Araştırma Makalesi / Research Article

Comparison of the $r - (k, d)$ class estimators to some estimators by the mean square error matrix criteria

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Abstract

The ordinary least squares, the principal components regression and the Liu-type estimators are special cases of the $r-(k,d)$ class estimators, for regression models with multicollinearity. In this article we derived conditions for the superiority of the $r-(k,d)$ class estimator over other estimators such as ordinary least squares, principal component and Liu-type estimator based on the mean square error matrix ($MSEM$) criterion. Finally, a numerical example and a Monte Carlo simulation are also given to show the theoretical results.

Keywords Liu-type estimators; $r-(k,d)$ class estimator, Principal Component Regression, Mean square error matrix, Multicollinearity

Öz

$r-(k,d)$ sınıf tahmin edicisinin, ortalama karesel hata kriterine göre bazı yanlı tahmin ediciler ile karşılaştırılması

En küçük kareler, temel bileşenler ve Liu-tipi tahmin ediciler, çok değişkenli regresyon modelleri için $r-(k,d)$ sınıf tahmin edicilerin özel durumlarındadır. Bu makalede $r-(k,d)$ sınıf tahmin edicisini, en küçük kareler, temel bileşenler ve Liu-tipi tahmin ediciler ile Matris Hata kareler ortalaması kriterine göre karşılaştırılmıştır. Son olarak teorik sonuçları göstermek için sayısal bir örnek ve bir Monte Carlo simülasyonu verilmektedir.

Anahtar sözcükler Liu-tipi tahmin edici; $r-(k,d)$ Sınıf Tahmin Edici; Temel bileşenler regresyonu; Matris Hata Kareler Ortalaması; Çoklu İç İlişki

1. Introduction

It is a common knowledge that variance of regression coefficients could be inflated, thus making the ordinary least squares (OLS) estimation unsuitable in the presence of multicollinearity. Through the years a lot of research has been dedicated to overcome this hurdle. While early studies have proposed new estimators to tackle the multicollinearity (e.g. the principal component regression (PCR) estimator and the
ordinary ridge regression (ORR), in later years researchers combined various estimators to obtain better results (e.g. the \((r - k)\) class estimator, which combines the ORR and PCR; the Liu estimator (LE), which combines Stein [1] and ORR estimators; the \((r - d)\) class estimator, which combines the LE and PCR estimators, and the Liu-type estimator (LTE), which combines the ORR and LE estimators.

One of the latest additions to such efforts is the \(r - (k, d)\) estimator, which is a combination of the Liu-type and PCR estimators. Inan [2], who proposed this approach, have also determined that this estimator was superior to OLS, ORR, Liu-type, PCR and \((r - k)\) class estimators based on the strong mean square error (SMSE) criterion. However, they did not compare the \(r - (k, d)\) estimator with other estimators by using the MSEM criterion, which is often used to measure the performance of an estimator and is stronger than the SMSE criterion. We choose MSEM criterion instead of SMSE criteria. Since the diagonal elements of the MSEM are neglected when calculating the SMSE, we have set the MSEM as the criterion. But calculation of MSEM is more difficult than SMSE. In this study, we aimed to compare the \(r - (k, d)\) estimator with OLS, PCR and Liu-type estimators based on the MSEM criteria. Furthermore, necessary and sufficient conditions for the \(r - (k, d)\) class estimator to dominate the OLS, PCR and Liu-type estimators’ sense are derived. A Monte Carlo simulation and and real data have been conducted to show the performance of estimators.

The rest of the article is organized as follows. In the second section, we described the model and defined the \(r - (k, d)\) estimator. In the third section, we compared the \(r - (k, d)\) estimator with OLS, PCR and Liu-type estimators based on the MSEM criterion. Furthermore, a numerical example that illustrates some of the theoretical results is given in section 4 and the results of the Monte Carlo simulation are presented in section 5. The paper is wrapped up with some concluding remarks in section 6.

