Hacettepe Journal of Mathematics and Statistics
Volume 47 (3) (2018), 513 – 519

On super edge-magic deficiency of certain Toeplitz graphs

Ali Ahmad*, Muhammad Faisal Nadeem† and Ashok Gupta‡§

Abstract

A graph G is called *edge-magic* if there exists a bijective function $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $\phi(x) + \phi(xy) + \phi(y) = c(\phi)$ is a constant for every edge $xy \in E(G)$, called the valence of ϕ. Moreover, G is said to be *super edge-magic* if $\phi(V(G)) = \{1, 2, \ldots, |V(G)|\}$. The *super edge-magic deficiency* of a graph G, denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$, has a super edge-magic labeling, if such integer does not exist we define $\mu_s(G)$ to be $+\infty$. In this paper, we study the super edge-magic deficiency of some Toeplitz graphs.

Keywords: edge-magic, super edge-magic, Toeplitz graphs

2000 AMS Classification: 05C78

Received : 02.01.2017 *Accepted* : 11.05.2017 *Doi* : 10.15672/HJMS.2017.465

1. Introduction and preliminary results

In this paper, we consider only finite, simple and undirected graphs $G = (V, E)$. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively, where $|V(G)| = p$ and $|E(G)| = q$. An edge-magic labeling of a graph G is a bijection $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$, where there exists a constant $c(\phi)$ such that $f(x) + f(xy) + f(y) = c(\phi)$, for every edge $xy \in E(G)$, $c(\phi)$ is called the valence of the graph and a graph with an edge-magic labeling is called *edge-magic*. An edge-magic labeling ϕ is called a super edge-magic labeling of G, and G is said to be super edge-magic if $\phi(V(G)) = \{1, 2, \ldots, p\}$. It is with to mention that super edge-magic labelings where independently defined by Acharia and Hegde in [1].

*College of Computer Science & Information Systems, Jazan University, Jazan, KSA.
Email : ahmadsms@gmail.com

†Department of Mathematics, Comsat Institute of Information Technology, Lahore, Pakistan.
Email : m.fayysal@gmail.com

‡College of Computer Science & Information Systems, Jazan University, Jazan, KSA.,
Email : kgupta.ashok@gmail.com

§Corresponding Author.

In [9], Kotzig and Rosa proved that for any graph G there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This fact leads to the concept of edge-magic deficiency of a graph G, which is the minimum nonnegative integer n such that $G \cup nK_1$ is edge-magic and it is denoted by $\mu(G)$. In particular,

$$\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic}\}.$$

In the same paper, Kotzig and Rosa gave an upper bound for the edge-magic deficiency of a graph G with n vertices, $\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1)$, where F_n is the n^{th} Fibonacci number. Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno et al. [7] defined a similar concept for super edge-magic labelings. The super edge-magic deficiency of a graph G, which is denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic labeling or $+\infty$ if there exists no such n.

Let $M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic graph}\}$, then

$$\mu_s(G) = \begin{cases} min\ M(G), & \text{if } M(G) \neq \phi \\ +\infty, & \text{if } M(G) = \phi. \end{cases}$$

As a consequence of the above two definitions, for every graph G, $\mu(G) \leq \mu_s(G)$. In [7, 8], Figueroa-Centeno et al. provided the exact values for the super edge-magic deficiency of several classes of graphs, such as cycles, complete graphs, 2-regular graphs, and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. In [10], A. A. G. Ngurah et al. proved some upper bounds for the super edge-magic deficiency of fans, double fans, and wheels. In [2], A. Ahmad et al. found the super edge-magic deficiency of some families related to ladder graphs. In this paper, we discuss the super edge-magic deficiency of some families of Toeplitz graphs.

In proving the results, we frequently use the following lemmas.

