Sakarya University Journal of Science, 22 (6), 1774-1786, 2018.

SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ
SAKARYA UNIVERSITY JOURNAL OF SCIENCE

e-ISSN: 2147-835X
Dergi sayfası: http://www.saujs.sakarya.edu.tr

Received
02-11-2017
Accepted
21-05-2018

Doi
10.16984/saufenbilder.348912

Approximation By Three-Dimensional q-Bernstein-Chlodowsky Polynomials

Nazmiye Gönül Bilgin[1], Merve ÇETİNKAYA

ABSTRACT

In the present paper we introduce positive linear three-dimensional Bernstein-Chlodowsky polynomials on a non-tetrahedron domain and we get their q-analogue. We obtain aproximation properties for these positive linear operators and their generalizations in this work. The rate of convergence of these operators is calculated by means of the modulus of continuity.

Keywords: Bernstein-Chlodowsky Polynomials, q- Bernstein-Chlodowsky Polynomials, linear positive operators, modulus of continuity.

1. INTRODUCTION

In recent years, many generalizations of well-known linear positive operators, based on $q-$calculus were introduced and studied by several authors. In 1996, Philips by using the q-binomial coefficients and the q-binomial theorem introduced a generalization of the Bernstein operators called q-Bernstein Operators [1]. q-Bernstein-Chlodowsky polynomials defined by Karsli Gupta in the one-dimensional case [2]. Buyukyazici introduced the two-dimensional q-analogue of Bernstein-Chlodowsky polynomial in [3]. He give these polynomials on a domain $D_{ab} = [0,a] \times [0,b]$. In this paper we define three-dimensional Bernstein-Chlodowsky and q-Bernstein-Chlodowsky polynomials. Then we compute the rate of convergence of these operators by means of the modulus of continuity. The aim of this paper is to prove Korovkin type theorems and to give some examples of numerical solutions for the three-dimensional q- Bernstein-Chlodowsky polynomials.

Firstly, we give some notions about q-integers. Let $q > 0$. For each non-negative integer n, we define the q-integer $[n]_q$ as

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & if\ q \neq 1 \\ n, & if\ q = 1 \end{cases}$$

and the q-factorial $[n]_q!$ as

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n = 1,2,\ldots \\ 1 \qquad , & n = 0 \end{cases}$$

For integers n and k, with $0 \le k \le n$, q-binomial coefficients are then defined as follow

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

q-based Bernstein-Chlodowsky type polynomials for a function f of two variables as follows in [1].

Let (α_n) and (β_m) be increasing sequences of positive real numbers;

$\lim_{n\to\infty} \alpha_n = \lim_{m\to\infty} \beta_m = \infty$, $\lim_{n\to\infty} \frac{\alpha_n}{[n]_{q_n}} = 0$ and

[1] Zonguldak Bülent Ecevit University, Faculty of Sciences-Department of Mathematics, nazmiyegonulbilgin@hotmail.com

$$\lim_{m\to\infty}\frac{\beta_m}{[m]_{q_m}}=0.$$

For any $\alpha_n>0,\ \beta_m>0$ where

$(x,y)\in D_{\alpha_n\beta_n}=\{(x,y):0\le x\le\alpha_n,0\le y\le\beta_m\}.$

The two-dimensional q-Bernstein-Chlodowsky operators;

$$\breve{B}_{n,m}^{q_n,q_m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m\right)\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\breve{B}_{n}^{q_n}(f;x,y)=\sum_{k=0}^{n}f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,y\right)\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\left(\frac{x}{\alpha_n}\right)^k$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)$$

$$\breve{B}_{m}^{q_m}(f;x,y)=\sum_{j=0}^{m}f\left(x,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m\right)\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\left(\frac{y}{\beta_m}\right)^j\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

Buyukyazici gave approximation properties these operators in [3].

Theorem1.1. Let $e_{ij}:D_{ab}\to D_{ab}$,

$e_{ij}(x,y)=x^iy^j, i,j=0,1,2$ and for any $(x,y)\in D_{ab}$;

i. $\breve{B}_{n,m}^{q_n,q_m}(e_{00};x,y)=1$

ii. $\breve{B}_{n,m}^{q_n,q_m}(e_{10};x,y)=x$

iii. $\breve{B}_{n,m}^{q_n,q_m}(e_{01};x,y)=y$

iv. $\breve{B}_{n,m}^{q_n,q_m}(e_{20};x,y)=x^2+\frac{x(\alpha_n-x)}{[n]_{q_n}}$

v. $\breve{B}_{n,m}^{q_n,q_m}(e_{02};x,y)=y^2+\frac{y(\beta_m-y)}{[m]_{q_m}}$.

Theorem1.2. Let $f\in C(D_{ab})$, for any sufficiently large fixed positive real a and b, $(a\le\alpha_n,b\le\beta_m)$ then

$\breve{B}_{n,m}^{q_n,q_m}(f;x,y)$ linear positive operators sequence satisfy next equalities.

$$\lim_{n,m\to\infty}\left\|\breve{B}_{n,m}^{q_n,q_m}(e_{00};x,y)-1\right\|_{C(D_{ab})}=0\qquad(1)$$

$$\lim_{n,m\to\infty}\left\|\breve{B}_{n,m}^{q_n,q_m}(e_{10};x,y)-x\right\|_{C(D_{ab})}=0\qquad(2)$$

$$\lim_{n,m\to\infty}\left\|\breve{B}_{n,m}^{q_n,q_m}(e_{01};x,y)-y\right\|_{C(D_{ab})}=0\qquad(3)$$

$$\lim_{n,m\to\infty}\left\|\breve{B}_{n,m}^{q_n,q_m}(t^2+\tau^2;x,y)-(x^2+y^2)\right\|_{C(D_{ab})}=0\qquad(4)$$

Using Korovkin type theorem we get

$$\lim_{n,m\to\infty}\left\|\breve{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right\|_{C(D_{ab})}=0\qquad(5)$$

2. CONSTRUCTION OF OPERATORS

Definition 2.1.

