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UNITS OF THE GROUP ALGEBRA OF THE GROUP $C_n \times D_6$ OVER ANY FINITE FIELD OF CHARACTERISTIC 3

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Dedicated to the memory of Professor John Clark

Abstract. In this paper, we establish the structure of the unit group of the group algebra $\mathbb{F}_{3^t}(C_n \times D_6)$ for $n \geq 1$.

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1. Introduction

Let KG denote the group algebra of the group G over the field K. Let $\mathcal{U}(KG)$ be the set of invertible elements of KG. The homomorphism $\varepsilon : KG \longrightarrow K$ given by $\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$ is called the augmentation mapping of KG. It is a well known fact that $\mathcal{U}(KG) \cong \mathcal{U}(K) \times V(KG)$ where $V(KG) = \{u \in \mathcal{U}(KG) \,|\, \varepsilon(u) = 1\}$.

Let G be a finite p-group and K a field of characteristic p, it is well known that $|V(KG)| = |K|^{|G|-1}$. Sandling in [8], provides a basis for $V(\mathbb{F}_p G)$ where G is an abelian p-group and $\mathbb{F}_p$ is the Galois field of p elements. In [10], it is shown that $Z(V_1)$ and $V_1/Z(V_1)$ are elementary abelian 3-groups where $V_1 = 1 + J(\mathbb{F}_{3^k} D_6)$, $J(\mathbb{F}_{3^k} D_6)$ is the Jacobson radical of $\mathbb{F}_{3^k} D_6$ and $Z(V_1)$ is the center of V_1. The structure of $\mathcal{U}(\mathbb{F}_{3^k} D_6)$ was determined in terms of split extensions of elementary abelian groups in [4]. The structure of FA_4 and FS_4 were established in [7,9] where F is any finite field, A_4 is the alternating group of degree 4 and S_4 is the symmetric group of degree 4. Additionally, the structure of $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_6))$ and $\mathcal{U}(\mathbb{F}_{3^k} D_{12})$ was established in [5,6] respectively. Consult [1] for an overview of modular group algebras.

The map $*: KG \longrightarrow KG$ defined by $\left(\sum_{g\in G} a_g g\right)^* = \sum_{g\in G} a_g g^{-1}$ is an antiautomorphism of KG of order 2. An element v of $V(KG)$ satisfying $v^{-1} = v^*$ is called unitary. We denote by $V_*(KG)$ the subgroup of $V(KG)$ formed by the unitary elements of KG. In [3] a basis for $V_*(KG)$ is constructed for any field of characteristic $p > 2$ and any finite abelian p-group. Additionally the order of $V_*(\mathbb{F}_{2^k}G)$ is determined for special cases of G in [2]. Let $\hat{g} = \sum_{h\in\langle g\rangle} h \in RG$. Our main results are:

Theorem 1.1.

$$\mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6)) \cong (C_3^{3nt} \rtimes C_3^{nt}) \rtimes \mathcal{U}(\mathbb{F}_{3^t}(C_n \times C_2)).$$

Corollary 1.2.

$$\mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6)) \cong \begin{cases} (C_3^{3nt} \rtimes C_3^{nt}) \rtimes C_{3^t-1}^{2n} & \text{if } n|(3^t-1) \\ (C_3^{3nt} \rtimes C_3^{nt}) \rtimes \left(C_3^{2f_1(V)} \times C_{3^2}^{2f_2(V)} \times \cdots \times C_{3^m}^{2f_m(V)} \times C_{3^m-1}^2\right) & \text{if } n = 3^m \end{cases}$$

where $f_i(V) = t(|{C_{3^m}}^{3^{i-1}}| - 2|{C_{3^m}}^{3^i}| + |{C_{3^m}}^{3^{i+1}}|)$.

