International Electronic Journal of Algebra
Volume 24 (2018) 129-152
DOI: 10.24330/ieja.440241

SOME STUDIES ON GZI RINGS

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Received: 7 December 2017; Accepted: 21 February 2018
Communicated by Burcu Üngör

Dedicated to the memory of Professor John Clark

Abstract. A ring R is called generalized ZI (or GZI for short) if for any $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRba = 0$, which is a proper generalization of ZI rings. In this paper, many properties of GZI rings are introduced, some known results are extended. Further, we introduce generalized GZI rings as a generalization of GZI rings, and quasi-abel rings as a generalization of generalized GZI rings. Some important results on *Abel* rings are extended to generalized GZI rings and quasi-abel rings.

Mathematics Subject Classification (2010): 16D80
Keywords: ZI ring, GZI ring, quasi-abel ring, generalized GZI ring, reduced ring, *Abel* ring

1. Introduction

All rings considered in this paper are associative with identity, and all modules are unital. Let R be a ring, write $J(R)$, $E(R)$, $Z(R)$, $U(R)$ and $N(R)$ denote the Jacobson radical, the set of all idempotents, the center, the set of all units and the set of all nilpotents of R, respectively. For any nonempty subset X of R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of X and the set of left annihilators of X, respectively. Especially, if $X = a$, we write $l(X) = l(a)$ and $r(X) = r(a)$.

Recall that a ring R is *zero commutative* [11] if R satisfies the condition: $ab = 0$ implies $ba = 0$ for $a, b \in R$, while Cohn [6] used the term reversible for what is called zero commutative. A generalization of a reversible ring is a ZI ring. A ring R is ZI if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Historically, some of the earliest results known to us about ZI rings was due to Shin [15]. He showed that a ring R is ZI if and only if $r_R(a)$ is an ideal of R for each $a \in R$. In [4], ZI property is called the insertion-of-factors property, or IFP. In [12], Mohammadi, Moussavi and Zahiri introduce nil-semicommutative rings (that is, $ab = 0$ implies $aRb = 0$ for any $a, b \in N(R)$) as a generalization of ZI rings. The other studies of ZI rings also can be found in [2,3].

Supported by the National Natural Science Foundation of China (No.11471282).

In this note, we call a ring R a *generalized ZI ring* (or, *GZI ring* for short) if $ab = 0$ implies $aRba = 0$ for each $a \in N(R)$ and $b \in R$. Clearly, ZI rings are GZI, but the converse is not true by Example 2.2. By Theorem 2.3 and Proposition 2.9, we constructed a lot of GZI rings which are not ZI. By Proposition 2.10 and Corollary 2.13, we know that GZI rings inherit many properties of ZI rings.

A ring R is called a *generalized* GZI ring if $ae = 0$ implies $aRea = 0$ for each $a \in N(R)$ and $e \in E(R)$. Example 2.6 implies that generalized GZI rings are proper generalization of GZI rings. In fact, generalized GZI rings are also proper generalization of quasi-normal rings by Proposition 2.3(3) and [21, P1858]. Theorem 3.7 shows that a ring R is a quasi-normal ring if and only if $V_2(R) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in R\}$ is a generalized GZI ring. Theorem 3.3 shows that R is an *Abel* ring if and only if $T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is a quasi-normal ring.

A ring R is called *quasi-abel* if $ea(1-e)Rea(1-e) = 0$ for each $e \in E(R)$ and $a \in R$. Proposition 3.11 points out that quasi-abel rings are proper generalization of generalized GZI rings. Some characterizations of quasi-abel rings are given by Propositions 4.1, 4.2 and 4.3. In fact, in Section 4, many properties of quasi-normal rings appeared in [21] are extended to quasi-abel rings.

2. Some examples of GZI rings

Definition 2.1. A ring R is called *generalized ZI ring* (or, *GZI ring* for short) if for each $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRba = 0$.

Clearly, ZI rings are GZI. But the following example illustrates that the converse is not true in general.

Example 2.2. *Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, the upper triangular matrix ring over F. Then $N(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is an ideal of R with $N(R)^2 = 0$, this implies that for each $A \in N(R)$ and $B \in R$, $ARBA = 0$. Hence R is GZI, but R is not ZI.*

Example 2.2 inspires us to think about the following problems.