Let $\{b_n\},\{c_m\},\{d_r\}$ be incresing sequences of real numbers and let them satisfy the next properties. Let

$$\lim_{n\to\infty}b_n=\lim_{m\to\infty}c_m=\lim_{r\to\infty}d_r=\infty\ ,$$

$$\lim_{n\to\infty}\left(\frac{b_n}{n}\right)=\lim_{m\to\infty}\left(\frac{c_m}{m}\right)=\lim_{r\to\infty}\left(\frac{d_r}{r}\right)=0$$

and for $b_n,c_m,d_r>0$;

$$\widetilde{D}_3:=D_{b_n,c_m,d_r}=\{(x,y,z):0\le x\le b_n,0\le y\le c_m,0\le z\le d_r\}$$

is defined.

We can introduce the Bernstein-Chlodowsky type polynomials for a function f of three variables same as [4], [5];

$$B_{n,m,r}(f;x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}f\left(\frac{k}{n}b_n,\frac{j}{m}c_m,\frac{l}{r}d_r\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k$$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j$$

$$\text{x}\left(1-\frac{y}{c_m}\right)^{m-j}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}\qquad(6)$$

Lemma 2.1. Let $B_{n,m,r}(f;x,y,z)$ defined in (6) and $B_{n,m,r}(f;x,y,z)$, $C_{\widetilde{D}_3}\to C_{\widetilde{D}_3}$ where

$e_{i_1,i_2,i_3}=x^{i_1}y^{i_2}z^{i_3}\ ,\ \ i_1+i_2+i_3\le 2$

for $i_1,i_2,i_3\in\{0,1,2\}$. We have the following equalities:

i. $B_{n,m,r}(e_{0,0,0};x,y,z)=1$

ii. $B_{n,m,r}(e_{1,0,0};x,y,z)=x$

iii. $B_{n,m,r}(e_{0,1,0};x,y,z)=y$

iv. $B_{n,m,r}(e_{0,0,1};x,y,z)=z$

v. for $g(x,y,z)=e_{2,0,0}(x,y,z)+e_{0,2,0}(x,y,z)+e_{0,0,2}(x,y,z)$

$$B_{n,m,r}(g;x,y,z)=x^2+\frac{x(b_n-x)}{n}+y^2+\frac{y(c_m-y)}{m}+z^2+\frac{z(d_r-z)}{r}.$$

Proof.

i) $B_{n,m,r}(e_{0,0,0};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\binom{n}{k}\left(\frac{x}{b_n}\right)^k$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\sum_{l=0}^{r}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\sum_{l=0}^{r}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=1$$

ii) $B_{n,m,r}(e_{1,0,0};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{k}{n}b_n\binom{n}{k}\left(\frac{x}{b_n}\right)^k$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{k}{n}b_n\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j$$

$$\text{x}\left(1-\frac{y}{c_m}\right)^{m-j}\sum_{l=0}^{r}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{k}{n}b_n\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=\sum_{k=0}^{n}\frac{k}{n}b_n\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=\sum_{k=0}^{n}\frac{k}{n}b_n\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=B_n(e_1,x)=x$$

iii) $B_{n,m,r}(e_{0,1,0};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{j}{m}c_m\binom{n}{k}\left(\frac{x}{b_n}\right)^k$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{j}{m}c_m\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j$$

$$\text{x}\left(1-\frac{y}{c_m}\right)^{m-j}\sum_{l=0}^{r}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\sum_{j=0}^{m}\frac{j}{m}c_m\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}B_m(e_1,y)$$

$$=y\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=y$$

iv) $B_{n,m,r}(e_{0,0,1};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{l}{r}d_r\binom{n}{k}\left(\frac{x}{b_n}\right)^k$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\sum_{l=0}^{r}\frac{l}{r}d_r\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}B_r(e_1,z)$$

$$=z\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=z\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=z$$

v) for

$$g(x,y,z):=e_{2,0,0}(x,y,z)+e_{0,2,0}(x,y,z)+e_{0,0,2}(x,y,z)$$

$$B_{n,m,r}(g;x,y,z)=x^2+\frac{x(b_n-x)}{n}+y^2+\frac{y(c_m-y)}{m}+z^2+\frac{z(d_r-z)}{r}$$

$$B_{n,m,r}(e_{2,0,0};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{k^2}{n^2}b_n^2\binom{n}{k}\left(\frac{x}{b_n}\right)^k$$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{k^2}{n^2}b_n^2\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\sum_{l=0}^{r}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\frac{k^2}{n^2}b_n^2\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=\sum_{k=0}^{n}\frac{k^2}{n^2}b_n^2\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$=B_n(e_2,x)=x^2+\frac{x(b_n-x)}{n}$$

$$B_{n,m,r}(e_{0,2,0};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{j^2}{m^2}c_m^2\binom{n}{k}\left(\frac{x}{b_n}\right)^k$$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{j^2}{m^2}c_m^2\binom{n}{k}\left(\frac{x}{b_n}\right)^k$$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\sum_{l=0}^{r}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\sum_{j=0}^{m}\frac{j^2}{m^2}c_m^2\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}B_m(e_2,y)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}y^2+\frac{y(c_m-y)}{m}$$

$$=y^2+\frac{y(c_m-y)}{m}$$

$$B_{n,m,r}(e_{0,0,2};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{l^2}{r^2}d_r^2\binom{n}{k}\left(\frac{x}{b_n}\right)^k$$

$$\text{x}\left(1-\frac{x}{b_n}\right)^{n-k}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\sum_{l=0}^{r}\frac{l^2}{r^2}d_r^2\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}B_r(e_2,z)$$

$$=\left(z^2+\frac{z(d_r-z)}{r}\right)\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=\left(z^2+\frac{z(d_r-z)}{r}\right)\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$=z^2+\frac{z(d_r-z)}{r}$$