2. Model

We considered the linear regression model given as

\[
y = X\beta + \varepsilon
\]

Where \(y\) is \((n \times 1)\) observable random vector, \(X\) is a \((n \times p)\) matrix of non-stochastics variables of rank \(p\); \(\beta\) is \((p \times 1)\) vector of unknown parameters associated with \(X\), and \(\varepsilon\) is a \((n \times 1)\) vector of error terms. Let \(T = [t_1, t_2, ..., t_p]\) be an orthogonal matrix that consist of the eigenvalues of \(X'X\).

\[T'X'XT = \Lambda\]

where \(\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_p)\) ranked according to the magnitude. Further let \(T_r = [t_{r+1}, t_{r+2}, ..., t_p]\) be remaining columns of \(T\) having deleted \(r\) columns where \(r \leq p\). Obviously, \(T_r'X'XT_r = \Lambda_r = diag(\lambda_1, \lambda_2, ..., \lambda_r)\) and \(T_p'X'XT_p' = \Lambda_p = diag(\lambda_1, \lambda_2, ..., \lambda_p)\) where \(T_p' = [t_1, t_2, ..., t_p]\). The \(r - (k, d)\) class estimator for \(\beta\) as proposed by Inan [2] is

\[
\hat{\beta}_{r}(k,d) = T_r(\hat{\beta}_{r}T_r'X'XT_r + kI)^{-1}(T_r'X'XT_r - dI)T_r'\hat{\beta}_{PCR}k \geq 0, -\infty < d < +\infty
\]

(2)

where \(\hat{\beta}_{PCR} = T_r(\hat{\beta}_{r}T_r'X'XT_r)^{-1}T_r'X'y\) is the PCR estimator. The \(r - (k, d)\) class estimator is a general estimator, which includes the OLS, PCR, \(r - k\) and Liu-type estimators as special cases:

\[
\hat{\beta}_{p}(0,0) = \hat{\beta} = (X'X)^{-1}Xy \text{ is the OLS estimator,}
\]
\( \hat{\beta}_p(k, -k) = T_r(T'_rX'Xr)^{-1} T'_rX'y \) is the PCR estimator,
\( \hat{\beta}_p(k, 0) = \hat{\beta}_p(k) = T_r(T'_rX'Xr + kI)^{-1} T'_rX'y \) is the \( r - k \) class estimator, and
\( \hat{\beta}_p(k, d) = \hat{\beta}(k, d) = (X'X + kI)^{-1} (X'X - dI) \hat{\beta} \) is the Liu-type estimator. \( PCR, r - k, \) and Liu-type estimators are all methods that address the multicollinearity. When proposing \( r - (k, d) \) class estimator, Inan [2] showed that the MSE was increased by jointly utilizing Liu-type and PCR estimators. In other words, \( r - (k, d) \) class estimator has smaller MSE than other estimators. Therefore, \( r - (k, d) \) class estimator can be used as an alternative estimator for combating multicollinearity.

3. Superiority of the proposed estimators

In this section, we have discussed the superiority of the \( r - (k, d) \) class estimator over some other estimators based on the MSEM criterion. We first listed some notations and definitions that will be needed in the following discussions. For a matrix \( A, A', A'', \) \( \text{rank}(A), \text{R}(A), \text{and} \ N(A) \) stand for the transpose, Moore-Penrose inverse, rank, column space and null space of \( M \), respectively.

If we denote the covariance matrix of an estimator of \( \hat{\beta} \) by \( \text{Cov}(\hat{\beta}) \)
Let \( \text{Cov}(\hat{\beta}) = E[(\hat{\beta} - E(\hat{\beta}))(\hat{\beta} - E(\hat{\beta}))'] \)
and the bias by \( \text{Bias}(\hat{\beta}) \)
\[ \text{Bias}(\hat{\beta}) = E(\hat{\beta}) - \beta \]
then
\[ \text{MSEM}(\hat{\beta}) = \text{Cov}(\hat{\beta}) = \text{Bias}(\hat{\beta}) = \text{Bias}(\hat{\beta}) \]

**Lemma 1.** Let \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) be two competing estimators. The estimator \( \hat{\beta}_2 \) is said to be better than \( \hat{\beta}_1 \) by the \( \text{MSEM} \) criterion if and if only \( \text{MSEM}(\hat{\beta}_1) - \text{MSEM}(\hat{\beta}_2) \geq 0 \)