1.1. Lemma. [6] *A graph G with p vertices and q edges is super edge-magic if and only if there exists a bijective function $\psi : V(G) \to \{1, 2, \cdots, p\}$ such that the set $S = \{\phi(x) + \phi(y) : xy \in E(G)\}$ consists of q consecutive integers. In such a case, ψ extends to a super edge-magic total labeling of G with magic constant $c(\psi) = p + q + s$, where $s = \min(S)$ and*

$$S = \{c(\psi) - (p+1), c(\psi) - (p+2), \ldots, c(\psi) - (p+q)\}.$$

1.2. Lemma. [5] *If a graph G with p vertices and q edges is super edge-magic, then $q \leq 2p - 3$.*

2. Results and Discussion

A simple undirected graph T of order p is called a *Toeplitz graph* if its adjacency matrix $A(T)$ is Toeplitz. A *Toeplitz matrix* $A(T) = (a_{i,j})$, is a $(p \times p)$ symmetric matrix which has constant values along all diagonals parallel to the main diagonal, i.e. $a_{i,j} = a_{i+1,j+1}$ for each $i, j = 1, 2, \ldots, p-1$. The p distinct diagonals of a $(p \times p)$ symmetric Toeplitz adjacency matrix will be labeled $0, 1, 2, \ldots, p-1$. Diagonal 0 is the main diagonal and it contains only zeros, i.e. $a_{ii} = 0$ for all $i = 1, 2, \ldots, p$ so that there are no loops in the Toeplitz graph. A Toeplitz graph T is uniquely defined by the first row of $A(T)$, a $(0-1)$-sequence. Let $t_1, t_2, \ldots, t_k$ be the diagonals containing ones, $0 < t_1 < t_2 < \cdots < t_k < p$. Then, the corresponding Toeplitz graph will be denoted by $T_p\langle t_1, \ldots, t_k\rangle$. That is, $T_p\langle t_1, \ldots, t_k\rangle$ is the graph with the vertex set $V(T) = \{v_i : i = 1, 2, \ldots, p\}$ in which two vertices u, v of T being connected by an edge if and only if $|u - v| \in \{t_1, t_2, \ldots, t_k\}$. If t_j, $j = 1, 2, \ldots, k$, is a diagonal containing ones then the diagonal elements a_{i,t_j+i}, $i = 1, 2, \ldots, p - t_j$, determining the edges $v_i v_{t_j+i}$ in the Toeplitz graph. Thus the edge set is $E(T) = \bigcup_{j=1}^{k}\{v_i v_{t_j+i} : i = 1, 2, \ldots, p - t_j\}$, $|V(T)| = p$ and $|E(T)| = pk - \sum\limits_{j=1}^{k} t_j$. In [4] Bača *et al.* determined super edge-antimagic total labeling of Toeplitz graphs and Ahmad *et al.* [3] determined the edge irregularity strength of Toeplitz graphs. A Toeplitz graph is not necessarily connected, see Figure 1.

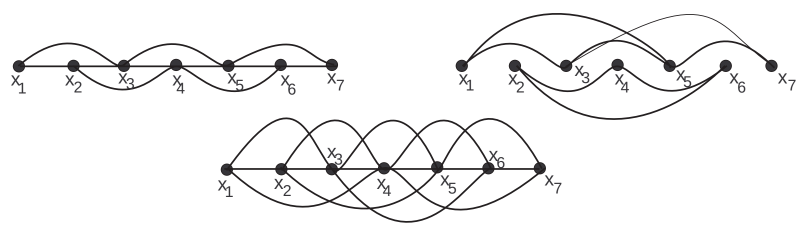

Figure 1. Toeplitz graphs $T_7\langle 1,2\rangle$, $T_7\langle 2,4\rangle$ and $T_7\langle 1,2,3\rangle$

The following Theorem shows a lower bound for the super edge-magic deficiency for the Toeplitz graph $T_p\langle t_1, \ldots, t_k\rangle$.

2.1. Theorem. *The lower bound for the super edge-magic deficiency for Toeplitz graph* $T_p\langle t_1, \ldots, t_k\rangle$ *is* $\frac{p(k-2)+3-\sum\limits_{j=1}^{k} t_j}{2}$.