We get

$$B_{n,m,r}(g;x,y,z)=x^2+\frac{x(b_n-x)}{n}+y^2$$

$$+\frac{y(c_m-y)}{m}+z^2+\frac{z(d_r-z)}{r}.$$

Theorem 2.1 Let $f\in C(\breve{D}_3)$, then for any sufficiently large fixed positive real numbers b,c and $(b<b_n, c<c_m,\ d<d_r)$ then we get

$$\lim_{n,m,r\to\infty}\max_{(x,y,z)\in\widetilde{D_3}}\left|B_{n,m,r}(f;x,y,z)-f(x,y,z)\right|=0.$$

Proof. Using Lemma 2.1

$$\left\|B_{n,m,r}(e_{0,0,0};x,y,z)-e_{0,0,0}(x,y,z)\right\|_{C(\widetilde{D_3})}=0$$

$$\left\|B_{n,m,r}\left(e_{1,0,0};x,y,z\right)-e_{1,0,0}(x,y,z)\right\|_{C(\widetilde{D_3})}=0$$

$$\left\|B_{n,m,r}\left(e_{0,1,0};x,y,z\right)-e_{0,1,0}(x,y,z)\right\|_{C(\widetilde{D_3})}=0$$

$$\left\|B_{n,m,r}\left(e_{0,0,1};x,y,z\right)-e_{0,0,1}(x,y,z)\right\|_{C(\widetilde{D_3})}=0$$

$$\left\|B_{n,m,r}\left(\left(e_{2,0,0};x,y,z\right)+\left(e_{0,2,0};x,y,z\right)+\left(e_{0,0,2};x,y,z\right)\right)-(x^2+y^2+z^2)\right\|_{C(\widetilde{D_3})}$$

$$\le b\frac{b_n}{n}+c\frac{c_m}{m}+d\frac{d_r}{r}.$$

The proof is completed using

$$\lim_{n\to\infty}\frac{b_n}{n}=\lim_{m\to\infty}\frac{c_m}{m}=\lim_{r\to\infty}\frac{d_r}{r}=0.$$

We can show the uniform approximation of the three dimensional Bernstein-Chlodowsky polynomials in next example.

Example2.1. The convergence of

$B_{n,m,r}(f;x,y,z)$ to $f(x,y,z)=x^6+y^6+z^6-\left(\frac{1}{6}\right)^{1/8}$, $b_n=\sqrt{n}$, $c_m=\sqrt{m}$, $d_r=\sqrt{r}$ is illustrated in Figure 2.1.

$n=m=r=50$ (yellow), $n=m=r=30$ (red), $n=m=r=15$(gray) , $n=m=r=10$ (magenta).

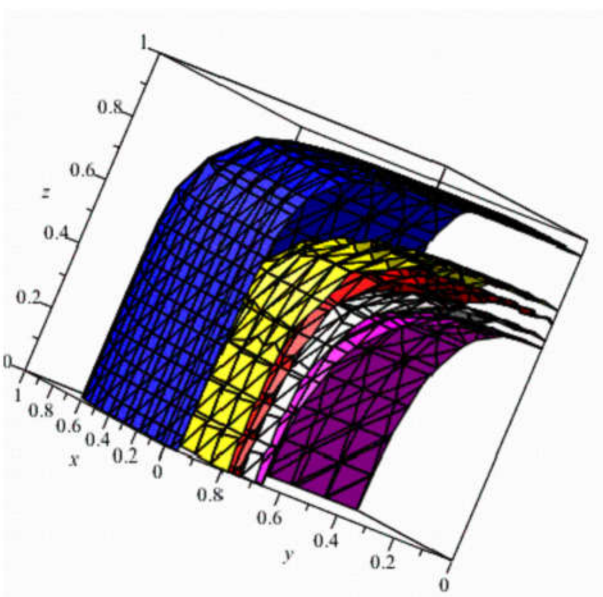

Figure 2.1. Approximation of $f(x,y,z)=x^6+y^6+z^6-\left(\frac{1}{6}\right)^{1/8}$ (blue) by $B_{n,m,r}(f;x,y,z)$.

Definition 2.2.

Let $\{\alpha_n\},\{\beta_m\},\{\gamma_r\}$ be increasing sequences of positive real numbers and let them satisfy the following properties:

$$\lim_{n\to\infty}\alpha_n=\lim_{m\to\infty}\beta_m=\lim_{r\to\infty}\gamma_r=\infty$$

that the sequences

$$\lim_{n\to\infty}\frac{\alpha_n}{[n]_{q_n}}=\lim_{m\to\infty}\frac{\beta_m}{[m]_{q_m}}=\lim_{r\to\infty}\frac{\gamma_r}{[r]_{q_r}}=0$$

where $q>0$, $\{q_n\}$ is a sequences of real numbers such that $0<q_n\le 1$ for all n and $\lim_{n\to\infty}q_n=1$.