From (1) and (2), we find that,
\[ \hat{\beta}_p(k, d) = T_rS_r(k)^{-1}S_r(-d)T'_r\hat{\beta}_{PCR} \]
\[ \text{Bias}(\hat{\beta}_p(k, d)) = E(\hat{\beta}_p(k, d)) - \beta = T_rS_r(k)^{-1}S_r(-d)T'_r\beta - \beta \]
\[ = \left( -k - d \right) T_rS_r(k)^{-1}T'_r - T_{p-r}T_{p-r}^' \]
\[ = \text{Cov}(\hat{\beta}_p(k, d)) = T_rS_r(k)^{-1}S_r(-d)T'_r \text{Cov}(\hat{\beta}_{PCR})T'_rT_rS_r(-d)S_r(k)^{-1} \]
\[ = \sigma^2 T_rS_r(k)^{-1}S_r(-d)A_r^{-1}S_r(-d)S_r(k)^{-1}T_r' \]
Where \( \text{Cov}(\hat{\beta}_{PCR}) = \sigma^2 T_rA_r^{-1}T_r' \), \( S_r(k) = (A_r + kI) \). Thus \( \text{MSEM} (\hat{\beta}_p(k, d)) \) is
\[ \text{MSEM}(\hat{\beta}_p(k, d)) = \sigma^2 T_rS_r(k)^{-1}S_r(-d)A_r^{-1}S_r(-d)S_r(k)^{-1}T_r' \]
\[ + \left[ (-k - d)T_rS_r(k)^{-1}T'_r - T_{p-r}T_{p-r}^' \right] \beta \]
\[ \times \beta' \left[ (-k - d)T_rS_r(k)^{-1}T'_r - T_{p-r}T_{p-r}^' \right] \]

\[ (3) \]

3.1. Comparison of the \( r - (k, d) \) class estimator to the OLS estimator
In this section, we have compared the \( r-(k,d) \) class estimator with the OLS estimator. In the following theorem, we have obtained a necessary and sufficient condition for the \( r-(k,d) \) class estimator to be superior to the OLS estimator in the MSEM sense.

**Theorem 1.** The \( r-(k,d) \) class estimator \( \hat{\beta},(k,d) \) is superior to the OLS estimator \( \hat{\beta} \) if and only if 
\[ \beta^\prime T_r(k+d)^2[2(k+d)L_r+(k^2-d^2)A_r^{-1}]^{-1} T_r^\prime \beta + \beta^\prime T_{p-r}A_{p-r}^{-1} T_{p-r}^\prime \beta \leq \sigma^2 \]

**Proof.** We first compared the \( r-(k,d) \) class estimator with the OLS estimator of \( \beta \) by the matrix mean squares criterion. We obtain the MSEM matrix of the OLS estimator \( \left( \hat{\beta} \right) \), by substituting \( k = 0 \) and \( r = p \) in (3). We thus have
\[ MSEM(\hat{\beta}) = \sigma^2 S^{-1} \text{ where } S = X^\prime X. \]
If we write \( A = \begin{pmatrix} A_r & 0 \\ 0 & A_{p-r} \end{pmatrix} \) and
\[ T = \begin{pmatrix} T_r & 0 \\ 0 & T_{p-r} \end{pmatrix} \text{ then } \]
\[ MSEM(\hat{\beta}) = \sigma^2 \left( T_r A_r^{-1} T_r^\prime + T_{p-r} A_{p-r}^{-1} T_{p-r}^\prime \right) \]
By using (4),
\[ MSEM(\hat{\beta}) - MSEM(\hat{\beta},(k,d)) \]
can be expressed as
\[ MSEM(\hat{\beta}) - MSEM(\hat{\beta},(k,d)) = \sigma^2 T_r S_r(k)^{-1}[2(k+d)L_r+(k^2-d^2)A_r^{-1}] S_r(k)^{-1} T_r^\prime \\
+ T_{p-r} \left[ \sigma^2 A_{p-r}^{-1} - T_{p-r}^\prime \beta \beta^\prime T_{p-r} \right] T_{p-r}^\prime \\
- (k+d)^2 T_r S_r(k)^{-1} T_r^\prime \beta \beta^\prime T_r S_r(k)^{-1} T_r^\prime \\
- (k+d) T_{p-r} S_r(k)^{-1} T_{p-r}^\prime \beta \beta^\prime T_{p-r} T_{p-r}^\prime \\
- (k+d) T_{p-r} T_{p-r}^\prime \beta \beta^\prime T_r S_r(k)^{-1} T_r^\prime \]
(5)
Let us define \( S^*(k,d) \) and \( A^*(k,d)^{-1} \) as follows
\[ S^*(k,d) = \begin{pmatrix} \frac{1}{k+d} S_r(k) & 0 \\ 0 & I_{p-r} \end{pmatrix} \]
and
\[ A^*(k,d)^{-1} = \begin{pmatrix} \frac{1}{(k+d)^2} \left[ 2(k+d)L_r+(k^2-d^2)A_r^{-1} \right] & 0 \\ 0 & A_{p-r}^{-1} \end{pmatrix} \]
Then the expression in (5) equals
\[ MSEM(\hat{\beta}) - MSEM(\hat{\beta},(k,d)) = T S^*(k,d)^{-1} \left[ \sigma^2 A^*(k,d)^{-1} - T^\prime \beta \beta^\prime T \right] S^*(k,d)^{-1} T^\prime \]
(6)
So that
\[ MSEM(\hat{\beta}) - MSEM(\hat{\beta},(k,d)) \]
is a ndd matrix if and only if \( \sigma^2 A^*(k,d)^{-1} - T^\prime \beta \beta^\prime T \) is ndd matrix. \( \sigma^2 A^*(k,d)^{-1} - T^\prime \beta \beta^\prime T \) is ndd if and only if \( \beta^\prime T A^*(k,d) T^\prime \beta \leq \sigma^2 \) (Rao and Toutenburg, [3]). By substituting the expression for \( A^*(k+d) \) and simplifying this condition, we conclude the proof.