Proof. The vertex set of $T_p\langle t_1, \ldots, t_k\rangle$ is $V(T_p\langle t_1, \ldots, t_k\rangle) = \{v_i : i = 1, 2, \ldots, p\}$ and the edge set of $T_p\langle t_1, \ldots, t_k\rangle$ is $q = E(T_p\langle t_1, \ldots, t_k\rangle) = \bigcup\limits_{j=1}^{k}\{v_i v_{t_j+i} : i = 1, 2, \ldots, p - t_j\}$.

By Lemma 1.2, we know that the size of any super edge-magic graph is bounded above by two times its order minus 3. Now we know that for the Toeplitz graph $T_p\langle t_1, \ldots, t_k\rangle$, $p = n$ and $q = pk - \sum\limits_{j=1}^{k} t_j$, which implies that we need $x \in \mathbb{N}$ such that $pk - \sum\limits_{j=1}^{k} t_j \leq 2(p+x)-3$ or $p(k-2)+3-\sum\limits_{j=1}^{k} t_j \leq 2x$. This implies that we need at least $\frac{p(k-2)+3-\sum\limits_{j=1}^{k} t_j}{2}$ isolated vertices. □

In [4], Bača *et al.* found the super edge-antimagic total labeling of $T_p\langle 1,2\rangle$. In the next theorem, we showed that the super edge-magic deficiency of Toeplitz graph $T_p\langle 1,t\rangle$ is zero for $t \geq 4$ even.

2.2. Theorem. *Let Toeplitz graph* $T_p\langle 1,t\rangle$ *with* p *vertices,* q *edges and* $t \geq 4$ *even admits a super edge-magic labeling i.e* $\mu_s(T_p\langle 1,t\rangle) = 0$

Proof. Let $x_1, x_2, \ldots, x_p$ be the vertex sequence of $T_p\langle 1,t\rangle$. The vertex set and edge set of $T_p\langle 1,t\rangle$ are defined as:

$$V(T_p\langle 1,t\rangle) = \{x_i : 1 \leq i \leq p\}.$$

$$E(T_p\langle 1,t\rangle) = \{x_i x_{i+1} : 1 \leq i \leq p-1\} \cup \{x_i x_{t+i} : 1 \leq i \leq p-t\}.$$

We define the labeling for $T_p\langle 1,t\rangle$ as follows:

Case 1: When $p \equiv 0 \pmod 2$

$$\phi(x_i) = \begin{cases} \frac{i+1}{2}, & \text{if } 1 \le i \le t+1, \text{ and } i \equiv 1 \pmod 2 \\ i - \frac{t}{2}, & \text{if } t+3 \le i \le p-1, \text{ and } i \equiv 1 \pmod 2 \\ i + \frac{t}{2}, & \text{if } 2 \le i < p-t+2, \text{ and } i \equiv 0 \pmod 2 \\ i + \frac{a}{2}, & \text{if } p-t+2 \le i \le p, i = p-a, \\ & \text{where } 0 \le a \le t-2, \text{ and } a, i \text{ are even} \end{cases}$$

Case 2: When $p \equiv 1 \pmod 2$

$$\phi(x_i) = \begin{cases} \frac{i+1}{2}, & \text{if } 1 \le i \le t+1, \text{ and } i \equiv 1 \pmod 2 \\ i - \frac{t}{2}, & \text{if } t+3 \le i \le p, \text{ and } i \equiv 1 \pmod 2 \\ i + \frac{t}{2}, & \text{if } 2 \le i \le p-t+2, \text{ and } i \equiv 0 \pmod 2 \\ i + \frac{a+1}{2}, & \text{if } p-t+2 \le i \le p, i = p-a, \\ & \text{where } 1 \le a \le t-3, \text{ and } a \text{ odd}, i \text{ even} \end{cases}$$

The set of all edge-sums generated by the above formula forms q consecutive integer sequences. Therefore by using Lemma 1.1, ϕ can be extended to a super edge-magic labeling. Since there is no isolated vertex, it follows that $\mu_s(T_p\langle 1,t\rangle) = 0$. Which completes the proof. □

2.3. Open Problem. Consider the graph $T_p\langle 1,t\rangle$ with t odd. Show that either $T_p\langle 1,t\rangle$ is super edge-magic or calculate its super edge-magic deficiency.