For any $\alpha_n>0,\beta_m>0,\gamma_r>0$ we denote by $D_{\widetilde{3}}$:

$$D_{\widetilde{3}}:=D_{\alpha_n,\beta_m,\gamma_r}=\{(x,y,z):0\le x\le\alpha_n,\ 0\le y\le\beta_m,\quad 0\le z\le\gamma_r\}$$

We can define the q- Bernstein- Chlodowsky type polynomials for a function f of three variables as follows:

$$\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m,\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r\right)$$

$$\text{x}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}$$

$$\text{x}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)$$

$$\text{x}\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

Remark2.1. We use following notations next section.

$$\tilde{B}_n^{q_n}(f;x,y,z)=\sum_{k=0}^{n}f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,y,z\right)\begin{bmatrix}n\\k\end{bmatrix}_{q_n}$$

$$\text{x}\left(\frac{x}{\alpha_n}\right)^k\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)$$

$$\tilde{B}_m^{q_m}(f;x,y,z)=\sum_{j=0}^{m}f\left(x,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m,z\right)\begin{bmatrix}m\\j\end{bmatrix}_{q_m}$$

$$\text{x}\left(\frac{y}{\beta_m}\right)^j\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\tilde{B}_r^{q_r}(f;x,y,z)=\sum_{l=0}^{r}f\left(x,y,\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r\right)\begin{bmatrix}r\\l\end{bmatrix}_{q_r}$$

$$\text{x}\left(\frac{z}{\gamma_r}\right)^l \prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

Lemma2.2. Let $e_{i_1,i_2,i_3}: D_{\widetilde{3}} \to D_{\widetilde{3}}$, $e_{i_1,i_2,i_3}(x,y,z) = x^{i_1}y^{i_2}z^{i_3}$, $i_1, i_2, i_3 \in \{0,1,2\}$ for any $x,y,z \in D_{\widetilde{3}}$ then we get

i. $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{0,0,0};x,y,z) = 1$

ii. $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{1,0,0};x,y,z) = x$

iii. $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{0,1,0};x,y,z) = y$

iv. $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{0,0,1};x,y,z) = z$

v. for $g(x,y,z) \coloneqq e_{2,0,0}(x,y,z) + e_{0,2,0}(x,y,z) + e_{0,0,2}(x,y,z)$

$$\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(g;x,y,z) = x^2 + \frac{x(\alpha_n - x)}{[n]_{q_n}} + y^2 + \frac{y(\beta_m - y)}{[m]_{q_m}} + z^2 + \frac{z(\gamma_r - z)}{[r]_{q_r}}.$$

Proof: We calculate using definition of $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)$;

i) $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{0,0,0};x,y,z)$

$$= \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \begin{bmatrix} m \\ j \end{bmatrix}_{q_m} \begin{bmatrix} r \\ l \end{bmatrix}_{q_r} \left(\frac{x}{\alpha_n}\right)^k \left(\frac{y}{\beta_m}\right)^j \left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x} \prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right) \prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x} \prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$= \left\{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \left(\frac{x}{\alpha_n}\right)^k \prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\right\}$$

$$\text{x}\left\{\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_{q_m} \left(\frac{y}{\beta_m}\right)^j \prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)\right\}$$

$$\text{x}\left\{\sum_{l=0}^{r} \begin{bmatrix} r \\ l \end{bmatrix}_{q_r} \left(\frac{z}{\gamma_r}\right)^l \prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)\right\}$$

$$= \tilde{B}_n^{q_n}(e_0,x)\tilde{B}_m^{q_m}(e_0,y)\tilde{B}_r^{q_r}(e_0,z) = 1$$

ii) $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{1,0,0};x,y,z) = \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r} \frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n \begin{bmatrix} n \\ k \end{bmatrix}_{q_n}$

$$\text{x}\begin{bmatrix} m \\ j \end{bmatrix}_{q_m} \begin{bmatrix} r \\ l \end{bmatrix}_{q_r} \left(\frac{x}{\alpha_n}\right)^k \left(\frac{y}{\beta_m}\right)^j \left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x} \prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right) \prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x} \prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$= \left\{\sum_{k=0}^{n} \frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \left(\frac{x}{\alpha_n}\right)^k \prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\right\}$$

$$\text{x}\left\{\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_{q_m} \left(\frac{y}{\beta_m}\right)^j \prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)\right\}$$

$$\text{x}\left\{\sum_{l=0}^{r} \begin{bmatrix} r \\ l \end{bmatrix}_{q_r} \left(\frac{z}{\gamma_r}\right)^l \prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)\right\}$$

$$= \tilde{B}_n^{q_n}(e_1,x)\tilde{B}_m^{q_m}(e_0,y)\tilde{B}_r^{q_r}(e_0,z) = x$$

iii) $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{0,1,0};x,y,z) = \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r} \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m \begin{bmatrix} n \\ k \end{bmatrix}_{q_n}$

$$\text{x}\begin{bmatrix} m \\ j \end{bmatrix}_{q_m} \begin{bmatrix} r \\ l \end{bmatrix}_{q_r} \left(\frac{x}{\alpha_n}\right)^k \left(\frac{y}{\beta_m}\right)^j \left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x} \prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right) \prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x} \prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$= \left\{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \left(\frac{x}{\alpha_n}\right)^k \prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\right\}$$

$$\text{x}\left\{\sum_{j=0}^{m} \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m \begin{bmatrix} m \\ j \end{bmatrix}_{q_m} \left(\frac{y}{\beta_m}\right)^j \prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)\right\}$$

$$\text{x}\left\{\sum_{l=0}^{r} \begin{bmatrix} r \\ l \end{bmatrix}_{q_r} \left(\frac{z}{\gamma_r}\right)^l \prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)\right\}$$

$$= \tilde{B}_n^{q_n}(e_0,x)\tilde{B}_m^{q_m}(e_1,y)\tilde{B}_r^{q_r}(e_0,z) = y$$

iv) $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{0,0,1};x,y,z)$

$$= \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r} \frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \begin{bmatrix} m \\ j \end{bmatrix}_{q_m} \begin{bmatrix} r \\ l \end{bmatrix}_{q_r} \left(\frac{x}{\alpha_n}\right)^k \left(\frac{y}{\beta_m}\right)^j \left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x} \prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right) \prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x} \prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$= \left\{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} \left(\frac{x}{\alpha_n}\right)^k \prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\right\}$$

$$\text{x}\left\{\sum_{j=0}^{m}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\left(\frac{y}{\beta_m}\right)^j\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)\right\}$$