3.2. **Comparison of the \( r-(k,d) \) class estimator to the PCR estimator**

In this section we have compared the \( r-(k,d) \) class estimator with the PCR estimator. There are many different theorems that compare two biased estimators in terms of MSEM such as Trenkler [4], Trenkler and Toutenburg [5] and etc. We used the theorem by Baksalary and Trenkler [6].

**Theorem 2.** Let \( \varphi_{n \times p} \) be a set of \( n \times p \) complex matrices and let \( H_n \) be subset of \( \varphi_{n \times n} \) consisting of Hermitian matrices. Further, given \( L^* \in \varphi_{n \times n} \) the symbols \( L^* \), \( \mathbb{R}(L) \) and \( \mathbb{R}(L) \) stand for conjugate transpose, the range, and the set of all generalized inverses with respect to \( L \). Now, let \( A \in H_n, \alpha_1 \) and \( \alpha_2 \in \varphi_{n \times 1} \) be linearly independent,
\( f_{ij} = a_i^*A a_j \) for \( i, j = 1, 2 \) for \( A^* \in \varphi(A) \) and \( s = [a_1^*(I_n - AA^*)^*(I_n - AA^*)a_1] [/a_1^*(I_n - AA^*)^*(I_n - AA^*)a_1] \) provided that \( a_1 \notin \Re(A) \) then \( A + a_1 a_1^* - a_2 a_2^* \) is non-negative definite if and only if any one of the following sets of conditions hold.

a) \( A \) is non-negative definite, \( a_1 \in \Re(A), a_2 \in \Re(A) \) and \( (f_{11} + 1)(f_{22} - 1) \leq |f_{12}|^2 \)

b) \( A \) is non-negative definite, \( a_1 \in \Re(A), a_2 \in \Re(A) \) and \( (a_2 - a_1)^*A^*(a_2 - a_1) \leq 1 - |s|^2 \)

c) \( A = U \delta U^* - \delta U v^*, a_1 \in \Re(A), a_2 \in \Re(A), v^* a_1 \neq 0 \) and \( (f_{11} + 1) \leq 0 \)

\((f_{22} - 1) \leq 0 (f_{11} + 1)(f_{22} - 1) \geq |f_{12}|^2 \)

where \((U, v)\) (where \(U\) is possibly absent) is a subunitary matrix, \( \Delta \) is a positive definite diagonal matrix (occurring when \( U \) is present) and \( \delta \) is a positive scalar. Furthermore, the conditions (a)-(c) are all independent of the choice of \( A^* \in \varphi(A) \). In the following theorem, we have obtained a necessary and sufficient condition for the \( r - (k, d) \) class estimator to be superior to the PCR estimator in the MSEM sense.

**Theorem 3.** \( \text{MSEM}(\hat{\beta}_r) - \text{MSEM}(\hat{\beta}_r(k, d)) \) is nnd if and only if \( \beta \in N(W_k), \) where \( N(W_k) \) is the null space of \( W_k = \left[ \frac{k + d}{\sigma^2} \right] [2(k + d)I_r + (k^2 - d^2)A_r^{-1}] \)

