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An improved approach for bending vibrations of cantilever beam with tip mass at different end conditions

Farklı uç koşullarına sahip uç kütleli bir ankastre kirişin eğilme titreşimleri için geliştirilmiş yeni bir yöntem

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An Improved Approach for Bending Vibrations of Cantilever Beam with Tip Mass at Different End Conditions

Highlights

- *Yeni bir yaklaşım önerildi ve doğrulandı/A new improved approach was proposed and validated*
- *Birinci bouytsuz doğal frekans kolaylıkla bulunabilir/The first nondimensional bending natural frequency can be obtained easily*
- *Bu yöntem farklı durumlar için uygulanabilir/It can easily be applied to different end conditions*
- *Simulasyon sonuçları ideal sonuçlarla neredeyse aynıdır/The simulation results are nearly identical to ideal values*

Graphical Abstract

In this work, first natural frequencies of the cantilever beam with tip mass at different end conditions and cross sections were tried to obtain by reducing the system to basic single degree of freedom spring-mass system.

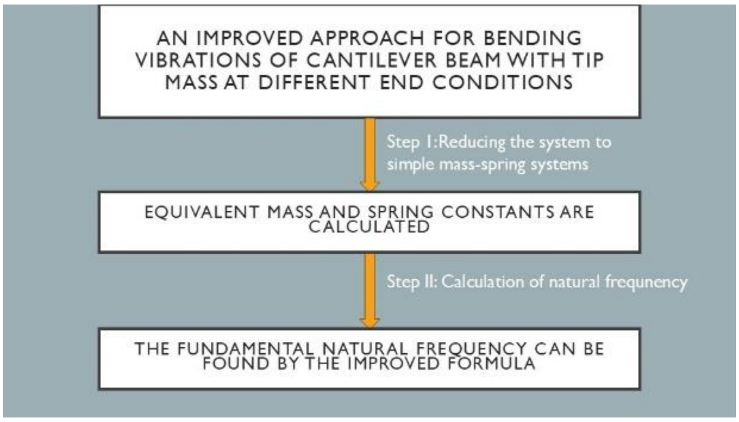

Figure: Graphical abstract of this work

Aim

The aim of this work is to obtain the first natural frequencies of the cantilever beam with tip mass at different end conditions and different tapers.

Design & Methodology

For this purpose, the whole system for each cases are reduced to single degree of freedom mass-spring system then the first natural frequencies are tried to obtain by selecting proper mass and spring constants.

Originality

In literature, there are some approaches to obtain the first natural frequnecy of the cantilever beam with and without tip mass and with some special cases but they can not be applied to all cases and their formulas are not explicit. In this work, a generalized formula is developed and this formula can be applied the all cases. For tapered beams, there is no this kind of approach.

Findings

The first natural frequnecies of the cantilever beam with tip mass at different end conditions and tapers which are height taper, width taper and double taper can be found easily by using this approach. The results are nearly identical to theoretical values so that the percentage of the error values are below the 7% for all cases.

Conclusion

The methods in literature are complex and can not be applied to other cases. A generalized and simple approach is developed to obtain the first natural frequencies. This approach can be applied to all cases easily. Since the calculation steps are simplified thus computational time is reduced.

Declaration of Ethical Standards

The authors of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

An Improved Approach for Bending Vibrations of a Cantilever Beam with Tip Mass at Different End Conditions

Araştırma Makalesi / Research Article

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ABSTRACT

Single dof lumped model is frequently utilized due to its simlicity, when studying bending vibration of cantilever beam with a tip mass. For this purpose, an equivalent single dof model is obtained with some assumptions. These assumptions effect the accuracy of the result. This study proposes a new formulation of proportion parameter η which is defined as a function of ratio of the tip mass to the beam mass and used to obtain the value of the first natural frequency. Depending on proposed formula, calculation of the first natural frequency with respect to the mass ratio was simplified and the results were improved. The new formulation of the proportion parameter was applied to the reference model studied in literature to verify its effectiveness. After it is verified, three complex new cases were considered and the corresponding equivalent spring constant and natural frequency parameter were obtained. Lastly, the first non-dimensional natural frequencies were obtained for three different tapered cases with and without tip mass consisting of only height taper, only width taper and double taper. One can see that the results are close to theoretical frequencies presented in the literature.

Keywords: Cantilever beam with tip mass, Natural frequency, Mass-spring system, Lateral Vibrations, Discrete vibration model.

Farklı Uç Koşullarına sahip Uç Kütleli bir Ankastre Kirişin Eğilme Titreşimleri için Geliştirilmiş Yeni Bir Yöntem

ÖZ

Tek serbestlik dereceli toplu model, uç kütlesi olan konsol kirişin eğilme titreşimini incelerken basitliği nedeniyle sıklıkla kullanılır. Bu amaçla, bazı varsayımlarla eşdeğer tek serbestlik dereceli bir model elde edilir. Bu varsayımlar sonucun doğruluğunu etkiler. Bu çalışma, uç kütlesinin kiriş kütlesine oranının bir fonksiyonu olarak tanımlanan ve ilk doğal frekans değerini elde etmek için kullanılan orantı parametresi η için yeni bir formülasyon önermektedir. Önerilen formüle bağlı olarak, ilk doğal frekansın kütle oranına göre hesaplanması basitleştirilmiş ve sonuçlar iyileştirilmiştir. Orantı parametresinin yeni formülasyonu, etkinliğini doğrulamak için literatürde incelenen referans modele uygulanmıştır. Doğrulandıktan sonra, üç yeni karmaşık durum ele alınmış ve karşılık gelen eşdeğer yay sabiti ve doğal frekans parametresi elde edilmiştir. Son olarak, yalnızca yükseklik konikliği, yalnızca genişlik konikliği ve çift koniklikten oluşan uç kütlesi olan ve olmayan üç farklı konik durum için ilk boyutsuz doğal frekanslar elde edilmiştir. Sonuçların literatürde sunulan teorik frekanslarına yakın olduğu görülebilir.