$$\text{x}\left\{\sum_{l=0}^{r}\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{z}{\gamma_r}\right)^l\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)\right\}$$

$$=z$$

v) $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{2,0,0};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{[k]_{q_n}^2}{[n]_{q_n}^2}\alpha_n^2\begin{bmatrix}n\\k\end{bmatrix}_{q_n}$

$$\text{x}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$=\left\{\sum_{k=0}^{n}\frac{[k]_{q_n}^2}{[n]_{q_n}^2}\alpha_n^2\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\left(\frac{x}{\alpha_n}\right)^k\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\right\}$$

$$\text{x}\left\{\sum_{j=0}^{m}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\left(\frac{y}{\beta_m}\right)^j\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)\right\}$$

$$\text{x}\left\{\sum_{l=0}^{r}\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{z}{\gamma_r}\right)^l\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)\right\}$$

$$=\tilde{B}_n^{q_n}(e_2,x)\tilde{B}_m^{q_m}(e_0,y)\tilde{B}_r^{q_r}(e_0,z)$$

$$=x^2+\frac{x(\alpha_n-x)}{[n]_{q_n}}$$

$$\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{0,2,0};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{[j]_{q_m}^2}{[m]_{q_m}^2}\beta_m^2\begin{bmatrix}n\\k\end{bmatrix}_{q_n}$$

$$\text{x}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$=\left\{\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\left(\frac{x}{\alpha_n}\right)^k\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\right\}$$

$$\text{x}\left\{\sum_{j=0}^{m}\frac{[j]_{q_m}^2}{[m]_{q_m}^2}\beta_m^2\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\left(\frac{y}{\beta_m}\right)^j\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)\right\}$$

$$\text{x}\left\{\sum_{l=0}^{r}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{z}{\gamma_r}\right)^l\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)\right\}$$

$$=\tilde{B}_n^{q_n}(e_0,x)\tilde{B}_m^{q_m}(e_2,y)\tilde{B}_r^{q_r}(e_0,z)$$

$$=y^2+\frac{y(\beta_m-y)}{[m]_{q_m}}$$

$$\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{0,0,2};x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\frac{[l]_{q_r}^2}{[r]_{q_r}^2}\gamma_r^2$$

$$\text{x}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$=\left\{\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\left(\frac{x}{\alpha_n}\right)^k\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\right\}$$

$$\text{x}\left\{\sum_{j=0}^{m}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\left(\frac{y}{\beta_m}\right)^j\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)\right\}$$

$$\text{x}\left\{\sum_{l=0}^{r}\frac{[l]_{q_r}^2}{[r]_{q_r}^2}\gamma_r^2\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{z}{\gamma_r}\right)^l\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)\right\}$$

$$=\tilde{B}_n^{q_n}(e_0,x)\tilde{B}_m^{q_m}(e_0,y)\tilde{B}_r^{q_r}(e_2,z)$$

$$=z^2+\frac{z(\gamma_r-z)}{[r]_{q_r}}$$

and so

$$\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(g;x,y,z)=x^2+\frac{x(\alpha_n-x)}{[n]_{q_n}}$$

$$+y^2+\frac{y(\beta_m-y)}{[m]_{q_m}}+z^2+\frac{z(\gamma_r-z)}{[r]_{q_r}}.$$

Theorem 2.2. Let $f \in C(D_{\tilde{3}})$, then for any sufficiently large fixed positive real numbers a,b,c and ($a \le \alpha_n$, $b \le \beta_m$, $c \le \gamma_r$) then we get

$$\lim_{n,m,r\to\infty} \max_{(x,y,z)\in D_{\tilde{3}}} \left|\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z) - f(x,y,z)\right| = 0.$$

Proof.

Using Lemma 2.2. for $e_{0,0,0}, e_{1,0,0}, e_{0,1,0}$;

$$\left\|\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{i_1,i_2,i_3};x,y,z) - e_{i_1,i_2,i_3}(x,y,z)\right\|_{C(D_{\tilde{3}})} = 0$$

$$\left\|\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(e_{2,0,0}+e_{0,2,0}+e_{0,0,2};x,y,z) - x^2+y^2+z^2\right\|_{C(D_{\tilde{3}})}$$

$$\le a\frac{\alpha_n}{[n]_{q_n}} + b\frac{\beta_m}{[m]_{q_m}} + c\frac{\gamma_r}{[r]_{q_r}}.$$

FromVolkov Theorem and $\{\alpha_n\},\{\beta_m\},\{\gamma_r\}$ this equations

$$\lim_{n,m,r\to\infty} \max_{(x,y,z)\in D_{\tilde{3}}} \left|\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z) - f(x,y,z)\right| = 0.$$

That is completed the proof.

Example 2.2. The convergence of $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)$ to

$f(x,y,z) = (2x)^3(3y)^{1/8}z + 1$, $\alpha_n = \sqrt{n}$, $\beta_m = \sqrt[3]{m}$, $\gamma_r = \sqrt[4]{r} + 1$, $q = \frac{1}{6}$ is illustrated in Figure2.2. $n = m = r = 2$ (yellow), $n = m = r = 5$(magenta).

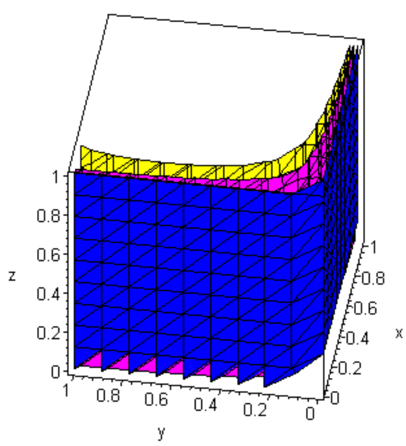

Figure 2.2 Approximation of

$f(x,y,z) = (2x)^3 + (3y)^{1/8}z + 1$ (blue) by $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)$.