**Proof.** The expression for \( \text{MSEM}(\hat{\beta}_r) \), the MSEM matrix of \( \hat{\beta}_r \), the PCR estimator of \( \beta \), can easily be obtained from (5) by putting \( d = -k \). Hence, we have

\[
\text{MSEM}(\hat{\beta}_r) = \sigma^2 T_r A_r^{-1} T_r' + (T_r T_r' - I_p) \beta \beta'(T_r T_r' - I_p)
\]

\[ \text{MSEM}(\hat{\beta}_r) - \text{MSEM}(\hat{\beta}_r(k, d)) \] can be expressed as

\[
\text{MSEM}(\hat{\beta}_r) - \text{MSEM}(\hat{\beta}_r(k, d)) = \sigma^2 T_r S_r(k)^{-1} [2(k + d)I_r + (k^2 - d^2)A_r^{-1}] S_r(k)^{-1} T_r'
\]

\[ + (T_r T_r' - I_p) \beta \beta'(T_r T_r' - I_p) + \left[-(k + d)T_r S_r(k)^{-1} T_r' - T_{p-r} T_{p-r}'\right] \beta \]

\[ \times \beta' [-(k + d)T_r S_r(k)^{-1} T_r' - T_{p-r} T_{p-r}'] \]

Using matrix to compare the \( r - (k, d) \) class estimator with PCR estimator based on the MSEM criterion would have been very difficult, so we used a theorem proposed by Baksalary and Trenkler [6].

We noted from (8) that in our case \( A = \sigma^2 T_r B T_r' \), where

\[ B = S_r(k)^{-1} [2(k + d)I_r + (k^2 - d^2)A_r^{-1}] S_r(k)^{-1} \]

Let us now consider the Moore-Penrose inverse \( A^+ \) of \( A \), which is given by,

\[ A^+ = \frac{1}{\sigma^2} T_r B^{-1} T_r' \]

And also \( A A^+ = T_r T_r' \). Hence \( a_1 = (T_r T_r' - I_p) \beta \) and \( a_2 = \left[-(k + d)T_r S_r(k)^{-1} T_r' - T_{p-r} T_{p-r}'\right] \beta \). \( \sigma^2 \in \Re(A) \)

if and only if \( a_1 = 0 \).

We cannot apply part (a) and (c) of the theorem, but part (b) is applicable. From the part (b) of theorem we can obtain the definition of \( s \):

\[ s = [a_1^*(I_n - AA^*)^*(I_n - AA^*)a_1] [/a_1^*(I_n - AA^*)^*(I_n - AA^*)a_1] = 1 \]

In addition, \( \sigma^2 - \alpha^2 = A \eta_1 \), where

\[ \eta_1 = \frac{(k + d)}{\sigma^2} T_r S_r(k) [2(k + d)I_r + (k^2 - d^2)A_r^{-1}]^{-1} T_r' \beta \]

which indicates \( \alpha^2 \in \Re(A) \). Thus, using this theorem a necessary and sufficient condition for superiority of the \( r - (k, d) \) class estimator over the PCR estimator is obtained as
Then, $r - (k, d)$ class estimator dominates the PCR estimator if and only if 

$$
\eta_1' A \eta_1 = \frac{(k + d)^2}{\sigma} \beta' T_r [2(k + d)I_r + (k^2 - d^2)A_r^{-1}]^{-1} T_r' \beta \leq 0.
$$

Under the assumption $(k - d + 2)(k + d) \geq 0$, we have

$$(a_2 - a_1)'A^{-1}(a_2 - a_1) \geq 0.$$ 

Accordingly, since $-\infty < d < \infty$ if we let $Q_1 = W_1 \beta$ where

$$W_1 = \left(\frac{k + d}{\sigma}\right)[2(k + d)I_r + (k^2 - d^2)A_r^{-1}]^{-1} T_r'.$$

MSEM$(\hat{\beta}_r) -$ MSEM$(\hat{\beta}_r(k, d))$ is nd and if and only if $Q_1 = 0$. $Q_1 = W_1 \beta = 0$ means that $\beta \in N(W_1)$, where $N(W_1)$ is the null space of $W_1$.

### 3.3. Comparison of the $r - (k, d)$ class estimator to the Liu-type estimator

In this section we have compared the $r - (k, d)$ class estimator with the Liu-type estimator. In the following theorem, we have obtained a necessary and sufficient condition for the $r - (k, d)$ class estimator to be superior to the Liu-type estimator in the MSEM sense.

**Theorem 4.** The $r - (k, d)$ class estimator dominates the Liu-type estimator if and only if $\beta \in N(W_2)$, where $N(W_2)$ is the null space of $W_2 = \left(\frac{1}{p}\right)(A_p - p)^{-\frac{1}{2}} T_{p-r}$. 