Keywords: Uç kütleli ankastre kiriş, Doğal frekans, Kütle-yay sistemi, Eğilme titreşimleri, Ayrık titreşim modeli

1. INTRODUCTION

Discrete vibration model of cantilever beam with tip mass has been studied by researches for many years. Usually, because of simplycity, beams with or without tip mass can be described as one-degree-of-freedom mass spring systems with equivalent mass and spring constants. For calculating the natural frequency, the assestment of equivalent mass and spring constants are so critical for that kind of systems[1-9]. The number of studies of this subject are limited and they can not be found easily. In literature, the value of 33/140 is widely used for the equivalent mass parameter (m_{eq}) [10,11]. Another critical parameter for the equivalent spring constant (k_{eq}) is taken as 3 in Gürgöze's study [12]. These values are derived from the deflection formula, which is under the force concentrated at the tip of the beam and always taken as constant. But, in reality the both parameters should be changed by changing of the mass ratio [12]. Where, first mode function of the beam was modelled as linear combination of deflection formula under uniformly distributed force and force concentrated at the tip of the beam. The author defined a new parameter η which effects degree of proportion of this combination. In Gürgöze's another study [13], another spring was added to end of the cantilever beam with tip mass. The system was modelled 2 DOF spring

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and mass system. The frequency formula was derived by Lagrange multipliers method and solved. Continous beam carrying elastically mounted masses was investigated by Ercoli and Laura [14] numerically and experimentally.

Obtaining the natural frequency of tapered beams is critical in different areas especially in aerospace and aeronautics engineerings. In Mabie and Rogers study [15], only height tapered and only width tapered beams with tip mass were investigated. Analytical formulations by using bessel series and exact solutions for first three non-dimensional natural frequencies were obtained for these cases. For double taper beams, the taper ratios were assumed as same. The exact solutions for first five natural frequencies were found in Mabie and Rogers [16].

It is observed from the literature that models of reducing the cantilever beam with and without tip mass are ineffective and they can not applied to other types of end conditions. For tapered beams, there are no reducing procedures and formulations in literature. Hence, the main aim of the this study is to represent a new model to improve and simplify the equations for deriving the first natural frequency of the cantilever beam with and without tip mass and for different end conditions and different type of tapered beams.

The organisation of this paper is as follows. The section 2 is about mathematical backround of the study. In section 3, the first natural frequency results are presented and section 4 includes discussion and suggestion about improving the results.

2. MATERIAL and METHOD

In this section, the model studied in [17] was considered as Case 1 first. The improved approach in this study was applied to this case. Then the results was compared to each other. By this way, the improved approach was validated. After the validation, Case 2 and Case 3 were considered. Their parameters and the first natural frequencies were obtained by the improved approach.

2.1. Case 1

The frequency equation of the first fundamental bending mode of the cantilever beam with the tip mass, shown in figure 1, is derived as [17],

$$1 + cos\beta_1 cosh\beta_1 + \beta_M\beta_1(cos\beta_1 sinh\beta_1 - sin\beta_1 cosh\beta_1) = 0 \qquad (1)$$

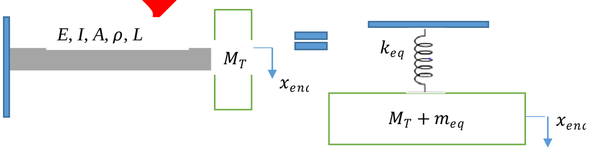

Figure 1. Cantilever Beam with tip Mass and Its Equivalent System [17]

Here,

$$\beta_M = \frac{M_{tip}}{m_b} = \frac{M_{tip}}{\rho AL}$$

Where, ρ, A, L presents density, cross section and length of the beam respectively. The β_1^2 parameter is the non-dimensional first natural frequency of the cantilever beam. The fundamental mode shape function of the cantilever beam with tip mass can be found as [17]

$$w(x) = \left(cosh\left(\frac{\beta_1}{L}x\right) - cos\left(\frac{\beta_1}{L}x\right)\right) - a_1\left(sinh\left(\frac{\beta_1}{L}x\right) - sin\left(\frac{\beta_1}{L}x\right)\right) \qquad (2)$$

Where,

$$a_1 = \frac{(sinh\beta_1 - sin(\beta_1)) + \beta_1\beta_M(cosh\beta_1 - cos(\beta_1))}{(cosh\beta_1 + cos(\beta_1)) + \beta_1\beta_M(sinh\beta_1 - sin(\beta_1))} \qquad (3)$$