(1) If R be a commutative ring or reduced ring, is the 2×2 upper triangular matrix ring $T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ over R GZI?

(2) Let R be a field and $n \geq 3$ a positive integer. Is the $n \times n$ upper triangular matrix ring $T_n(R) = \begin{pmatrix} R & R & R & \cdots & R \\ 0 & R & R & \cdots & R \\ 0 & 0 & R & \cdots & R \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & R \end{pmatrix}$ over R GZI?

Proposition 2.3. (1) *If R is a commutative ring, then $T_2(R)$ is GZI.*

(2) *If R is a reduced ring, then $T_2(R)$ is GZI.*

(3) *R is a reduced ring if and only if $T_3(R)$ is a GZI ring.*

(4) *Nil-semicommutative rings are GZI.*

(5) *If $T_2(R)$ is a GZI ring, then R is nil-semicommutative.*

Proof. (1) Assume that $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in N(T_2(R))$ and $B = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T_2(R)$ with $AB = 0$. Then

$$ax = 0 \tag{2.1}$$

$$cz = 0 \tag{2.2}$$

$$ay + bz = 0 \tag{2.3}$$

Now let $C = \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in T_2(R)$. Then

$ACBA = \begin{pmatrix} auxa & auxb + auyc + avzc + bwzc \\ 0 & cwzc \end{pmatrix}$.

Since R is commutative, by (2.1) $\sim$ (2.3), one gets

$$auxa = axua = 0 \tag{2.4}$$

$$cwzc = czwc = 0 \tag{2.5}$$

$$auxb = axub = 0 \tag{2.6}$$

$$avzc = czav = 0 \tag{2.7}$$

$$bwzc = bwcz = 0 \tag{2.8}$$

$$0 = uc(ay + bz) = auyc + ucbz = ubcz + auyc = auyc \tag{2.9}$$

all these imply that $ACBA = 0$. Thus $AT_2(R)BA = 0$ and so $T_2(R)$ is GZI.

(2) It is trivial.

(3) First we assume that $A = \begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \in N(T_3(R)) = \begin{pmatrix} 0 & R & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in T_3(R)$ with $AB = 0$. Then

$$a_1 b_4 = 0 \tag{2.10}$$

$$a_1 b_5 + a_2 b_6 = 0 \tag{2.11}$$

$$a_3 b_6 = 0 \tag{2.12}$$

Now let $C = \begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 \\ 0 & 0 & x_6 \end{pmatrix} \in T_3(R)$. Then $ACBA = \begin{pmatrix} 0 & 0 & a_1x_4b_4a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
Since R is reduced, by (2.10), $a_1Rb_4 = 0$, which implies $a_1x_4b_4a_3 = 0$, one gets $ACBA = 0$. Thus $T_3(R)$ is GZI.

Next we assume that $T_3(R)$ is GZI and $a \in R$ with $a^2 = 0$. Then we choose $A = \begin{pmatrix} a & 1 & 1 \\ 0 & a & 1 \\ 0 & 0 & 0 \end{pmatrix} \in N(T_3(R)), B = \begin{pmatrix} 0 & -1 & 1 \\ 0 & a & -a \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R)$. Since $T_3(R)$ is GZI and $AB = 0$, $AT_3(R)BA = 0$. Choose $C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R)$. Then $ACBA = 0$, this implies $\begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, so $a = 0$. Hence R is reduced.

(4) Assume that $a \in N(R)$ and $b \in R$ such that $ab = 0$. Then $ba \in N(R)$ and $a(ba) = 0$. Since R is nil-semicommutative, $aRba = 0$, this shows that R is GZI.

(5) Assume that $a \in N(R)$ and $x \in R$ such that $ax = 0$. Choose $A = \begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix} \in N(T_2(R))$ and $B = \begin{pmatrix} x & -1 \\ 0 & a \end{pmatrix} \in T_2(R)$. By computing, we have $AB = 0$. Since $T_2(R)$ is a GZI ring, $A \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} BA = 0$ for all $r \in R$, this gives $arx = 0$. Hence $aRx = 0$ and so R is nil-semicommutative. □

The following example illustrates that if R is only a GZI ring, then $T_2(R)$ need not be GZI.