**Proof.** We obtain $MSEM(\hat{\beta}(k, d))$ by substituting $r = p$ in (3). Hence, we have

$$MSEM(\hat{\beta}(k, d)) = \sigma^2 T_S(k)^{-1} S(-d)A^{-1} S(-d)S(k)^{-1} T' + (k + d)^2 T_S(k)^{-1} T' \beta' TS(k)^{-1} T'$$

where $S(k) = (A + kI_p)$. In consequence of (3) and (9), it can be seen that

$$MSEM(\hat{\beta}_r(k, d)) = \sigma^2 T_{p-r} S_p(-r)^{-1} S_p(-r)A_{p-r}^{-1} S_{p-r}(-d)S_{p-r}(-d)S_{p-r}(-r) T_{p-r}$$

$$+ (k + d)^2 T_S(k)^{-1} T' \beta' TS(k)^{-1} T'$$

$$- [(k - d)T_{p-r} T_{r'} T_{p-r}] \beta'$$

$$\times \beta$$. As in the previous case, we use the theorem by Baksalarly and Trenkler (6).

Letting $A = \sigma^2 T_{p-r} B T_{p-r}$, where $B = S_{p-r}(-d)A_{p-r}^{-1} S_{p-r}(-d)S_{p-r}(-d)S_{p-r}(-r)$

$$a_4 = (k + d) T_S(k)^{-1} T' \beta$$

and $a_2 = [(k + d)T_{p-r} T_S(k)^{-1} T_{p-r}] \beta$. The Moore-Penrose inverse of $A$, which is given by,

$$A^+ = \frac{1}{\sigma^2} T_{p-r} B^{-1} T_{p-r}$$

and $A^+ = T_{p-r} T_{p-r}'$. $a_1 \in \mathcal{H}(A)$ and $a_2 \in \mathcal{H}(A): a_1$ because $s = -1$, where

$$\eta_2 = -\frac{1}{\sigma^2} T_{p-r} S_{p-r}(-d)A_{p-r}^{-1} S_{p-r} T_{p-r} \beta.$$ 

also $(a_2 + a_1)'A(a_2 + a_1) = \eta_2' A \eta_2$ then the $r - (k, d)$ class estimator dominates the Liu-type estimator if and only if

$$\eta_2' A \eta_2 = \frac{1}{\sigma^2} \beta' T_{p-r} A_{p-r} T_{p-r} \beta \leq 0$$

However, it is obvious that $\eta_2' A \eta_2$ is always $\geq 0$ so the condition turns out to be $\eta_2' A \eta_2 = 0$. If we let $Q_2 = W_2 \beta$ where
\[ W_2 = \left( \frac{1}{n} \right) (A_{p-r})^{T} \mathbf{T}_{p-r} = Q_2 \]

we obtain \( \eta_2' A \eta_2 = Q_2' Q_2 \)

Then \( MSEM(\beta_1, (k)) - MSEM(\beta, (k, d)) \) is nd if and only if \( Q_2 = 0; Q_2 = W_2 \beta = 0 \) means that \( \beta \in N(W_2) \). where \( N(W_2) \) is the null space of \( W_2 \).

### 3.4. Criteria of the \( r - (k, d) \) class estimator to dominate both the PCR and Liu-type estimator

In the following theorem, we have obtained a necessary and sufficient condition for the \( r - (k, d) \) class estimator to be superior over the PCR and Liu-type estimator in the MSEM sense.

**Theorem 5.** A necessary and sufficient condition for the \( r - (k, d) \) class estimator to dominate both the PCR and Liu-type estimators simultaneously is given by \( \beta \in N(W_3) \), where \( N(W_3) \) is the null space of \( W_3 = \left( \frac{1}{a} \right) A'(k, d)^{1/2} T' \)

**Proof.** If we define partitioned matrix \( W_3 = \left( \begin{array}{c} W_1 \\ W_2 \end{array} \right)_{p \times p} \) then \( W_3 = \left( \frac{1}{a} \right) A'(k, d)^{1/2} T' \)

Multiplying \( W_3 \) by \( \beta \) from the right, we get \( W_3 \beta = 0 \) which is the proof the theorem.

