\ g(T,N) = g(T,W) = g(N,W) = 0 \end{matrix}\right\} \tag{2}$$

if α is a spacelike curve with a timelike principal normal N, where k_1 and k_2 are the first and second Frenet curvatures of α, respectively [4].
In $\mathbb{L}^3$, the curvature of a random curve X is derived as follows.

$$k_1 = \frac{\|X' \wedge X''\|}{\|X'\|^3},\quad k_2 = \frac{\det(X', X'', X''')}{\|X' \wedge X''\|^2},$$

where ∧ vector product in $\mathbb{L}^3$ [2].
We are now ready to express this in terms of harmonic curvatures of a spacelike focal curve in $\mathbb{L}^3$.

2.2. Method

We will derive the Frenet equations for the spatial focal curve in three-dimensional Lorentzian space using the literature, and then use these equations to derive certain definitions, theorems and results. Finally, we will examine the spacelike focal curve in terms of harmonic curvatures.

3. Results

Expression in terms of harmonic curvatures of spacelike focal curve in $\mathbb{L}^3$

Definition 1. [1] Let α be a spacelike curve in $\mathbb{L}^3$. The harmonic functions

$H_J : I \to \mathbb{R}$, $j = 0; 1$

defined by

$\{ H_0 = 0, H_1 = \frac{k_1}{k_2}, \ (k_2 \neq 0),$

These are known as the harmonic curvatures of α. Here, k_1 and k_2 are the Frenet curvatures of α.

Theorem 1. Let $\{T, N, W\}$ be a Frenet frame for the spacelike curve α and the timelike principal normal N in $\mathbb{L}^3$. Then for a spacelike curve α in $\mathbb{L}^3$, the following Frenet equations hold :

$$\left\{\begin{array}{c} T' = k_2 H_1 N, \\ N' = k_2 H_1 T + \frac{k_1}{H_1} W, \\ W' = \frac{k_1}{H_1} N, \end{array}\right\} \qquad (3)$$

where $H_1 \neq 0$ and $k_2 \neq 0$ are harmonic curvature and second Frenet curvatures of α, respectively.

Proof. By using the definition of harmonic curvatures, we obtain the theorem. ■

Theorem 2. Let α be a spacelike curve with a spacelike binormal in $\mathbb{L}^3$. $\{T, N, W\}$ is the Frenet frame of α. Where T is the tangent vector field, N is the principal normal vector field and W is the binormal vector field. Then

$$\left\{\begin{array}{c} g(T',T) = g(N',N) = g(W',W) = 0, \\ g(T',N) = -g(T,N'), \\ g(T',W) = -g(T,W'), \\ g(N',W) = -g(N,W'), \\ g(T'',T) = k_2^2 H_1^2 \\ g(N'',N) = -(1+H_1^2)k_2^2, \\ g(W'',W) = \frac{k_1^2}{H_1^2}, \\ g(T''N) = -g(T,N''), \\ g(T'',W) + g(T,W'') = 2H_1 k_2^2, \\ g(N'',W) = -g(N,W''). \end{array}\right\} \quad (4)$$

Here $k_2 \neq 0$ and $H_1 \neq 0$ are the second Frenet curvatures and the harmonic curvatures of α, respectively.

Proof. By using Equations (2) and (3), we obtain the proof of the theorem. ■

Definition 2. [5] Let (x_1, x_2, x_3) be semi-Euclidean coordinate system in $\mathbb{L}^3$.We take a Lorentzian sphere $g(F_\alpha - \alpha, F_\alpha - \alpha) = r^2$, with origin and radius F_α and r, respectively. Let

$$f(s) = \frac{1}{2}(\|F_\alpha - \alpha\|^2 - r^2). \quad (5)$$

If we have the following equations

$$f(s) = f'(s) = f''(s) = f'''(s) = 0, \quad (6)$$

then we say that the sphere passing through four consecutive points of the curve is called the osculating Lorentzian sphere.

Definition3. [6] Let $\alpha = \alpha(s) : I \rightarrow \mathbb{L}^3$ be any curve. The set of points F_α is called the focal curve of α if it is the set of centres of the osculating spheres of α.