Therefore, total bending deflection of the beam can be expressed as

$$w(x,t) = w_t(t)w(x) \qquad (4)$$

If the displacement of the tip of the beam is assumed as unity, Eq.4 can be normalized as follows

$$w(x,t) = w_t(t)\frac{w(x)}{w(L)} = w_t(t)\widehat{w}(x) \qquad (5)$$

The equivalent mass and stiffness of the cantilever beam with tip mass can be obtained as [17]

$$m_{eq} = \rho A\int_0^L \widehat{w}(x)^2 dx = \widehat{m}_{eq}\rho AL \qquad (6)$$

$$k_{eq} = EI\int_0^L \left(\frac{d^2\widehat{w}(x)}{dx^2}\right)^2 dx = \widehat{k}_{eq}\frac{EI}{L^3} \qquad (7)$$

Where, E, I are modulus of elasticity of the beam and moment of inertia of the beam respectively. Using Eq. (6) and Eq. (7) the first natural frequency can be determined as below.

$$\omega_1 = \sqrt{\frac{\widehat{k}_{eq}}{\widehat{m}_{eq}+\beta_M}}\sqrt{\frac{EI}{\rho AL^4}} = \beta_1^2\sqrt{\frac{EI}{\rho AL^4}} \qquad (8)$$

Here, β_1^2 can be defined as undimensional natural frequency. The important point of this approach is to determine $\widehat{w}(x)$. For the first mode of deflection, it can be modelled as linear combination of the deflection for distributed force $\widehat{w}_d$ and for concentrated tip force $\widehat{w}_t$. In Kim's study, η was defined as an interpolation parameter and its formula was not given. Moreover, when β_M is zero, the deflection terms due to the tip force should be zero. But in [17] when β_M is zero, the deflection terms which comes from the tip force is not zero. In this study, the formula of the first mode deflection is changed and the parameter η is redefined as follows.

$$w_1(x,t) = w_t(t)\widehat{w}_c(x) = w_t(t)[\eta\widehat{w}_d + (1-\eta)\widehat{w}_t] \qquad (9)$$

Where,

$$\widehat{w}_d = \frac{1}{3L^4}x^2(6L^2 - 4Lx + x^2) \qquad (10)$$

$$\widehat{w}_t = \frac{1}{2L^3}x^2(3L - x) \qquad (11)$$

$$\eta = \frac{1}{1+\beta_M}$$
(12)

For a given β_M, $\widehat{m}_{eq}$ and $\hat{k}_{eq}$ can be determined from the Eq. (6) and (7) by substituting the $\widehat{w}_c(x)$ term into these equations instead of $\widehat{w}(x)$. Then the undimensional first natural frequency can be obtained from Eq. (8).

2.2. Case 2

Now, if the end condition is changed as in figure 2, the springs behave as parallel connected springs. The general formula of the system remains same except for total k_{eqt} term. Its formula can be obtained as

$$k_{eqt} = k_{eq} + k_e$$
(13)

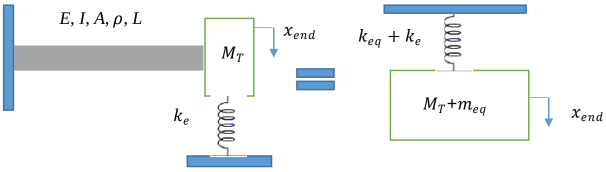

Figure 2. Cantilever Beam with tip mass and spring and Its Eqivalent System

Where, k_e is end-spring constant. If k_e term is normalized as below.

$$\hat{k}_{eqe} = k_e \frac{L^3}{3EI}$$
(14)

As a result, the fundemantal natural frequency of that system is becomes as

$$\omega_1 = \sqrt{\frac{\hat{k}_{eqt}}{\widehat{m}_{eq}+\beta_M}} \qquad (15)$$

Here,

$$\hat{k}_{eqt} = \hat{k}_{eq} + \hat{k}_{eqe}$$

2.3. Case 3

Then, if we change the end condition again such as cantilever beam with conected mass-spring attachment as shown in figure 3. The equivalent system is assumed as two masses-two springs system.

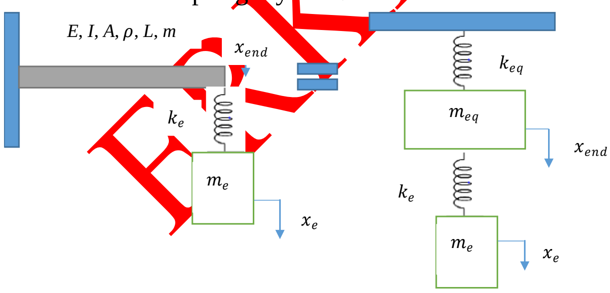

Figure 3. Cantilever beam with connected mass-spring system and Its Equivalent System

Here, the equivalent system's equation of motion is as follows [10].

$$\begin{bmatrix} m_{eq} & 0 \\ 0 & m_e \end{bmatrix}\begin{bmatrix} \ddot{x}_{end} \\ \ddot{x}_e \end{bmatrix} + \begin{bmatrix} k_{eq}+k_e & -k_e \\ -k_e & k_e \end{bmatrix}\begin{bmatrix} x_{end} \\ x_e \end{bmatrix} = 0$$
(16)

Where, x_{end} and x_e are deflection at the end of the beam and deflection of connected mass respectively.