The following theorem shows an upper bound for the super edge-magic deficiency of $T_p\langle 1,2,3\rangle$ for $p \ge 5$.

2.4. Theorem. *Consider the Toeplitz graph $T_p\langle 1,2,3\rangle$ with p vertices and q edges. The super edge-magic deficiency for $T_p\langle 1,2,3\rangle$ is upper bounded by*

$$\mu_s(T_p\langle 1,2,3\rangle) \le \begin{cases} \frac{p-1}{2}, & \text{if } p \equiv 1 \pmod 4 \\ \lceil \frac{p-3}{2} \rceil, & \text{otherwise} \end{cases}$$

Proof. Let $x_1, x_2, \ldots, x_p$ be the vertex sequence of $T_p\langle 1,2,3\rangle$ and the edges of $T_p\langle 1,2,3\rangle$ are $\{x_i x_{i+1} : 1 \le i \le p-1\} \cup \{x_i x_{i+2} : 1 \le i \le p-2\} \cup \{x_i x_{i+3} : 1 \le i \le p-3\}$.

Let p be a nonnegative integer. According to Lemma 1.1 it is sufficient to prove that there exists a vertex labeling with the property that the edge-sums under this labeling are consecutive integers. It is easy to see that the following labelings $\phi_1 : V(T_p\langle 1,2,3\rangle \cup (\frac{p-1}{2})K_1) \to \{1,2,\ldots,|V(T_p\langle 1,2,3\rangle)| + \frac{p-1}{2}\}$ and $\phi_2 : V(T_p\langle 1,2,3\rangle \cup (\lceil \frac{p-3}{2} \rceil)K_1) \to \{1,2,\ldots,|V(T_p\langle 1,2,3\rangle)| + \lceil \frac{p-3}{2} \rceil\}$ have the desired property, for $p \equiv 1 \pmod 4$ and $p \equiv 0,2,3 \pmod 4$, respectively. Here, for $p \equiv 1 \pmod 4$ we label $T_p\langle 1,2,3\rangle \cup (\frac{p-1}{2})K_1$ where $V((\frac{p-1}{2})K_1) = \{z_i\}$ for $1 \le i \le \frac{p-1}{2}$ and for $p \equiv 0,2,3 \pmod 4$, we label $T_p\langle 1,2,3\rangle \cup \lceil \frac{p-3}{2} \rceil K_1$ where $V(\lceil \frac{p-3}{2} \rceil K_1) = \{z_i\}$ for $1 \le i \le \lceil \frac{p-3}{2} \rceil$.

$$\phi_1(x_i) = \phi_2(x_i) = \begin{cases} \lfloor \frac{3i-1}{2} \rfloor, & \text{if } i \equiv 1,2 \pmod 4 \\ \lfloor \frac{3i-2}{2} \rfloor, & \text{if } i \equiv 0,3 \pmod 4 \end{cases}$$

The isolated vertices z_i, for $1 \le i \le \frac{p-1}{2}$ under labeling ϕ_1 and for $1 \le i \le \lceil \frac{p-3}{2} \rceil$ under labeling ϕ_2 are labeled as $\phi_1(z_i) = \phi_2(z_i) = 3i+1$ for i odd and $\phi_1(z_i) = \phi_2(z_i) = 3i$ for i even. The edge-sum under the labeling ϕ_1 and ϕ_2 are as follows:

$$w(x_ix_{i+1}) = \begin{cases} \lfloor\frac{3i-1}{2}\rfloor + \lfloor\frac{3i+1}{2}\rfloor, & \text{if } i \equiv 1,2 \pmod 4 \\ \lfloor\frac{3i-2}{2}\rfloor + \lfloor\frac{3i+1}{2}\rfloor, & \text{if } i \equiv 3 \pmod 4 \\ \lfloor\frac{3i-2}{2}\rfloor + \lfloor\frac{3i+2}{2}\rfloor, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