### 4. A numerical example

In order to illustrate our theoretical results, we use a widely popular Portland cement data analyzed in detail by Woods et al. [7] and Kaçıranlar et al. [8]. These data came from an experimental investigation of the heat created during the setting and hardening of Portland cements of varied composition and the dependence of this heat on the percentages of four compounds in the clinkers from which the cement was produced. The four compounds considered were tricalcium aluminate: 3CaO.Al\( _2 \)O\( _3 \), tricalcium silicate: 3CaO.SiO\( _2 \), tetracalcium aluminoferrite: 4CaO.Al\( _2 \)O\( _3 \).Fe\( _2 \)O\( _3 \), and \( \beta \)-dicalcium silicate: 2CaO.SiO\( _2 \), which are denoted by \( X_1, X_2, X_3, \) and \( X_4 \) respectively. The heat evolved after 180 days of curing, which is denoted by \( y \). The data set is given in Table 1.

<table>
<thead>
<tr>
<th>( y )</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( X_4 )</th>
</tr>
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<td>26</td>
<td>6</td>
<td>60</td>
</tr>
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<td>1</td>
<td>29</td>
<td>15</td>
<td>52</td>
</tr>
<tr>
<td>104.3</td>
<td>11</td>
<td>56</td>
<td>8</td>
<td>20</td>
</tr>
<tr>
<td>87.6</td>
<td>11</td>
<td>31</td>
<td>8</td>
<td>47</td>
</tr>
<tr>
<td>95.9</td>
<td>7</td>
<td>52</td>
<td>6</td>
<td>33</td>
</tr>
<tr>
<td>109.2</td>
<td>11</td>
<td>55</td>
<td>9</td>
<td>22</td>
</tr>
<tr>
<td>102.7</td>
<td>3</td>
<td>71</td>
<td>17</td>
<td>6</td>
</tr>
<tr>
<td>72.5</td>
<td>1</td>
<td>31</td>
<td>22</td>
<td>44</td>
</tr>
<tr>
<td>93.1</td>
<td>2</td>
<td>54</td>
<td>18</td>
<td>22</td>
</tr>
<tr>
<td>115.9</td>
<td>21</td>
<td>47</td>
<td>4</td>
<td>26</td>
</tr>
<tr>
<td>83.8</td>
<td>1</td>
<td>40</td>
<td>23</td>
<td>34</td>
</tr>
<tr>
<td>113.3</td>
<td>11</td>
<td>66</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>109.4</td>
<td>10</td>
<td>68</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

All the results below were computed by Matlab 7.13. We standardized the data so that \( X'X \) matrix is in the form of a correlation matrix. According to eigenvalues the standardized data are

\[ \lambda_1 = 2.2357, \lambda_2 = 1.5760, \lambda_3 = 0.1866, \lambda_4 = 0.001624 \]

And the condition number is
We can use the following formula to choose $k$ and optimal value of $d$ as given by Liu (9):

$$
\hat{k} = \frac{\lambda_1 - 100 \times \lambda_p}{99}
$$

$$
\hat{d}_{opt} = \frac{\sum_{i=1}^{p}(a_i^2 - kY_i^2)/(\lambda_i + k)^2}{\sum_{i=1}^{p}(a_i^2 + \sigma^2)/(\lambda_i + k)^2}
$$

Values of $d_{opt}$, $\hat{k}$ and MSE estimates were obtained by replacing the corresponding theoretical expressions in all unknown model parameters with their OLS estimates.

Numerical results of the comparison of $r - (k, d)$ class estimator with the $r - k$ estimator and PCR estimator are summarized in Table 2 and 3. The estimated MSE values for the PCR, $r - k$ and $r - (k, d)$ class estimator were obtained by replacing the corresponding theoretical MSE expressions in all unknown model parameters with their OLS.

Table 2. Values of estimates and MSE for $d_{opt}$ and various values of $k$

<table>
<thead>
<tr>
<th></th>
<th>$k = 0$</th>
<th>$k = 0.3$</th>
<th>$k = 0.5$</th>
<th>$k = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r - k$</td>
<td>0.1691</td>
<td>0.1691</td>
<td>0.1691</td>
<td>0.1691</td>
</tr>
</tbody>
</table>
| $r - (k, d)$ | 0.1510  | 0.1518    | 0.1521    | 0.1539  | 0.1599

Table 3. Values of estimates and MSE for $\hat{k}_{HK}$ and various values of $d$