Then, the first nondimensional natural frequency can be obtained from solution of eigen value of $K - \omega^2 M$ matrix wrt $\hat{k}_{eqe}$, β_{Me}, $\hat{k}_{eq}$. Here, K, M, ω are stiffness matrix, mass matrix and natural frequency respectively and $\beta_{Me} = \frac{m_e}{m_b}$. As a result, the nondimensional mass and stiffness matrices can be obtained as below.

$$M = \begin{bmatrix} \widehat{m}_{eq} & 0 \\ 0 & \beta_{Me} \end{bmatrix}, K = \begin{bmatrix} \hat{k}_{eq}+\hat{k}_{ege} & -\hat{k}_{ege} \\ -\hat{k}_{ege} & \hat{k}_{ege} \end{bmatrix}$$
(17)

2.4. Case 4

Next, a general model will be solved. This model's first bending natural frequency can be obtained exactly from the equation below. [12]

$$\left(\hat{k}_{ege} - \beta_{Me}\alpha^4\right)\left[\left(sin(\alpha) + sinh(\alpha)\right)r - \left(cos(\alpha) - cosh(\alpha)\right)g\right] + \hat{k}_{ege}\beta_{Me}\alpha^4\left[\left(sin(\alpha) + sinh(\alpha)\right)\left(cos(\alpha) - cosh(\alpha)\right) - \left(sin(\alpha) - sinh(\alpha)\right)\left(cos(\alpha) + cosh(\alpha)\right)\right] = 0 \qquad (18)$$

Here,

$$r = \beta_{Me}\alpha^4\left(cos(\alpha) - cosh(\alpha)\right) + \alpha^3\left(sin(\alpha) - sinh(\alpha)\right) \qquad (19)$$

$$g = \beta_{Me}\alpha^4\left(sin(\alpha) - sinh(\alpha)\right) + \alpha^3\left(cos(\alpha) + cosh(\alpha)\right) \qquad (20)$$

This system's first natural frequency can be calculated approximately by modelling system as two masses-two springs system as in figure 4. As a result, approximate natural frequency can be obtained by solving eigen-value of $K - \omega^2 M$ matrix as in case 3.

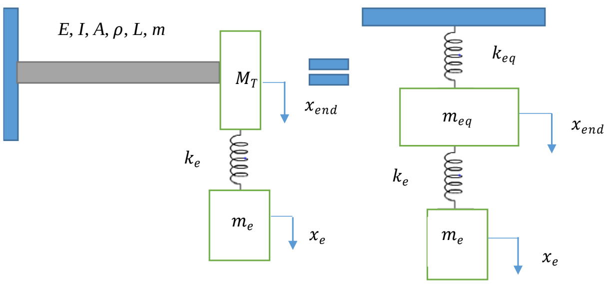

Figure 4.Cantilever Beam with Tip Mass and Mass-Spring System and Its Equivalent System

The equivalent system's equation of motion can be found as

$$\begin{bmatrix} m_{eq}+M_T & 0 \\ 0 & m_e \end{bmatrix}\begin{bmatrix} \ddot{x}_{end} \\ \ddot{x}_e \end{bmatrix} + \begin{bmatrix} k_{eq}+k_e & -k_e \\ -k_e & k_e \end{bmatrix}\begin{bmatrix} x_{end} \\ x_e \end{bmatrix} = 0 \qquad (21)$$

Then, the nondimensional mass and stiffness matrices can be found as

$$M = \begin{bmatrix} \widehat{m}_{eq}+\beta_M & 0 \\ 0 & \beta_{Me} \end{bmatrix}, K = \begin{bmatrix} \hat{k}_{eq}+\hat{k}_{ege} & -\hat{k}_{ege} \\ -\hat{k}_{ege} & \hat{k}_{ege} \end{bmatrix}$$
(22)

2.5. Case 5

Then, the beam is evaluated as a linearly tapered cantilever beam. Firstly, this case contains only height tapered, only width tapered and double tapered cantilever beam with and without tip mass. By using similar formulation, the first natural frequencies of all states can be found.

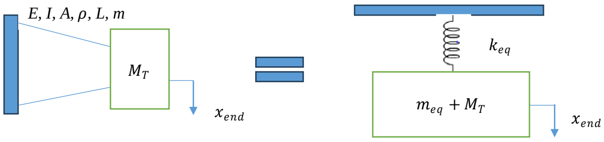

Figure 5. Linearly Tapered Cantilever Beam with Tip Mass and Its Equivalent Mass-Spring System

Eq. (6) and (7) should be modified as

$$\widehat{m}_{eq} = \frac{\int_0^L A(x)\widehat{w}(x)^2 dx}{A(L)}$$

(23)

$$\widehat{k}_{eq} = \frac{\int_0^L I(x)\left(\frac{d^2\widehat{w}(x)}{dL^2}\right)^2 dx}{I(L)}$$

(24)

Here, $A(x) = h(x)b(x)$, $I(x) = \frac{1}{12}b(x)h(x)^3$ and $w(x)$ can be obtained by integrating two times the Eq. (25).

$$\frac{d^2w(x)}{dx^2} = \frac{M}{EI(x)}$$

(25)

Where, M is bending moment derived from the distributed load effect. In this case, $\hat{b}(\zeta) = 1$ when the beam has only height taper, $\hat{h}(\zeta) = 1$ when the beam has only width taper.

$$\hat{h}(x) = \frac{h(x)}{H} = 1 - (1 - n)\frac{x}{L}$$

(26)

$$\hat{b}(x) = \frac{b(x)}{B} = 1 - (1 - a)\frac{x}{L}$$

(27)

Here, $n = \frac{h}{H}$ is height taper ratio. h and H are the heights at the right end and the left end of the beam respectively. $a = \frac{b}{B}$ is width taper ratio. b and B are the widths at the right end and the left end of the beam respectively. Lastly, $\hat{h}(x)$ and $\hat{b}(x)$ are non-dimensional heigth and width of the beam respectively. If the tip mass is added to the beam, the equivalent mass is obtained as follow.

$$\widehat{m}_{eq} = \frac{\int_0^L A(x)\widehat{w}(x)^2 dx + w_2\beta_M}{A(L)}$$

(28)

Here, $w_2 = \int_0^L A(x)dx$. The $\hat{k}_{eq}$ formulation is same. The non-dimensional deflection curves can be found by integrating Eq. (25) separetely and the total deflection curve can be obtained by using the Eq. (9).