$$w(x_ix_{i+2}) = \begin{cases} \lfloor\frac{3i-1}{2}\rfloor + \lfloor\frac{3i+4}{2}\rfloor, & \text{if } i \equiv 1,2 \pmod 4 \\ \lfloor\frac{3i-2}{2}\rfloor + \lfloor\frac{3i+5}{2}\rfloor, & \text{if } i \equiv 0,3 \pmod 4 \end{cases}$$

$$w(x_ix_{i+3}) = \begin{cases} \lfloor\frac{3i-1}{2}\rfloor + \lfloor\frac{3i+7}{2}\rfloor, & \text{if } i \equiv 1 \pmod 4 \\ \lfloor\frac{3i-1}{2}\rfloor + \lfloor\frac{3i+8}{2}\rfloor, & \text{if } i \equiv 2 \pmod 4 \\ \lfloor\frac{3i-2}{2}\rfloor + \lfloor\frac{3i+8}{2}\rfloor, & \text{if } i \equiv 3 \pmod 4 \\ \lfloor\frac{3i-2}{2}\rfloor + \lfloor\frac{3i+7}{2}\rfloor, & \text{if } i \equiv 0 \pmod 4 \end{cases}$$

It is easy to see that the edge-sums form q consecutive integer. Which completes the proof. □

From Theorem 2.1, the super edge-magic deficiency of $T_p\langle 1,2,3\rangle$ i.e $\mu_s(T_p\langle 1,2,3\rangle) \geq \lceil\frac{p-3}{2}\rceil$, combine with Theorem 2.4, which leads the following corollary and open problem.

2.5. Corollary. *Consider the graph $T_p\langle 1,2,3\rangle$ with p vertices and q edges. For $p \not\equiv 1 \pmod 4$, the super edge-magic deficiency for $T_p\langle 1,2,3\rangle$ is $\lceil\frac{p-3}{2}\rceil$ i.e $\mu_s(T_p\langle 1,2,3\rangle) = \lceil\frac{p-3}{2}\rceil$.*

2.6. Open Problem. Calculate the super edge-magic deficiency for the Toeplitz graph $T_p\langle 1,2,3\rangle$ for $p \equiv 1 \pmod 4$.

2.7. Theorem. *Consider the graph $T_p\langle 1,2,3,4\rangle$ with p vertices, q edges, then*

$$p-3 \leq \mu_s(T_p\langle 1,2,3,4\rangle) \leq \begin{cases} \lceil\frac{p-4}{5}\rceil + p - 3, & \text{if } p \equiv 3,4 \pmod 5 \\ \lceil\frac{p}{5}\rceil + p - 3, & \text{if } p \equiv 0,2 \pmod 5 \\ \lceil\frac{p}{5}\rceil + p - 2, & \text{if } p \equiv 1 \pmod 5 \end{cases}$$

Proof. Let $x_1, x_2, \ldots, x_p$ be the vertex sequence of $T_p\langle 1,2,3,4\rangle$ and the edges of $T_p\langle 1,2,3,4\rangle$ are $\{x_ix_{i+1} : 1 \leq i \leq p-1\} \cup \{x_ix_{i+2} : 1 \leq i \leq p-2\} \cup \{x_ix_{i+3} : 1 \leq i \leq p-3\} \cup \{x_ix_{i+4} : 1 \leq i \leq p-4\}$.