Theroem 3. Let $\alpha = \alpha(s)$ be a spacelike curve with a spacelike binormal in $\mathbb{L}^3$ and its Frenet frame be $\{T, N, W\}$. Then the focal curve F_α of α is

$$F_\alpha = \alpha + f_1 N + f_2 W$$

and the focal curvatures are

$$f_1 = -\frac{1}{H_1 k_2}, \quad f_2 = \frac{(H_1 k_2)'}{k_1 k_2^2 H_1},$$

where $H_1 \neq 0$ $k_1 \neq 0$ and $k_2 \neq 0$ harmonic curvature, first and second Frenet curvatures of α, respectively.

Proof. Let us suppose that $\alpha = \alpha$ (s) lies on S_1^2 with center F_α . By the definition, we have
$g(F_\alpha - \alpha, F_\alpha - \alpha) = r^2,$ (7)

for every s∈I ⊂ R. Differentiating (7), three times with respect to *s* using Frenet equations (3), we have, respectively, we get

$$\begin{cases} g(T, F_\alpha - \alpha) = 0\,, \\ g(N, F_\alpha - \alpha) = -\dfrac{1}{H_1\, k_2}, \\ g(W, F_\alpha - \alpha) = \dfrac{(H_1\, k_2\,)}{k_1 k_2^2 H_1}\,. \end{cases}$$

Finally, we may write the focal curve as

$$F_\alpha(s) = \alpha\,(s) + f_1\, N(s) + f_2\, W(s) = \alpha\,(s) - \frac{1}{H_1\, k_2} N(s) + \frac{(H_1\, k_2\,)'}{k_1 k_2^2 H_1} W(s), \quad (8)$$

we can easily say the focal curvatures are

$$f_1 = -\frac{1}{H_1\, k_2}\,,\; f_2 = \frac{(H_1\, k_2\,)'}{k_1 k_2^2\, H_1}\,,$$

where $H_1 \neq 0$, $k_1 \neq 0$ and $k_2 \neq 0$ are. ■
By Definition 2 and Theorem 3 we have the following result.

Corollary 4. The expression in terms of harmonic curvatures of f(s) in Definition 2 is

$$f(s) = \frac{1}{2} \left(-\left(-\frac{1}{H_1\, k_2}\right)^2 + \left(\frac{(H_1\, k_2\,)'}{k_1 k_2^2\, H_1}\right)^2 - r^2 \right).$$

Corollary 5. The radius r of the Lorentzian sphere in $\mathbb{L}^3$ is

$$r^2 = -\left(-\frac{1}{H_1\, k_2}\right)^2 + \left(\frac{(H_1\, k_2\,)'}{k_1 k_2^2\, H_1}\right)^2 = -(f_1)^2 + (f_2)^2.$$

If r is constant then the derivative of r is

$$f_1 \left[-(f_1\,)' - \left(\frac{(H_1\, k_2\,)'}{k_1 k_2}\right)(f_2\,)' \right] = 0.$$

Here $f_1 = 0$, $\left(\frac{(H_1\, k_2\,)'}{k_1 k_2^2 H_1}(f_2\,)'\right) \neq 0$, $H_1 \neq 0$, $k_1 \neq 0$ and $k_2 \neq 0$.

Proof. By using the equations (2), (7) and (8), the result is obtained. ■

Theorem 6. Let α = (s) be a spacelike curve with a spacelike binormal in $\mathbb{L}^3$. The curve α satisfies the following Frenet equations:

$$\begin{bmatrix} 1 \\ \left(-\frac{1}{H_1\,k_2}\right)' \\ \left(\frac{(H_1\,k_2\,)'}{k_1k_2^2\,H_1}\right)' - \frac{(r^2)'\,k_1k_2^2\,H_1}{2(H_1\,k_2\,)'} \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{H_1\,k_2} \\ \left(-\frac{1}{H_1\,k_2}\right)'\,\frac{H_1}{k_1} \end{bmatrix}$$