Example 2.4. *Let* $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. *Then by Proposition 2.3(1), R is GZI. Let*

$A = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$. *Then* $A \in N(T_2(R))$. *Let*

$B = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$ *and* $C = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$. *Then*

$AB = 0$ *and* $ACBA = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0$. *Thus* $T_2(R)$ *is not GZI.*

Remark 2.5. *It is well known that ZI rings are Abel, but paying attention to the ring R appeared in Example 2.2 is not Abel, one knows that GZI rings need not be Abel. The following example also illustrates that Abel rings need not be GZI.*

Example 2.6. *Let* $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}, a \equiv d (mod 2), b \equiv c \equiv 0 (mod 2)\}$. *Since* $E(R) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$, *R is Abel. Now let* $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in N(R)$ *and* $B = \begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$. *Then by computing, we have* $AB = 0$ *and* $ACBA = \begin{pmatrix} 0 & 16 \\ 0 & 0 \end{pmatrix} \neq 0$. *Thus R is not GZI.*

Example 2.7. *Let* $R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{Z}_2\}$. *Then R is commutative. Let*

$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in N(T_3(R))$,

$$B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$C = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in T_3(R).$$ *Then* $AB = 0$ *and*

$$ACBA = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$ *Thus* $T_3(R)$ *is not* GZI.

Example 2.7 illustrates that for a commutative ring R, $T_3(R)$ need not be GZI.

Example 2.8. *Let* F *be a field and* $R = T_4(F)$. *Choose* $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in N(R)$ *and* $B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$. *Then* $AB = 0$ *and*

$$ACBA = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0,$$ *so* R *is not* GZI.

Example 2.8 illustrates that for a field F, $T_4(F)$ need not be GZI.

Let R be a ring and $V_4(R) = \{\begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} | a, a_{ij} \in R\}$. Clearly, $V_4(R)$ is a subring of $T_4(R)$. The following proposition implies the converse of Proposition 2.3(4) is not true.

Proposition 2.9. *Let F be a field. Then $R = V_4(F)$ is a GZI ring, while R is not nil-semicommutative.*

Proof. Assume that $A = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in N(R) = \begin{pmatrix} 0 & F & F & F \\ 0 & 0 & F & F \\ 0 & 0 & 0 & F \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_1 & b_5 & b_6 \\ 0 & 0 & b_1 & b_7 \\ 0 & 0 & 0 & b_1 \end{pmatrix}, C = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ 0 & c_1 & c_5 & c_6 \\ 0 & 0 & c_1 & c_7 \\ 0 & 0 & 0 & c_1 \end{pmatrix} \in R$ with $AB = 0$. Then

$$a_1b_1 = 0 \tag{2.13}$$

$$a_1b_5 + a_2b_1 = 0 \tag{2.14}$$

$$a_1b_6 + a_2b_7 + a_3b_1 = 0 \tag{2.15}$$

$$a_4b_1 = 0 \tag{2.16}$$

$$a_4b_7 + a_5b_1 = 0 \tag{2.17}$$

$$a_6b_1 = 0 \tag{2.18}$$

and

$$ACBA = \begin{pmatrix} 0 & 0 & a_1c_1b_1a_4 & a_1c_1b_1a_5 + a_1c_1b_5a_6 + a_1c_5b_1a_6 + a_2c_1b_1a_6 \\ 0 & 0 & 0 & a_4c_1b_1a_6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ Clearly,}$$

$$a_1c_1b_1a_4 = (a_1b_1)(c_1a_4) = 0 \tag{2.19}$$

$$a_4c_1b_1a_6 = (a_4b_1)(c_1a_6) = 0 \tag{2.20}$$

$$a_1c_1b_1a_5 = (a_1b_1)(c_1a_5) = 0 \tag{2.21}$$

$$a_1c_5b_1a_6 = (a_1b_1)(c_5a_6) = 0 \tag{2.22}$$

$$a_1c_1b_5a_6 + a_2c_1b_1a_6 = (a_1b_5 + a_2b_1)c_1a_6 = 0 \tag{2.23}$$

Thus $ACBA = 0$ and so $R = V_4(F)$ is GZI.