From Theorem 2.1 the lower bound for the super edge-magic deficiency of $T_p\langle 1,2,3,4\rangle$ is $\mu_s(T_p\langle 1,2,3,4\rangle) \geq p-3$. For the upper bound let p be a nonnegative integer. According to Lemma 1.1 it is sufficient to prove that there exists a vertex labeling with the property that the edge-sums under this labeling are consecutive integers. It is easy to see that the following labelings $\phi_1 : V(T_p\langle 1,2,3,4\rangle \cup (\lceil\frac{p-4}{5}\rceil + p - 3)K_1) \to \{1,2,\ldots,|V(T_p\langle 1,2,3,4\rangle)| + \lceil\frac{p-4}{5}\rceil + p - 3\}$, $\phi_2 : V(T_p\langle 1,2,3,4\rangle \cup (\lceil\frac{p}{5}\rceil + p - 3)K_1) \to \{1,2,\ldots,|V(T_p\langle 1,2,3,4\rangle)| + \lceil\frac{p}{5}\rceil + p - 3\}$ and $\phi_3 : V(T_p\langle 1,2,3,4\rangle \cup (\lceil\frac{p}{5}\rceil + p - 2)K_1) \to \{1,2,\ldots,|V(T_p\langle 1,2,3,4\rangle)| + \lceil\frac{p}{5}\rceil + p - 2\}$ have the desired property, for $p \equiv 3,4 \pmod 5$, $p \equiv 0,2 \pmod 5$ and $n \equiv 1 \pmod 5$, respectively. Here, for $p \equiv 3,4 \pmod 5$ we label $T_p\langle 1,2,3,4\rangle \cup (\lceil\frac{p-4}{5}\rceil + p - 3)K_1$ where $V((\lceil\frac{p-4}{5}\rceil + p - 3)K_1) = \{z_i\}$ for $1 \leq i \leq \lceil\frac{p-4}{5}\rceil + p - 3$, for $p \equiv 0,2 \pmod 5$, we label $T_p\langle 1,2,3,4\rangle \cup (\lceil\frac{p}{5}\rceil + p - 3)K_1$ where $V(\lceil\frac{p}{5}\rceil + p - 3)K_1) = \{z_i\}$ for $1 \leq i \leq \lceil\frac{p}{5}\rceil + p - 3$ and for $p \equiv 1 \pmod 5$, we label $T_p\langle 1,2,3,4\rangle \cup (\lceil\frac{p}{5}\rceil + p - 2)K_1$ where $V(\lceil\frac{p}{5}\rceil + p - 2)K_1) = \{z_i\}$ for $1 \leq i \leq \lceil\frac{p}{5}\rceil + p - 2$.

$$\phi_1(x_i) = \phi_2(x_i) = \phi_3(x_i) = \begin{cases} 2i - 1 + \lceil\frac{i-1}{5}\rceil, & \text{if } i \equiv 1 \pmod 5 \\ 2i - 2 + \lceil\frac{i-2}{5}\rceil, & \text{if } i \equiv 2 \pmod 5 \\ 2i - 3 + \lceil\frac{i-3}{5}\rceil, & \text{if } i \equiv 3 \pmod 5 \\ 2i - 4 + \lceil\frac{i}{5}\rceil, & \text{if } i \equiv 4 \pmod 5 \\ 2i - 3 + \lceil\frac{i}{5}\rceil, & \text{if } i \equiv 0 \pmod 5 \end{cases}$$

The isolated vertices z_i, for $1 \le i \le \lceil \frac{p-4}{5} \rceil + p - 3$, $p \equiv 3, 4 \pmod 5$ under labeling ϕ_1 , for $1 \le i \le \lceil \frac{p}{5} \rceil + p - 3$, for $p \equiv 0, 2 \pmod 5$ under labeling ϕ_2 and for $1 \le i \le \lceil \frac{p}{5} \rceil + p - 2$ for $p \equiv 1 \pmod 5$ under labeling ϕ_3 are labeled as:

$$\phi_1(z_i) = \phi_2(z_i) = \phi_3(z_i) = \begin{cases} i + 2 + \lceil \frac{5i}{6} \rceil, & \text{if } i \equiv 1, 2 \pmod 6 \\ i + 1 + \lceil \frac{5i}{6} \rceil, & \text{if } i \equiv 3, 4 \pmod 6 \\ i + \lceil \frac{5i}{6} \rceil, & \text{if } i \equiv 0, 5 \pmod 6 \end{cases}$$