3. RESULTS AND DISCUSSION

So far, the nondimensional first natural frequency has been obtained for all the cases approximately. The advantage of our approach is to calculate this frequency easily with some small errors. In Kim's study [17], η value's formulation is not clear and it is seemed that the values are limited to those which are given in table. In present study, new η value is formulated with respect to β_M and formulation of beam deflection for the first mode i changed. For instance, η and $(1 - \eta)$ terms are replaced. These evolutions simplify the calculation's process and provide the adaptation to other cases easily.

3.1. Case 1

In table 1, one can see the equivalent system parameters of present study and Kim's work [17]. For small β_M values, there are small differences between natural frequency values of present study and reference [17]. But as β_M values gets bigger, the natural frequency values are converged to each other. One can see that these errors are small and acceptable. The main advantage of this approach is easy adaptation to the other cases. This means that computational time and difficulty of solving the equations are decreased. Table 2 presents natural frequency values of exact solution, our solution and errors between them. For small β_M values, there are some errors but these errors are less than 1%. This means that the error values are small and acceptable.

3.2. Case 2

From table 3, comparison of natural frequency values for case 2 can be seen. In each cell, first figure is the natural frequency which is obtained from our approach, the second one is exact natural frequency which is listed in Gürgöze's study [12] and third one, which is written in bold style, is error percentage. Again, when β_M value is small, the percentage error is relatively large but less than 1%. Moreover, as k_{ege} gets bigger, the percentage error gets lower except for β_M value is zero. It can be seen that the error percentage is acceptable and our approach is succesfull.

3.3. Case 3

In table 4, comparison of natural frequency between our approach and exact values can be seen. For each cell, arrangement of the values are same with each cell in the case 2. One can see that there is a rise of error percentage. But, maximum error percentage still less than 5% almost 3.17%. In spite of this increasing, our approach is still successfull.

3.4. Case 4

This system is the general case of case 3. Because, if β_M is taken as zero in case 4, the system behaves as in case 3. Here, β_M value is begun to vary from 0.5. This is because of that the maximum error in table 2 is seen as 0.678719 when β_M value is 0.5. It will be checked whether the increasing of the error percentage still proceeds or not. Ideal values comes from the solution of Eq. (18).

From table 5, for each cell, arrangement of the values are same with each cell in the case 2 and case 3. One can see that the increasing of the error percentages do not proceed and the error percentages decrease. The maximum error percentage is almost 1.5%. In table 6, the β_M value will be increased and it will be checked whether this decreasing trend proceeds or not.

From table 6, one can see that the error percentages are decreasing and when β_M value is increased our approach is converged to ideal values which comes from the solution of Eq. (18).

3.5. Case 5

Reducing the continous tapered beam with tip mass to SDOF mass-spring system has not studied in literature yet. It is one of contributions of present study to the literature. For all tapered cases, there is an assumption that the taper is linear. One can see comparison of the first non-dimensional natural frequencies between our results and [15] results from the table 7. It can also be seen that although the percentage of error values are increased, their maximum value is 3.06228% and it is still less than 5% hence it is plausible. The arrangement of the Tables 7, 8 and 9 is that the first column is taper ratio, the second, third, fourth and fifth columns are mass ratios, exact natural frequencies, our results and error percentages between exact results and our results respectively.

From table 8, one can see non-dimensional first natural frequencies variations wrt the taper and mass ratios and error percentages between exact values in [16] and present study. It can be seen that the value of the error percentages are still low and our formulation is succesfull. For double taper case, width and heigth taper ratios are assumed to equal to each other.

From Table 9, one can see the non-dimensional fundamental natural frequencies variations wrt mass and tapers ratios. The maximum error percentage is 6.38100 which is still plausible. When the mass ratio is equals to zero, the error percentages are decreased while the taper ratios are increased but it can be seen that when the mass ratio is not equal to zero, the percentages of error are increased while the taper ratios are increased.