3. RATES OF CONVERGENCE

In this section we want to find the rate of convergence of the sequence of operators $\{B_{n,m,r}\}$ and $\{\tilde{B}_{n,m,r}^{q_n,q_m,q_r}\}$.

Let $f \in C(\tilde{D}_3)$ be a continuous function and δ_n,δ_m and δ_r a positive number sequence. $w_1(f,\delta_n), w_2(f,\delta_m), w_3(f,\delta_r)$ are partial continuity modulus of the function $f(x,y,z)$.

It is also known that $\lim_{\delta_n\to 0} w_1(f,\delta_n) = \lim_{\delta_m\to 0} w_2(f,\delta_m) = \lim_{\delta_r\to 0} w_3(f,\delta_r) = 0$

Lemma 3.1. Let$x,y,z \in [0,A]$, then for any sufficiently large n and then we same as [5] get

$$\left|B_{n,m,r}(f;x,y,z) - f(x,y,z)\right| \le 2A\big(w_1(f,\delta_n) + w_2(f,\delta_m) + w_3(f,\delta_r)\big).$$

Proof.

$$\left|B_{n,m,r}(f;x,y,z) - f(x,y,z)\right| \le \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\left|f\left(\frac{k}{n}b_n,\frac{j}{m}c_m,\frac{l}{r}d_r\right) - f(x,y,z)\right|\binom{n}{k}\binom{m}{j}\binom{r}{l}$$

$$\text{x}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(\frac{y}{c_m}\right)^j$$

$$\text{x}\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\left|f\left(\frac{k}{n}b_n,\frac{j}{m}c_m,z\right) - f\left(x,\frac{j}{m}c_m,z\right)\right|$$

$$\text{x}\binom{n}{k}\binom{m}{j}\binom{r}{l}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$+ \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\left|f\left(x,\frac{j}{m}c_m,z\right) - f(x,y,z)\right|$$

$$\text{x}\binom{n}{k}\binom{m}{j}\binom{r}{l}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\left|f\left(\frac{k}{n}b_n,\frac{j}{m}c_m,\frac{l}{r}d_r\right) - f\left(\frac{k}{n}b_n,\frac{j}{m}c_m,z\right)\right|$$

$$\text{x}\binom{n}{k}\binom{m}{j}\binom{r}{l}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(\frac{y}{c_m}\right)^j$$

$$\text{x}\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$\le \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r} w_1\left(f;\left|\frac{k}{n}b_n - x\right|\right)$$

$$\text{x}\binom{n}{k}\binom{m}{j}\binom{r}{l}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$+\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}w_2\left(f;\left|\frac{j}{m}c_m-y\right|\right)\binom{n}{k}\binom{m}{j}\binom{r}{l}$$

$$x\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$+\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}w_3\left(f;\left|\frac{l}{r}d_r-z\right|\right)$$

$$x\binom{n}{k}\binom{m}{j}\binom{r}{l}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\psi_1(x,y,z)+\psi_2(x,y,z)+\psi_3(x,y,z)$$

is found. If calculated all of them respectively;

$$\psi_1(x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}w_1\left(f;\left|\frac{k}{n}b_n-x\right|\right)$$

$$x\binom{n}{k}\binom{m}{j}\binom{r}{l}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}w_1\left(f;\left|\frac{k}{n}b_n-x\right|\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$x\sum_{l=0}^{r}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}w_1\left(f;\left|\frac{k}{n}b_n-x\right|\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=\sum_{k=0}^{n}w_1\left(f;\left|\frac{k}{n}b_n-x\right|\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

At this condition $x\in[0,A]$and $A>1$ if take $t=\frac{k}{n}b_n$ when

$|f(t)-f(x)|\le w_1(f,\delta_n)\left(1+\frac{|t-x|}{\delta_n}\right)$ obtain by using Cauchy-Schwarz inequality;

$$\psi_1(x,y,z)\le$$

$$w_1(f,\delta_n)\left\{1+\frac{1}{\delta_n}\left[\sum_{k=0}^{n}\left(\frac{k}{n}b_n-x\right)^2\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\right]^{1/2}\right\}$$

$$\le w_1(f,\delta_n)\left\{1+\frac{1}{\delta_n}\sqrt{A\frac{b_n}{n}}\right\}$$

is found. By choosing $\delta_n=\sqrt{\frac{b_n}{n}}$;

$$\psi_1(x,y,z)\le w_1\left(f,\sqrt{\frac{b_n}{n}}\right)\{1+\sqrt{A}\}$$

$$\le 2Aw_1\left(f,\sqrt{\frac{b_n}{n}}\right).$$

Similarly

$$\psi_2(x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}w_2\left(f;\left|\frac{j}{m}c_m-y\right|\right)$$

$$x\binom{n}{k}\binom{m}{j}\binom{r}{l}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\sum_{j=0}^{m}w_2\left(f;\left|\frac{j}{m}c_m-y\right|\right)\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$x\sum_{l=0}^{r}\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1-\frac{z}{d_r}\right)^{r-l}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$x\sum_{j=0}^{m}w_2\left(f;\left|\frac{j}{m}c_m-y\right|\right)\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

$$=\sum_{j=0}^{m}w_2\left(f;\left|\frac{j}{m}c_m-y\right|\right)\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1-\frac{y}{c_m}\right)^{m-j}$$

At this condition $y \in [0, A]$and $A > 1$ if take $t = \frac{j}{m} c_m$ then for any sufficiently large n and when $|f(t) - f(y)| \leq w_2(f, \delta_m)\left(1 + \frac{|t-y|}{\delta_m}\right)$ obtain by using Cauchy-Schwarz inequality;