2. The structure of $\mathcal{U}(\mathbb{F}_{3^k}(C_n \times D_6))$

Let $G = C_n \times D_6 = \langle x, y, z \,|\, x^3 = y^2 = z^n = 1,\, x^y = x^{-1},\, xz = zx,\, yz = zy\rangle$ where $n \geq 1$. The natural group homomorphism $G \longrightarrow G/\langle x\rangle$ extends linearly to the algebra homomorphism $\theta : \mathbb{F}_{3^t}(C_n \times D_6) \longrightarrow \mathbb{F}_{3^t}(C_n \times C_2)$ where

$$\sum_{i=1}^{3} x^{i-1}(\alpha_i + \alpha_{i+3}z + \cdots + \alpha_{i+3n}z^{n-1} + \alpha_{i+3n+3}y + \alpha_{i+3n+6}yz + \cdots + \alpha_{i+6n}yz^{n-1}) \mapsto$$
$$\sum_{i=1}^{3}(\alpha_i + \alpha_{i+3}b + \cdots + \alpha_{i+3n}b^{n-1} + \alpha_{i+3n+3}a + \alpha_{i+3n+6}ab + \cdots + \alpha_{i+6n}ab^{n-1})$$

and $C_n \times C_2 = \langle a, b \,|\, a^2 = b^n = 1,\, ab = ba\rangle$. If we restrict θ to $\mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6))$, we can construct the group epimorphism $\theta' : \mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6)) \longrightarrow \mathcal{U}(\mathbb{F}_{3^t}(C_n \times C_2))$. Consider the group homomorphism $\psi : \mathcal{U}(\mathbb{F}_{3^t}(C_n \times C_2)) \longrightarrow \mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6))$ by

$$\gamma_1 + \gamma_2 b + \cdots + \gamma_n b^{n-1} + \delta_1 a + \delta_2 ab + \cdots + \delta_n ab^{n-1} \mapsto$$
$$\gamma_1 + \gamma_2 z + \cdots + \gamma_n z^{n-1} + \delta_1 y + \delta_2 yz + \cdots + \delta_n yz^{n-1}$$

where $\gamma_i, \delta_j \in \mathbb{F}_{3^t}$. Clearly $\theta' \circ \psi$ is the identity map of $\mathcal{U}(\mathbb{F}_{3^k}(C_n \times C_2))$. Therefore $\mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6))$ is a split extension of $\mathcal{U}(\mathbb{F}_{3^t}(C_n \times C_2))$ by $ker(\theta')$ and $\mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6)) \cong H \rtimes \mathcal{U}(\mathbb{F}_{3^t}(C_n \times C_2))$ where $H \cong ker(\theta')$. Now, $\theta : \mathbb{F}_{3^t}(C_n \times D_6) \longrightarrow \mathbb{F}_{3^t}((C_n \times D_6)/\langle x\rangle) \cong \mathbb{F}_{3^t}(C_n \times D_6)/\mathcal{J}(\langle x\rangle)$ where $\mathcal{J}(\langle x\rangle)$ is the ideal of $\mathbb{F}_{3^t}(C_n \times D_6)$ generated by all $x - 1$ where $x \in \langle x\rangle$. Additionally, $\theta' : \mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6)) \longrightarrow$

$\mathcal{U}(\mathbb{F}_{3^t}((C_n \times D_6)/\langle x \rangle)) \cong \mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6))/1 + \mathcal{J}(\langle x \rangle)$. As the characteristic of $\mathbb{F}_{3^t}$ is 3 and x is of order 3, $\mathcal{J}(\langle x \rangle)$ is nilpotent of index 3. Therefore H has exponent 3.

Lemma 2.1. *$C_H(x) \cong C_3^{3nt}$ where $C_H(x) = \{h \in H \mid xh = hx\}$.*

Proof. Let $h = 1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{k=1}^{n} \mathfrak{B}_k y \in H$ where

$$\mathfrak{A}_j = \sum_{i=1}^{2} \alpha_{i+2(j-1)} z^{j-1}(x^i - 1) \text{ and } \mathfrak{B}_k = \sum_{i=1}^{2} \alpha_{i+2(k+n-1)} z^{k-1}(x^i - 1)$$

and $\alpha_j \in \mathbb{F}_{3^t}$. Now

$$\begin{aligned} xh - hx &= x\left(1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{k=1}^{n} \mathfrak{B}_k y\right) - \left(1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{k=1}^{n} \mathfrak{B}_k y\right) x \\ &= x\left(\sum_{k=1}^{n} \mathfrak{B}_k y\right) - \left(\sum_{k=1}^{n} \mathfrak{B}_k y\right) x. \end{aligned}$$

Now,

$$\begin{aligned} x\mathfrak{B}_k y - \mathfrak{B}_k y x &= z^{k-1}[(\alpha_{2k+2n-1}(x^2 - x) + \alpha_{2k+2n}(1 - x)) - (\alpha_{2k+2n-1}(1 - x^2) + \alpha_{2k+2n}(x - x^2))]y \\ &= \hat{x} y z^{k-1}(\alpha_{2k+2n} - \alpha_{2k+2n-1}). \end{aligned}$$