Table 1. The equivalent system parameters for varying β_M

	Our Results				Reference [17]			
β_M	η	m_{eq}	k_{eq}	ω_1	η	m_{eq}	k_{eq}	ω_1
0	1	0.256790	3.2	3.530090	0.339577	0.249445	3.087232	3.518008
0.001	0.999001	0.256768	3.199601	3.523167	0.342358	0.249386	3.086499	3.510977
0.01	0.990099	0.256573	3.196059	3.462575	0.366397	0.248874	3.080291	3.449467
0.1	0.909091	0.254805	3.165259	2.986840	0.537003	0.245264	3.042873	2.968700
0.5	0.666667	0.249579	3.088889	2.029984	0.790717	0.239986	3.008760	2.016424
1	0.5	0.246043	3.05	1.564528	0.876033	0.238236	3.003074	1.557332
5	0.166667	0.239110	3.005556	0.757414	0.970941	0.236303	3.000169	0.756938
10	0.090909	0.237561	3.001653	0.541479	0.985153	0.236015	3.000044	0.541375
25	0.038462	0.236494	3.000296	0.344800	0.993982	0.235836	3.000007	0.344788
50	0.019608	0.236111	3.000077	0.244376	0.996978	0.235775	3.000002	0.244374
100	0.009901	0.235915	3.000020	0.173002	0.998486	0.235745	3.000000	0.173001
500	0.001996	0.235755	3.000001	0.077441	0.999697	0.235720	3.000000	0.077441
1000	0.000999	0.235735	3.000000	0.054766	0.999848	0.235717	3.000000	0.054766
5000	0.0002	0.235718	3.000000	0.024494	0.999910	0.235716	3.000000	0.024494
10000	0.00001	0.235716	3.000000	0.017320	0.999605	0.235722	3.000000	0.017320

Table 2. Comparison of natural frequency values between exact values and our values

β_M	ω_{1exact} [17]	ω_1	% errors
0	3.516015	3.530090	0.400311
0.001	3.509003	3.523167	0.403647
0.01	3.447658	3.462575	0.432671
0.1	2.967838	2.986840	0.640264
0.5	2.016299	2.029984	0.678719
1	1.557298	1.564528	0.464266
5	0.756937	0.757414	0.063017
10	0.541375	0.541479	0.019210
25	0.344788	0.344800	0.003480
50	0.244374	0.244376	0.000818
100	0.173001	0.173002	0.000578
500	0.077441	0.077441	0
1000	0.054766	0.054766	0
5000	0.024494	0.024494	0
10000	0.017320	0.017320	0

Table 3. Comparison of natural frequency values for case 2

	$\hat{k}_{ege}$						
β_M	0	0.5	1	1.5	2	2.5	3
0	3.530090 3.516015 **0.400311**	3.795873 3.789758 **0.161364**	4.044227 4.0401 **0.102151**	4.278187 4.274680 **0.082034**	4.500000 4.494824 **0.115154**	4.711381 4.702566 **0.187457**	4.913680 4.899582 **0.287734**
0.2	2.631853 2.612749 **0.731185**	2.833727 2.820048 **0.485045**	3.022147 3.012550 **0.318556**	3.199489 3.192940 **0.205105**	3.367505 3.363152 **0.129416**	3.527527 3.524706 **0.080039**	3.680599 3.678724 **0.050969**
0.4	2.183572 2.167991 **0.718674**	2.352879 2.340900 **0.511726**	2.510983 2.501680 **0.371871**	2.659617 2.652501 **0.268282**	2.800372 2.795016 **0.191643**	2.934384 2.930430 **0.134915**	3.062537 3.059666 **0.093845**
0.6	1.904445 1.892468 **0.632880**	2.053303 2.043699 **0.469933**	2.192076 2.184366 **0.352963**	2.322572 2.316423 **0.265447**	2.446116 2.441250 **0.199324**	2.563714 2.560000 **0.145078**	2.676148 2.673160 **0.111793**
0.8	1.709878 1.700651 **0.542573**	1.844218 1.836675 **0.410663**	1.969415 1.963221 **0.315486**	2.087116 2.082047 **0.243463**	2.198524 2.194398 **0.188761**	2.304553 2.301198 **0.145794**	2.405914 2.403213 **0.112389**
1	1.564528 1.557304 **0.463858**	1.687904 1.681924 **0.355565**	1.802856 1.797906 **0.275359**	1.910906 1.906774 **0.216683**	2.013164 2.009760 **0.169380**	2.110474 2.107636 **0.134647**	2.203490 2.201158 **0.105945**
4	0.842299 0.841549 **0.089077**	0.909614 0.908972 **0.070678**	0.972280 0.971723 **0.057344**	1.031144 1.030651 **0.047800**	1.086825 1.086389 **0.040106**	1.139789 1.139407 **0.033543**	1.190399 1.190063 **0.028250**
8	0.603711 0.603543 **0.027836**	0.652044 0.651895 **0.022893**	0.697034 0.696908 **0.018118**	0.739291 0.739170 **0.016361**	0.779259 0.779159 **0.012797**	0.817276 0.817180 **0.011767**	0.853601 0.853517 **0.009807**
12	0.495227 0.495151 **0.015349**	0.534891 0.534829 **0.011603**	0.571810 0.571763 **0.008251**	0.606486 0.606436 **0.008247**	0.639284 0.639248 **0.005598**	0.670480 0.670433 **0.007010**	0.700287 0.700251 **0.005144**