$\psi_2(x, y, z) \leq$

$$w_2(f, \delta_m)\left\{1 + \frac{1}{\delta_m}\left[\sum_{j=0}^{m}\left(\frac{j}{m}c_m - y\right)^2 \binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1 - \frac{y}{c_m}\right)^{m-j}\right]^{1/2}\right\}$$

$$\leq w_2(f, \delta_m)\left\{1 + \frac{1}{\delta_m}\sqrt{A\frac{c_m}{m}}\right\}$$

founded. By choosing $\delta_m = \sqrt{\frac{c_m}{m}}$;

$$\psi_2(x, y, z) \leq w_2\left(f, \sqrt{\frac{c_m}{m}}\right)\{1 + \sqrt{A}\}$$

$$\leq 2Aw_2\left(f, \sqrt{\frac{c_m}{m}}\right)$$

is found. Finally

$$\psi_3(x, y, z) = \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r} w_3\left(f; \left|\frac{l}{r}d_r - z\right|\right)$$

$$\text{x}\binom{n}{k}\binom{m}{j}\binom{r}{l}\left(\frac{x}{b_n}\right)^k\left(1 - \frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\left(\frac{y}{c_m}\right)^j\left(1 - \frac{y}{c_m}\right)^{m-j}\left(\frac{z}{d_r}\right)^l\left(1 - \frac{z}{d_r}\right)^{r-l}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1 - \frac{x}{b_n}\right)^{n-k}$$

$$\text{x}\sum_{j=0}^{m}\binom{m}{j}\left(\frac{y}{c_m}\right)^j\left(1 - \frac{y}{c_m}\right)^{m-j}$$

$$\text{x}\sum_{l=0}^{r} w_3\left(f; \left|\frac{l}{r}d_r - z\right|\right)\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1 - \frac{z}{d_r}\right)^{r-l}$$

$$= \sum_{l=0}^{r} w_3\left(f; \left|\frac{l}{r}d_r - z\right|\right)\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1 - \frac{z}{d_r}\right)^{r-l}$$

At this conditions $z, t \in [0, A]$ ve $A > 1$ if take $t = \frac{l}{r}d_r$ when $|f(t) - f(z)| \leq w_3(f, \delta_r)\left(1 + \frac{|t-z|}{\delta_r}\right)$ obtain by using Cauchy-Schwarz inequality;

$\psi_3(x, y, z) \leq$

$$w_3(f, \delta_r)\left\{1 + \frac{1}{\delta_r}\left[\sum_{l=0}^{r}\left(\frac{l}{r}d_r - z\right)^2\binom{r}{l}\left(\frac{z}{d_r}\right)^l\left(1 - \frac{z}{d_r}\right)^{r-l}\right]^{1/2}\right\}$$

$$\leq w_3(f, \delta_r)\left\{1 + \frac{1}{\delta_r}\sqrt{A\frac{d_r}{r}}\right\}$$

founded. By choosing $\delta_r = \sqrt{\frac{d_r}{r}}$;

$$\psi_3(x, y, z) \leq w_3\left(f, \sqrt{\frac{d_r}{r}}\right)\{1 + \sqrt{A}\}$$

$$\leq 2Aw_3\left(f, \sqrt{\frac{d_r}{r}}\right)$$

is obtained.

$$|B_{n,m,r}(f; x, y, z) - f(x, y, z)| \leq 2A\big(w_1(f, \delta_n) + w_2(f, \delta_m) + w_3(f, \delta_r)\big)$$

Then proof is completed.

Example 3.1. The error bound of the function $f(x, y, z) = \frac{x^2+y^2+z^2}{2+\exp(7)}$,

$b_n = \sqrt{n}\,, c_m = \sqrt[3]{m}\,, d_r = \sqrt[4]{r} + 1.$

Table 3.1. The error bound of

$f(x, y, z) = \frac{x^2+y^2+z^2}{2+\exp(7)}$.

n,m,r	Error bound for full modulus of continuity of function $f(x, y, z)$
10	0.0218237378
10^2	0.0073671615
10^3	0.0027753196
10^4	0.0011048136
10^5	0.0004521637
10^6	0.0001876932
10^7	0.0000784867
10^8	0.0000329479
10^9	0.0000138598

Using the q-modulus of continuity we get the rate of convergence following.

Lemma 3.2. For any $f \in D_{\mathfrak{z}}$, the following inequality hold.

a) $\left|\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)-f(x,y,z)\right| \leq 3\left[w_1\left(f;\sqrt{\frac{a\alpha_n}{[n]_{q_n}}}\right)+w_2\left(f;\sqrt{\frac{b\beta_m}{[m]_{q_m}}}\right)+\left(f;\sqrt{\frac{c\gamma_r}{[r]_{q_r}}}\right)\right]$

b) $\left|\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)-f(x,y,z)\right| \leq 3w\left(f;\sqrt{\frac{a\alpha_n}{[n]_{q_n}}}+\sqrt{\frac{b\beta_m}{[m]_{q_m}}}+\sqrt{\frac{c\gamma_r}{[r]_{q_r}}}\right)$

Proof. Using next equality

$$\sum_{k=0}^{n} f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n\right)=\sum_{j=0}^{m} f\left(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m\right)=\sum_{l=0}^{r} f\left(\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r\right)$$

we estimate the difference between $\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)$ and $f(x,y,z)$;

$$\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)-f(x,y,z)=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\left[f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m,\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r\right)-f(x,y,z)\right]$$

$$\text{x}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\left[f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m,\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r\right)-f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,y,z\right)+f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,y,z\right)-f(x,y,z)\right]$$

$$\text{x}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

then

$$\left|\tilde{B}_{n,m,r}^{q_n,q_m,q_r}(f;x,y,z)-f(x,y,z)\right|\leq$$

$$\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\left[f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m,\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r\right)-f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,y,z\right)\right]$$

$$\text{x}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$=\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}\left[f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,y,z\right)-f(x,y,z)\right]$$

$$\text{x}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$\leq\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}w_2\left(f;\left|\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y\right|\right)$$

$$\text{x}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$\leq\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r}w_1\left(f;\left|\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x\right|\right)$$

$$\text{x}\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\begin{bmatrix}m\\j\end{bmatrix}_{q_m}\begin{bmatrix}r\\l\end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\text{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\text{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$\leq \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r} w_3\left(f;\left|\frac{[l]_{q_r}}{[r]_{q_r}}\gamma_r - z\right|\right)$$

$$\mathrm{x}\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\begin{bmatrix} m \\ j \end{bmatrix}_{q_m}\begin{bmatrix} r \\ l \end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\mathrm{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\mathrm{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$= \breve{\psi}_2(x,y,z)+\breve{\psi}_1(x,y,z)+\breve{\psi}_3(x,y,z)$$