Where if the curve α is spherical, then $(r^2)' = 0$

Example 1. Let us consider a spacelike curve in $\mathbb{L}^3$ with the equation

$\alpha(s) = \left(\sinh\ \frac{s}{\sqrt{3}}\,,\cosh\,\frac{s}{\sqrt{3}}\,,\frac{s}{\sqrt{3}}\right).$

$T(s) = \alpha'(s) = \left(\frac{1}{\sqrt{3}}\,\cosh\,\frac{s}{\sqrt{3}}\,,\frac{1}{\sqrt{3}}\,\sinh\,\frac{s}{\sqrt{3}}\,,\frac{1}{\sqrt{3}}\right),$

where $g(T(s), T(s)) = \frac{2}{3}$. Also,

$\alpha'' = \left(\frac{1}{3}\,\sinh\,\frac{s}{\sqrt{3}}\,,\frac{1}{3}\,\cosh\,\frac{s}{\sqrt{3}}\,,0\right),$

$\alpha''' = \left(\frac{1}{3\sqrt{3}}\,\cosh\,\frac{s}{\sqrt{3}}\,,\frac{1}{3\sqrt{3}}\,\sinh\,\frac{s}{\sqrt{3}}\,,0\right)$

$N(s) = \left(\sinh\ \frac{s}{\sqrt{3}}\,,\cosh\,\frac{s}{\sqrt{3}}\,,0\right),$

$W(s) = \left(\frac{1}{\sqrt{3}}\,\cosh\,\frac{s}{\sqrt{3}}\,,\frac{1}{\sqrt{3}}\,\sinh\,\frac{s}{\sqrt{3}}\,,-\frac{1}{\sqrt{3}}\right).$

Morever, we have the curvatures $k_1 = \frac{1}{2}$, $k_2 = -\frac{1}{2}$ and $H_1 = -1$. Then, we obtain the Frenet equations of spacelike curve α in $\mathbb{L}^3$

$$\left\{\begin{array}{c} T'(s) = \frac{1}{2}\,N(s)\,, \\ N'(s) = \frac{1}{2}\,T(s)\, - \frac{1}{2}\,W(s), \\ W'(s) = -\,\frac{1}{2}\,N(s), \end{array}\right\}$$

and questions 4 are :

$$\left\{\begin{array}{c} g(T',T) = g(N',N) = g(W',W) = g(T',W) = g(T\,,W')\ = 0, \\ g(T'',N) = g(T,N'') = g(N'',W) = g(N,W'') = 0, \\ g(T',N) = g(N',W) = \ g(N'',N) =\ -\,\frac{1}{2}\,, \\ g(T'',W)\ +\ g(T,W'')\ \ = -\,\frac{1}{2}\,, \\ g(T,N') =\ g(N,W')\ =\ \frac{1}{2}\,, \\ g(T'',T) = g(W'',W) =\ \frac{1}{4}. \end{array}\right\}$$

Additionally, the focal coefficients of α . $f_1 = -2$ and $f_2 = 0$. Then the focal curve F_α of α is

$$F_\alpha = \alpha(s) - 2N(s) = \left(-\sinh \frac{s}{\sqrt{3}}, -\cosh \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right).$$

Since $\frac{k_2}{k_1}$ is constant, the curve α is the Lorentzian circular helix in $\mathbb{L}^3$.

4. Conclusion

This study examined spacelike focal curves in three-dimensional Lorentz space with index 1 within the framework of the Frenet equations, investigating their geometric properties. First, the Frenet equations were rearranged using the tangent, normal and binormal vectors, and fundamental definitions and theorems for spacelike focal curves were established. Then, these curves were analyzed in terms of harmonic curvatures to provide a different perspective. The results obtained reveal that the theory of curves in Lorentz space has a rich structure, both theoretically and in terms of potential applications. Furthermore, presenting explicit examples of spacelike focal curves and the Lorentz circular helix concretized the theoretical analyses. In this regard, the study lays the groundwork for further research in differential geometry and the geometric structures of Lorentz spaces.

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Conflict of interest

There is no conflict of interest.

Author contribution

Author2 was responsible for the literature review and compilation of information, while Author1 contributed to the writing and editing of the manuscript.

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