Table 4. Comparison of natural frequency values for case 3

	$\hat{k}_{ege}$					
β_{Me}	0.5	1	5	10	50	100
0.5	0.925379 0·921122 **0.462154**	1.215135 1·205409 **0.806863**	1.794126 1·764572 **1.674854**	1.918361 1·883690 **1.840589**	2.027779 1·988822 **1.958798**	2.041991 2·002508 **1.971678**
1	0.656012 0·653037 **0.455564**	0.866288 0·859420 **0.799144**	1.331198 1·307155 **1.839338**	1.449604 1·419384 **2.129093**	1.563988 1·527404 **2.395175**	1.579669 1·542201 **2.429515**
5	0.293947 0·292628 **0.450743**	0.389788 0·386724 **0.792296**	0.618744 0·606746 **1.977434**	0.686179 0·670001 **2.414623**	0.758506 0·736996 **2.918605**	0.769146 0·746678 **3.009061**
10	0.207901 0·206969 **0.450309**	0.275825 0·273659 **0.791596**	0.439620 0·431021 **1.995030**	0.488761 0·477054 **2.454020**	0.542284 0·526491 **2.999671**	0.550547 0·533782 **3.140795**
50	0.092994 0·092578 **0.449351**	0.123425 0·122457 **0.790482**	0.197357 0·193471 **2.008570**	0.219870 0·214535 **2.486774**	0.244701 0·237417 **3.068020**	0.248431 0·240821 **3.160023**
100	0.065758 0·065464 **0.449102**	0.087281 0·086596 **0.791030**	0.139619 0·136867 **2.010711**	0.155585 0·151805 **2.490037**	0.173225 0·168056 **3.075760**	0.175878 0·170475 **3.169380**

Table 5. Comparison of natural frequency values for case 4 when $\beta_M = 0.5$

	$\hat{k}_{ege}$					
β_{Me}	0.5	1	5	10	50	100
0.5	0.911814 0.909421 **0.263134**	1.161898 1.156186 **0.494038**	1.489559 1.476420 **0.889923**	1.532018 1.517907 **0.929635**	1.564409 1.549611 **0.954949**	1.568340 1.553467 **0.957407**
1	0.650830 0.649317 **0.233014**	0.847552 0.843964 **0.425137**	1.198010 1.187080 **0.920747**	1.262134 1.249484 **1.012418**	1.315319 1.301264 **1.080104**	1.322019 1.307793 **1.087673**
5	0.292940 0.292315 **0.213810**	0.386919 0.385471 **0.375644**	0.600420 0.594945 **0.920253**	0.658120 0.650935 **1.103797**	0.716313 0.707146 **1.296338**	0.724523 0.715059 **1.323527**
10	0.207294 0.206854 **0.212710**	0.274229 0.273217 **0.370402**	0.430729 0.426814 **0.917261**	0.475370 0.470124 **1.115876**	0.522220 0.515319 **1.339170**	0.529013 0.521852 **1.372228**
50	0.092759 0.092564 **0.210665**	0.122862 0.122409 **0.370071**	0.194855 0.193087 **0.915649**	0.216306 0.213896 **1.126716**	0.239623 0.236369 **1.376661**	0.243090 0.239690 **1.418499**
100	0.065596 0.065452 **0.220009**	0.086896 0.086579 **0.366140**	0.137981 0.136725 **0.918632**	0.153285 0.151576 **1.127487**	0.169998 0.167677 **1.384209**	0.172492 0.170069 **1.424716**

Table 6. Comparison of natural frequency values for case 4 when $\beta_M = 5$

	$\hat{k}_{ege}$					
β_{Me}	0.5	1	5	10	50	100
0.5	0.695785 0.695374 **0.059105**	0.713153 0.712697 **0.063982**	0.721942 0.721466 **0.065977**	0.722825 0.722345 **0.066450**	0.723503 0.723026 **0.065972**	0.723586 0.723109 **0.065965**
1	0.591876 0.591647 **0.038706**	0.655593 0.655204 **0.059371**	0.688316 0.687854 **0.067165**	0.691292 0.690826 **0.067455**	0.693527 0.693054 **0.068249**	0.693798 0.693323 **0.068511**
5	0.289306 0.289251 **0.019015**	0.372796 0.372675 **0.032468**	0.502053 0.501716 **0.067169**	0.522053 0.521675 **0.072459**	0.537891 0.537482 **0.076096**	0.539845 0.539427 **0.077490**
10	0.205879 0.205841 **0.018461**	0.269114 0.269034 **0.029736**	0.391121 0.390869 **0.064472**	0.416374 0.416072 **0.072583**	0.438389 0.438031 **0.081729**	0.441237 0.440871 **0.083017**
50	0.092494 0.092480 **0.015138**	0.122097 0.122062 **0.028674**	0.189829 0.189709 **0.063255**	0.208514 0.208358 **0.074871**	0.227693 0.227490 **0.089235**	0.230432 0.230222 **0.091216**
100	0.065439 0.065423 **0.024456**	0.086480 0.086459 **0.024289**	0.135614 0.135533 **0.059764**	0.149698 0.149582 **0.077549**	0.164576 0.164427 **0.090618**	0.166745 0.166594 **0.090452**

Table 7. Comparison of natural frequency values for case of the height tapered beam

n	β_M	ω_{1exact} [15]	ω_1	Error %
1.2	0	4.30907	4.3267	0.40914
	0.2	3.13864	3.1654	0.85260
	0.4	2.58502	2.6061	0.81541
	0.6	2.24640	2.2633	0.75232
	0.8	2.01359	2.0266	0.46186
	1.0	1.84072	1.8509	0.55304
	2.0	1.36359	1.3674	0.27941
	3.0	1.13186	1.1337	0.16256
	4.0	0.98849	0.9896	0.11229
	5.0	0.88870	0.8894	0.07877
	10.0	0.63497	0.6351	0.02047
2.0	0	7.6474	7.6776	0.39491
	0.2	5.1998	5.2692	1.33467
	0.4	4.1786	4.2358	1.36888
	0.6	3.5888	3.6323	1.21210
	0.8	3.1937	3.2268	1.03642
	1.0	2.9053	2.9311	0.88803
	2.0	2.1290	2.1383	0.43682
	3.0	1.7600	1.7645	0.25568
	4.0	1.5339	1.5364	0.16298
	5.0	1.3772	1.3788	0.11618
	10.0	0.9813	0.9817	0.04076
5.0	0	21.4630	21.5108	0.22271
	0.2	12.347	12.7211	3.02989
	0.4	9.4864	9.7769	3.06228
	0.6	7.9840	8.1964	2.66032
	0.8	7.0246	7.1818	2.23785
	1.0	6.3423	6.4636	1.91256
	2.0	4.5745	4.6166	0.93255
	3.0	3.7605	3.7802	0.52387
	4.0	3.2674	3.2787	0.34584
	5.0	2.9289	2.9355	0.22534
	10.0	2.0794	2.0807	0.06252