<table>
<thead>
<tr>
<th></th>
<th>$d = -0.5$</th>
<th>$d = -0.3$</th>
<th>$d_{opt}$</th>
<th>$d = 0$</th>
<th>$d = 0.5$</th>
<th>$d = 0.7$</th>
<th>$d = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r - k$</td>
<td>0.1621</td>
<td>0.1621</td>
<td>0.1621</td>
<td>0.1621</td>
<td>0.1621</td>
<td>0.1621</td>
<td>0.1621</td>
</tr>
<tr>
<td>$r - (k, d)$</td>
<td>0.1517</td>
<td>0.1509</td>
<td>0.1518</td>
<td>0.1519</td>
<td>0.1598</td>
<td>0.1652</td>
<td>0.1757</td>
</tr>
</tbody>
</table>

We observed that under some conditions on $d$ and $k$, $r - (k, d)$ class estimator performed well compared to others. We can see that $r - (k, d)$ class estimator is better than the $r - k$ and PCR, except for big values of $d$ ($d = 0.70, 1$). We see that in practice we can choose small $d$ and regulate $k$. At the same time, we can choose big $k$ and regulate $d$. Therefore, we believe that our estimator is meaningful in practice.

5. Monte Carlo simulation

In order to further the MSE performances of the PCR, $r - k$ and $r - (k, d)$ we performed a Monte Carlo simulation study by considering different levels of multicollinearity. Following Liu [9] and Kibria [10], we obtain the explanatory variables while the response variables were generated using the following equations:

$$
x_{ij} = (1 - \gamma^2)^{1/2} z_{ij} + \gamma z_{ip}, \quad y_i = (1 - \gamma^2)^{1/2} z_{ij} + \gamma z_{ip} \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., p
$$

where $z_{ij}$ are independent standard normal pseudo-random numbers and $p$ is specified so that correlation between any two explanatory variables is given by $\gamma^2$. In this simulation, three different sets of correlations namely, $\gamma = 0.90, 0.95$ and $0.99$ were considered to show collinearity between the explanatory variables. By applying the variance inflation factors and condition indices it can easily be shown that the explanatory variables are...
weak, strong and severely collinear when \( y = 0.90, 0.95 \) and \( 0.99 \), respectively. In this experiment, we selected \( p = 4 \) for \( n = 30, 50, 100 \). Then, the experiment was replicated 1000 times by generating new error terms. Let us consider the PCR, \( r - k \) and \( r - (k,d) \) and compute their respective estimated MSE values with the different levels of multicollinearity. Based on the simulation results shown in Tables 4, we can see that with the increase of the levels of multicollinearity, the estimated MSE values of the PCR, \( r - k \) and \( r - (k,d) \) increase in general. For fixed \( k \) and \( d \) the estimated MSE of estimators increase with the increasing level of multicollinearity. We can see that \( r - (k,d) \) is much better than the competing estimators when the explanatory variables are severely collinear.

**Table 4.** MSE values for three estimator

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( n )</th>
<th>( PCR )</th>
<th>( r - k )</th>
<th>( (r - k,d) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90</td>
<td>30</td>
<td>0.1454</td>
<td>0.1304</td>
<td>0.1181</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.1860</td>
<td>0.1824</td>
<td>0.1818</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.0800</td>
<td>0.0809</td>
<td>0.0645</td>
</tr>
<tr>
<td>0.95</td>
<td>30</td>
<td>0.3462</td>
<td>0.4318</td>
<td>0.1627</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.1054</td>
<td>0.0821</td>
<td>0.0816</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.2587</td>
<td>0.2530</td>
<td>0.2530</td>
</tr>
<tr>
<td>0.99</td>
<td>30</td>
<td>0.1543</td>
<td>0.3523</td>
<td>0.1955</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>0.2721</td>
<td>0.1754</td>
<td>0.2542</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.2659</td>
<td>0.1695</td>
<td>0.1688</td>
</tr>
</tbody>
</table>

6. Conclusion

In this paper, we used the MSEM criterion to compare the \( r - (k,d) \) class estimator with the OLS, PCR and Liu-type estimators. We obtained necessary and sufficient conditions for the \( r - (k,d) \) class estimator that showed it being superior over the other three estimators. Furthermore, we saw that the conditions obtained on the comparisons of the \( r - (k,d) \) class estimator with OLS, PCR and Liu-type estimators depend on the unknown parameters. We constructed tests to decide whether or not these conditions hold in given situations. Moreover, we also mentioned that if we use unknown parameters for the comparisons, we should use unbiased estimates or priori information for those parameters in order to obtain practical results. Finally, we illustrated our findings with a numerical example and a Monte Carlo simulation. Both numerical example and simulation results which agrees with our theoretical findings.

References