Now choose $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in N(R)$ and $C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$. Then $AB = 0$ while $ACB = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$, this shows that R is not nil-semicommutative. □

Let R be a ring and write $ME_l(R) = \{e \in E(R) |\ Re$ is a minimal left ideal of $R\}$. A ring R is called *left min-abel* if every element of $ME_l(R)$ is left semicentral in R, a ring R is said to be *strongly left min-abel* if every element of $ME_l(R)$ is central, and a ring R is said to be *left* $MC2$ if $aRe = 0$ implies $eRa = 0$ for each $e \in ME_l(R)$ and $a \in R$. [18, Theorem 1.8] showed that R is a strongly left min-abel ring if and only if R is a left min-abel left $MC2$ ring. A ring R is called *left quasi-duo* if every maximal left ideal of R is ideal, and R is said to be *MELT* if every essential maximal left ideal of R is an ideal. Clearly, left quasi-duo rings are $MELT$. In [18, Theorem 1.2], it is shown that a ring R is a left quasi-duo ring if and only if R is a left min-abel $MELT$ ring. Recall that a ring R is *directly finite* if

$ab = 1$ implies $ba = 1$, and R is said to be *NCI* if either $N(R) = 0$ or $N(R)$ contains a nonzero ideal of R. By [9, Example 1.2], one knows that NCI rings need not be directly finite. Hence the following proposition implies NCI rings need not be GZI.

Proposition 2.10. *If R is a GZI ring, then*

(1) *R is NCI;*
(2) *R is directly finite;*
(3) *R is left min-abel;*
(4) *R is left $MC2$ if and only if R is strongly left min-abel;*
(5) *R is left quasi-duo if and only if R is $MELT$.*

Proof. (1) If $N(R) = 0$, we are done. Now assume that $N(R) \neq 0$. Then there exists $0 \neq a \in N(R)$ such that $a^2 = 0$. Since R is GZI and $a(ar) = 0$, $aR(ar)a = 0$ for each $r \in R$, this gives $aRaRa = 0$ and $(RaR)^3 = 0$. Thus R is NCI.

(2) Let $ab = 1$ and write $e = ba$. Then $ae = a$ and $eb = b$. Let $h = a - ea$. Then $he = h$, $eh = 0$ and $h^2 = 0$. By the proof of (1), one has $hRhRh = 0$, this gives $hbhbh = 0$. Since $hb = 1 - e$, $hbhbh = (1 - e)h = h$. Thus $h = 0$ and $a = ea$, this leads to $1 = ab = eab = e = ba$. Hence R is directly finite.

(3) Let $e \in ME_l(R)$ and $a \in R$. If $h = ae - eae \neq 0$, then $Rh = Re$ and $h^2 = 0$. Let $e = ch$ for some $c \in R$. Then $h = he = hch$. By the proof of (1), one has $hRhRh = 0$, this gives $Re = (Re)^3 = (Rh)^3 = 0$, which is a contradiction. Hence $h = 0$ and $ae = eae$ for each $a \in R$, this implies R is left min-abel.

(4) and (5) are immediate results of (1) and [18, Theorem 1.2 and Theorem 1.8]. □

Since *Abel* rings are directly finite and left min-abel, Example 2.6 illustrates that neither directly finite rings nor left min-abel rings need be GZI.

Example 2.11. *Let F be a field and $R = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix}$. Then $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in ME_l(R)$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} Re = 0$, but $e \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$. Thus R is not left $MC2$. By Example 2.1, one knows that R is GZI. Hence GZI rings need not be left $MC2$, and so GZI rings need not be strongly left min-abel.*

Corollary 2.12. *Let R be a GZI ring and $e \in ME_l(R)$. Then ${}_RRe$ is injective if and only if $aRe = 0$ implies $eRa = 0$ for each $a \in R$.*

Proof. First we assume that $aRe = 0$ implies $eRa = 0$ for each $a \in R$. Since R is GZI, R is left min-abel by Proposition 2.10(3), this implies $(1 - e)Re = 0$, by hypothesis, $eR(1 - e) = 0$. Hence e is central in R. By [22, Lemma 2.2], ${}_RRe$ is injective.