Table 8. Comparison of natural frequency values for case of the width tapered beam

a	β_M	ω_{1exact} [15]	ω_1	Error %
1.2	0	3.7168	3.7307	0.37398
	0.2	2.7202	2.7412	0.77200
	0.4	2.2440	2.2612	0.76649
	0.6	1.9527	1.9660	0.68111
	0.8	1.7517	1.7617	0.57087
	1.0	1.6017	1.6098	0.50571
	2.0	1.1879	1.1908	0.24413
	3.0	0.9862	0.9878	0.16224
	4.0	0.8616	0.8624	0.09285
2.0	0	4.3152	4.3273	0.28040
	0.2	3.0088	3.0354	0.88407
	0.4	2.4395	2.4617	0.91002
	0.6	2.1045	2.1214	0.80304
	0.8	1.8777	1.8907	0.69234
	1.0	1.7111	1.7213	0.59611
	2.0	1.2589	1.2625	0.28596
	3.0	1.0422	1.0440	0.17271
	4.0	0.9090	0.9100	0.11001
5.0	0	5.3977	5.4029	0.09634
	0.2	3.4210	3.4584	1.09325
	0.4	2.6948	2.7267	1.18376
	0.6	2.2934	2.3175	1.05084
	0.8	2.0301	2.0484	0.90143
	1.0	1.8406	1.8548	0.77149
	2.0	1.3386	1.3438	0.38847
	3.0	1.1038	1.1062	0.21743
	4.0	0.9606	0.9619	0.13533

Table 9. Comparison of natural frequency values for case of the double tapered beam

$n = a$	β_M	ω_{1exact} [16]	ω_1	Error %
1.2	0	4.54997	4.5673	0.38088
	0.2	3.25843	3.2879	0.90442
	0.4	2.66608	2.6904	0.91220
	0.6	2.31036	2.3290	0.80680
	0.8	2.06697	2.0813	0.69329
	1.0	1.88708	1.8982	0.58927
	2.0	1.39383	1.3979	0.29200
	5.0	0.90655	0.9073	0.08273
	10.0	0.64725	0.6474	0.02317
2.0	0	9.25030	9.2725	0.23999
	0.2	5.72408	5.8261	1.78230
	0.4	4.47965	4.5615	1.82715
	0.6	3.80088	3.8617	1.60016
	0.8	3.35894	3.4046	1.35936
	1.0	3.04210	3.0772	1.15381
	2.0	2.20735	2.2198	0.56402
	5.0	1.41865	1.4208	0.15155
	10.0	1.00862	1.0091	0.04759
5.0	0	30.9820	31.0152	0.10716
	0.2	12.5670	13.3689	6.38100
	0.4	9.19450	9.7211	5.72734
	0.6	7.59518	7.9534	4.71641
	0.8	6.61634	6.8725	3.87163
	1.0	5.93881	6.1295	3.21091
	2.0	4.22934	4.2927	1.49812
	5.0	2.68636	2.6967	0.38491
	10.0	1.90227	1.9045	0.11723

4. CONCLUSION

This work proposed a improved and simplified approach by comparison of Kim's method [17] to reduce cantilever beam with different boundary conditions to simply mass-spring system. For reducing mass-spring systems, assesstment of equivalent mass and spring parameters are very important and should be selected properly. Firstly, we improved and simplified the formulation of η in [17] because the calculations of η are not clear and simple in addition that, this model can not be applied to other types of end conditions. Our approach can be applied to other end conditions. Then, first bending natural frequencies were obtained and compared to [17] with and without tip mass. After that, for different boundary conditions in [12], the systems are reduced to simple mass-spring system, which is also a new approach and slightly different from the models in [12], and the first bending natural frequencies are presented and compared to results in [12]. The exact values can be found by solving the Eq. 18, whose formulation is given in [12]. The solutions of the Eq. (18) was not taken from the [12], It is solved by using graphical approach in this work. For the last case, the beam is assumed as linearly tapered cantilever beam and the first non-dimensional natural frequencies are obtained by our approach and compared to results in [15] and [16]. It is the one of contributions of our work. Reducing the continous systems to discrete systems made the solution simple but sometimes caused some errors. In present study, error percanteges are under 7% and while tip mass is increased, error percentages are decreased and nearly identical to exact values. This work can be extended to beams with other the types of boundary conditions.

DECLARATION OF ETHICAL STANDARDS

The authors of this article declare that the materials and methods they used in their work do not require ethics committee approval or legal-specific permission.

AUTHORS' CONTRIBUTIONS

Orçun BİÇER: Calculations and Verification
Mustafa YILDIZ: Calculations and Verification
Sadettin ORHAN: Verification and Writing

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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