By using Lemma 2.2.(i) and properties continuity, we get

$$\breve{\psi}_2(x,y,z) = \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r} w_2\left(f;\left|\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m - y\right|\right)$$

$$\mathrm{x}\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}\begin{bmatrix} m \\ j \end{bmatrix}_{q_m}\begin{bmatrix} r \\ l \end{bmatrix}_{q_r}\left(\frac{x}{\alpha_n}\right)^k\left(\frac{y}{\beta_m}\right)^j\left(\frac{z}{\gamma_r}\right)^l$$

$$\mathrm{x}\prod_{s_1=0}^{n-k-1}\left(1-q_n^{s_1}\frac{x}{\alpha_n}\right)\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\mathrm{x}\prod_{s_3=0}^{r-l-1}\left(1-q_r^{s_3}\frac{z}{\gamma_r}\right)$$

$$= \sum_{j=0}^{m} w_2\left(f;\left|\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m - y\right|\right)\begin{bmatrix} m \\ j \end{bmatrix}_{q_m}\left(\frac{y}{\beta_m}\right)^j$$

$$\mathrm{x}\prod_{s_2=0}^{m-j-1}\left(1-q_m^{s_2}\frac{y}{\beta_m}\right)$$

$$\leq w_2(f;\delta_m)\left\{1+\frac{1}{\delta_m}\left[\sum_{j=0}^{m}\left(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m - y\right)^2\begin{bmatrix} m \\ j \end{bmatrix}_{q_m}\left(\frac{y}{\beta_m}\right)^j\prod_{s_2=0}^{m-j-1}\left(1 - q_m^{s_2}\frac{y}{\beta_m}\right)\right]^{1/2}\right\}$$

Expending the squared term and making use of Lemma 2.2 (i), (iii) and (v) we have

$$\breve{\psi}_2(x,y,z) \leq w_2(f;\delta_m)\left\{1+\frac{1}{\delta_m}\sqrt{\frac{y(\beta_m - y)}{[m]_{q_m}}}\right\}$$

$$\leq w_2(f;\delta_m)\left\{1+\frac{1}{\delta_m}\sqrt{\frac{b\beta_m}{[m]_{q_m}}}\right\}.$$

By choosing $\delta_m = \sqrt{\frac{b\beta_m}{[m]_{q_m}}}$ we get

$$\breve{\psi}_2(x,y,z) \leq 3w_2\left(f;\sqrt{\frac{b\beta_m}{[m]_{q_m}}}\right).$$

In the some way we have

$\breve{\psi}_1(x,y,z) \leq 3w_1\left(f;\sqrt{\frac{a\alpha_n}{[n]_{q_n}}}\right)$ and

$\breve{\psi}_3(x,y,z) \leq 3w_3\left(f;\sqrt{\frac{c\gamma_r}{[r]_{q_r}}}\right)$. We define

$A \coloneqq \max\{a,b,c\}$ then we get

$\left|B_{n,m,r}(f;x,y,z) - f(x,y,z)\right| \leq$
$3A\left(w_1(f,\delta_n)+w_2(f,\delta_m)+w_3(f,\delta_r)\right).$

Example3.2. The error bound of the function $f(x,y,z) = \frac{xy+10-z}{10}$, $\alpha_n = \sqrt{n}$, $\beta_m = \sqrt[3]{m}$, $\gamma_r = \sqrt[4]{r}+1$ and $q = 1$.

Table3.2. The error bound of $f(x,y,z) = \frac{xy+10-z}{10}$.

n, m, r	Error bound for q-modulus of continuity of function $f(x,y,z)$
10	0.7244296242
10^2	0.2210754467
10^3	0.0792069915
10^4	0.0308396231
10^5	0.0125035752
10^6	0.0051698321
10^7	0.0021582801

4. CONCLUSION

We give Bernstein-Chlodowsky and q generalized this operators so researchers can compare their approximation we have beter approach result for q-Bernstein-Chlodowsky operators means of modulus of continuity.

REFERENCES

[1] G.M.Philips, "On Generalized Bernstein Polynomials", in *Numerical Analysis: D.F. Griffits*, G.A. Watson Eds, World Scientific Singapore, pp. 263-269, 1996.

[2] H. Karsli and V. Gupta, "Some Approximation Properties of q-Chlodowsky Operators",

Applied Mathematics and Computation, vol. 195, pp. 220–229, 2008.

[3] I. Buyukyazici, "One the Approximation Properties of Two-Dimensional q-Bernstein-Chlodowsky Polynomials", *Mathematical Communications* vol. 14, no. 2, pp. 255-269, 2009.

[4] I. Buyukyazici and E. Ibikli, "The Approximation Properties of Generalized Bernstein-Chlodowsky Polynomials of Two Variables", *Applied Mathematics and Computation* vol. 156, pp. 367–380, 2004.

[5] A.K. Gazanfer, "Weighted Approximation Of Continuous Functions Of Three Variables in a Tetrahedron With Variable Boundary By Bernstein-Chlodowsky Polynomials", Ph. D. Thesis, Graduate School of Natural and Applied Sciences, Bulent Ecevit Univ., Zonguldak Turkey, 2015.