Therefore, every element of $C_H(x)$ takes the form

$$1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{l=1}^{n} \alpha_{l+2n} \hat{x} y z^{l-1}$$

where $\mathfrak{A}_j = \sum_{i=1}^{2} \alpha_{i+2(j-1)} z^{j-1}(x^i - 1)$ and $\alpha_i \in \mathbb{F}_{2^t}$. Clearly $(\hat{x})^2 = 3\hat{x} = 0$ and $\hat{x}\mathfrak{A}_j = \mathfrak{A}_j \hat{x}$. Therefore $C_H(x)$ is an abelian group of order $3^{2nt} \cdot 3^{nt} = 3^{3nt}$. □

Next, consider a subset S of H where the elements of S take the form:

$$1 + \sum_{j=1}^{n} \mathfrak{R}_j$$

where $\mathfrak{R}_j = \sum_{i=1}^{2} i r_j x^i (1 + y) z^{j-1}$ and $r_i \in \mathbb{F}_{3^t}$.

Lemma 2.2. *$S \cong C_3^{nt}$.*

Proof. Let $s_1 = 1 + \sum_{j=1}^{n} \mathfrak{R}_j \in S$ and $s_2 = 1 + \sum_{j=1}^{n} \mathfrak{T}_j \in S$ where

$\mathfrak{R}_j = \sum_{i=1}^{2} i r_j x^i (1+y) z^{j-1}$, $\mathfrak{T}_j = \sum_{i=1}^{2} i t_j x^i (1+y) z^{j-1}$ and $r_i, t_j \in \mathbb{F}_{3^t}$. Now

$$\begin{aligned} s_1 s_2 &= \left(1 + \sum_{j=1}^{n} \mathfrak{R}_j\right)\left(1 + \sum_{j=1}^{n} \mathfrak{T}_j\right) \\ &= 1 + \sum_{j=1}^{n} (\mathfrak{R}_j + \mathfrak{T}_j) + \left(\sum_{j=1}^{n} \mathfrak{R}_j\right)\left(\sum_{j=1}^{n} \mathfrak{T}_j\right) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{R}_j \mathfrak{T}_k &= \left(\sum_{i=1}^{2} i r_j x^i (1+y) z^{j-1}\right)\left(\sum_{i=1}^{2} i t_k x^i (1+y) z^{k-1}\right) \\ &= (r_j x + r_j xy + 2 r_j x^2 + 2 r_j x^2 y)(t_k x + t_k xy + 2 t_k x^2 + 2 t_k x^2 y) z^{j+k-2} \\ &= \sum_{i=1}^{3} (12 - 3i) r_j t_k x^{i-1} (1+y) z^{j+k-2} \\ &= 0. \end{aligned}$$

Clearly $s_1 s_2 \in S$ and S is abelian, therefore $S \cong C_3^{nt}$. □

Theorem 2.3.

$$\mathcal{U}(\mathbb{F}_{3^t}(C_n \times D_6)) \cong (C_3^{3nt} \rtimes C_3^{nt}) \rtimes \mathcal{U}(\mathbb{F}_{3^t}(C_n \times C_2)).$$

Proof. Let $c = 1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{l=1}^{n} \alpha_{l+2n} \hat{x} y z^{l-1} \in C_H(x)$ and $s = 1 + \sum_{j=1}^{n} \mathfrak{R}_j \in S$ where $\mathfrak{A}_j = \sum_{i=1}^{2} \alpha_{i+2(j-1)} z^{j-1} (x^i - 1)$, $\mathfrak{R}_j = \sum_{i=1}^{2} i r_j x^i (1+y) z^{j-1}$ and $\alpha_i, r_j \in \mathbb{F}_{3^t}$. Now

$$\begin{aligned} c^s &= s^2 c s \\ &= \left(1 + \sum_{j=1}^{n} \mathfrak{R}_j\right)^2 \left(1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{l=1}^{n} \alpha_{l+2n} \hat{x} y z^{l-1}\right)\left(1 + \sum_{j=1}^{n} \mathfrak{R}_j\right) \\ &= \left(1 + 2\sum_{j=1}^{n} \mathfrak{R}_j\right)\left(1 + \sum_{j=1}^{n} \mathfrak{A}_j + \sum_{l=1}^{n} \alpha_{l+2n} \hat{x} y z^{l-1}\right)\left(1 + \sum_{j=1}^{n} \mathfrak{R}_j\right). \end{aligned}$$