It is easy to see that the edge-sums form q consecutive integer. (See Figure 2 for illustration.) Therefore by using Lemma 1.1, ϕ_1, ϕ_2, ϕ_3 can be extended to a super edge-magic labeling. It follows that

$$\mu_s(T_p\langle 1, 2, 3, 4\rangle) \le \begin{cases} \lceil \frac{p-4}{5} \rceil + p - 3, & \text{if } p \equiv 3, 4 \pmod 5 \\ \lceil \frac{p}{5} \rceil + p - 3, & \text{if } p \equiv 0, 2 \pmod 5 \\ \lceil \frac{p}{5} \rceil + p - 2, & \text{if } p \equiv 1 \pmod 5 \end{cases}$$

Combining with lower bound, which completes the proof. □

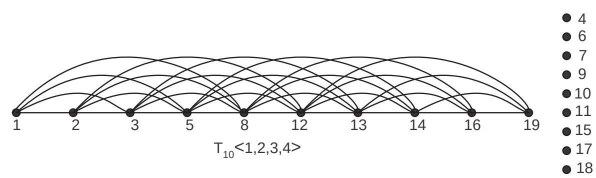

Figure 2. An illustration for the labeling given in the proof of Theorem 2.7.

2.8. Open Problem. Improve, if possible, the bounds for the super edge-magic deficiency for the Toeplitz graph $T_p\langle 1, 2, 3, 4\rangle$ or show that the bounds provided are sharp.

3. Closing Remarks

We have studied that the lower bounds for the super edge-magic deficiency of the Toeplitz graph $T_p\langle t_1, \ldots, t_k\rangle$. For t even we determined that the super edge-magic deficiency of Toeplitz graph $\mu_s(T_p\langle 1, t\rangle) = 0$. We also studied the upper bounds for the super edge-magic deficiency for Toeplitz graph $T_p\langle 1, 2, 3\rangle$ and $T_p\langle 1, 2, 3, 4\rangle$. It would be interesting to find the super edge-magic deficiency of $T_p\langle 1, t\rangle)$ for t odd. We encourage researchers to try to determine the super edge-magic deficiency of other graphs for further research. In fact, it seems to be a very challenging problem to find the exact value for the super edge-magic deficiency of families of graphs.

Acknowledgement:

The authors wish to thank the anonymous referee for his/her valuable comments. We also thank the Deanship of Research, Jazan University for supporting this project under the Grant No. 000114/7/37.

References

[1] B. D. Acharya, S. M. Hegde, *Strongly indexable graphs*, Discrete Math., **93** (1991), 123–129.

[2] A. Ahmad, I. Javaid , M. F. Nadeem and R. Hasni, *On Super edge-magic deficiency of some families related to ladder graphs*, Australas. J. Combin., **51**, 201–208, 2011.

[3] A. Ahmad, M. Bača, M. F. Nadeem, *On edge irregularity strength of Toeplitz graphs*, U.P.B. Sci. Bull., Series A, **78(4)**, 155–162, 2016.

[4] M. Bača, Y. Bashir, M. F. Nadeem, A. Shabbir, On super edge-antimagic total labeling of Toeplitz graphs, *Springer Proceedings in Mathematics & Statistics*, **98** (2015), 1-11.

[5] H. Enomoto, A. Lladó, T. Nakamigawa, and G. Ringel, *Super edge-magic graphs*, SUT J. Math., **34**, 105–109, 1998.

[6] R. M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, *The place of super edge-magic labeling among other classes of labeling*, Discrete Math., **231**, 153–168, 2001.

[7] R. M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, *On the super edge-magic deficiency of graphs*, Electron. Notes Discrete Math., **11**, 2002.

[8] R. M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, *On the super edge-magic deficiency of graphs*, Ars Combin., **78**, 2006.

[9] A. Kotzig and A. Rosa, *Magic valuaton of finite graphs*, Canad. Math. Bull., **13(4)**, 451–461, 1970.

[10] A. A. G. Ngurah, R. Simanjuntak and E.T. Baskoro, *On the super edge-magic deficiencies of graphs*, Australas. J. Combin., **40**, 3–14, 2008.

[11] J.Y. Park, J.H. Choi and J.H. Bae, *On super edge-magic labeling of some graphs*, Bull. Korean Math. Soc., **45**, No.1, 11–21, 2008.