Now ${\mathfrak{R}_j}^2 = 0$ and $\hat{x}\mathfrak{R}_j = 3\hat{x}r_j(1+y)z^{j-1} = 0 = \mathfrak{R}_j\hat{x}$, therefore

$$c^s = 1 + \sum_{j=1}^{n}\mathfrak{A}_j + \sum_{l=1}^{n}\alpha_{l+2n}\hat{x}yz^{l-1} + 2\left(\sum_{j=1}^{n}\mathfrak{R}_j\right)\left(\sum_{j=1}^{n}\mathfrak{A}_j\right) + \left(\sum_{j=1}^{n}\mathfrak{A}_j\right)\left(\sum_{j=1}^{n}\mathfrak{R}_j\right)$$
$$+ 2\left(\sum_{j=1}^{n}\mathfrak{R}_j\right)\left(\sum_{j=1}^{n}\mathfrak{A}_j\right)\left(\sum_{j=1}^{n}\mathfrak{R}_j\right).$$

Now, $\mathfrak{R}_j\mathfrak{A}_k = r_j(\alpha_{2k}-\alpha_{2k-1})\hat{x}(1-y)z^{j+k-2}$, $\mathfrak{A}_k\mathfrak{R}_j = r_j(\alpha_{2k}-\alpha_{2k-1})\hat{x}(1+y)z^{j+k-2}$ and

$$\begin{aligned} 2\mathfrak{R}_j\mathfrak{A}_k + \mathfrak{A}_k\mathfrak{R}_j &= r_j(\alpha_{2k}-\alpha_{2k-1})\hat{x}[2(1-y)+(1+y)]z^{j+k-2} \\ &= r_j(\alpha_{2k-1}-\alpha_{2k})\hat{x}yz^{j+k-2}. \end{aligned}$$

Additionally, $\mathfrak{R}_j\mathfrak{A}_k\mathfrak{R}_l = 0$ since $\hat{x}\mathfrak{R}_j = 0$. Therefore $c^s \in C_H(x)$ and consequently $C_H(x)$ is a normal subgroup of H. Note that $|H| = 3^{4nt}$ and that $C_H(x)\cap S = \{1\}$. By the Second Isomorphism Theorem, $H = C_H(x).S$. Thus, $H \cong C_H(x) \rtimes S \cong C_3^{3nt} \rtimes C_3^{nt}$. □

Corollary 2.4.

$$\mathcal{U}(\mathbb{F}_{3^t}(C_n\times D_6)) \cong \begin{cases} (C_3^{3nt}\rtimes C_3^{nt})\rtimes C_{3^t-1}^{2n} & \text{if } n|(3^t-1) \\ (C_3^{3nt}\rtimes C_3^{nt})\rtimes\left(C_3^{2f_1(V)}\times C_{3^2}^{2f_2(V)}\times\cdots\times C_{3^m}^{2f_m(V)}\times C_{3^m-1}^2\right) & \text{if } n=3^m \end{cases}$$

where $f_i(V) = t(|{C_{3^m}}^{3^{i-1}}| - 2|{C_{3^m}}^{3^i}| + |{C_{3^m}}^{3^{i+1}}|)$.

Proof. It is well known that $\mathbb{F}_{3^t}(C_2\times C_n) \cong (\mathbb{F}_{3^t}C_2)C_n \cong (\mathbb{F}_{3^t}\oplus\mathbb{F}_{3^t})C_n \cong \mathbb{F}_{3^t}C_n \oplus \mathbb{F}_{3^t}C_n$. It is well known that if $n|(3^t-1)$, then $\mathbb{F}_{3^t}C_n \cong \oplus_{i=1}^n\mathbb{F}_{3^t}$. Therefore $\mathcal{U}(\mathbb{F}_{3^t}(C_2\times C_n)) \cong C_{3^t-1}^{2n}$ when $n|(3^t-1)$. When $n=3^m$, the number of cyclic groups $f_i(V)$ of order 3^i in the direct product of $V(\mathbb{F}_{3^t}G)$ is $f_i(V) = t(|{C_{3^m}}^{3^{i-1}}| - 2|{C_{3^m}}^{3^i}| + |{C_{3^m}}^{3^{i+1}}|)$ ([8]). □

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