Conversely, assume that $aRe = 0$. If $eRa \neq 0$, then there exists $b \in R$ such that $eba \neq 0$. Since $l(e) = l(eba)$, ${}_RReba \cong_R Re$. Since ${}_RRe$ is injective, ${}_RReba$ is injective, this leads to $Reba = Rg$ for some $g \in E(R)$. Thus $Reba = (Reba)^2 = 0$, which is a contradiction. Hence $eRa = 0$. □

It is well known that a ring R is a reduced ring if and only if R is a semiprime ZI ring. By the proof of Proposition 2.10(1), one has the following corollary.

Corollary 2.13. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced ring;*
(2) *R is a semiprime nil-semicommutative ring;*
(3) *R is a semiprime GZI ring.*

The following proposition is a direct result of the definition of GZI ring.

Proposition 2.14. (1) *Every subring of GZI rings is GZI;*
(2) *If R is a GZI ring and $e \in E(R)$, then eRe is GZI.*

Recall that an ideal I of a ring R is reduced if $N(R) \cap I = 0$. With the help of reduced ideal, one has the following proposition.

Proposition 2.15. *Let I be a reduced ideal of R. If R/I is a GZI ring, then R is GZI.*

Proof. Let $a \in N(R)$ and $b \in R$ satisfy $ab = 0$. Then $\bar{a} \in N(\bar{R})$ and $\bar{a}\bar{b} = \bar{0}$ where $\bar{R} = R/I$. Since $\bar{R}$ is GZI, $\bar{a}\bar{x}\bar{b}\bar{a} = \bar{0}$ for each $x \in R$, this gives $axba \in I$. Clearly, $(baxba)^2 = 0$ and $baxba \in I$. Since I is reduced, $baxba = 0$ for each $x \in R$. For each $y \in R$, $(ayba)^2 = (ay)(ba(ay)ba) = 0$, so $ayba = 0$ because $ayba \in I$. Thus $aRba = 0$ and R is GZI. □

A ring R is called *left WNV* if every singular simple left R-module is $Wnil$-injective ([19]). Clearly, left V-rings and reduced rings are left WNV. The following proposition generalizes [10, Lemma 3].

Proposition 2.16. *The following conditions are equivalent for a left $MC2$ ring R:*

(1) *R is a reduced ring;*
(2) *R is a ZI left WNV ring;*
(3) *R is a nil-semicommutative left WNV ring;*
(4) *R is a GZI left WNV ring.*

Proof. We only need to show (4) ⇒ (1). Let $a \in R$ with $a^2 = 0$. If $a \neq 0$, then there exists a maximal left ideal M of R containing $l(a)$. We claim that M is essential in ${}_RR$. If not, then $M = l(e)$ for some $e \in ME_l(R)$. Since R is GZI, R is strongly left min-abel by Proposition 2.10(4). Thus $e \in Z(R)$, this gives $ea = ae = 0$ because $a \in l(a) \subseteq M = l(e)$, so $e \in l(a) \subseteq l(e)$, which is a contradiction. Therefore M is essential in ${}_RR$, R/M is singular simple left

R-module, by (4), R/M is $Wnil$-injective. Clearly, the map $f : Ra \longrightarrow R/M$ defined by $f(ra) = r + M$ is a well-defined left R-homomorphism, this illustrates that there exists $c \in R$ such that $f(ra) = rac + M$ for each $r \in R$, especially, $1 + M = f(a) = ac + M$, so $1 - ac \in M$. Since R is GZI and $a^2 = 0$, by the proof of Proposition 2.10(1), $(aR)^3 = 0$, this implies $1 - ac \in U(R)$, which is a contradiction. Thus $a = 0$ and so R is reduced. □

A ring R is called biregular if for every $a \in R$, RaR is generated by a central idempotent of R. A ring R is called weakly regular if for any $a \in R$, $a \in RaRa \cap aRaR$. Clearly, biregular rings are weak regular, but the converse is not true, in general. Certainly, reduced weakly regular rings are biregular. In [10, Theorem 4], it is proved that if R is a ZI ring whose every singular simple left module is YJ-injective, then R is a reduced weakly regular ring. Hence, by Proposition 2.16, we have the following corollary.

Corollary 2.17. *Let R be a GZI ring. If every singular simple left R-module is YJ-injective, then R is a reduced biregular ring.*

In [17, Theorem 16], it is proved that R is a strongly regular ring if and only if R is a ZI $MELT$ ring whose singular simple left modules are YJ-injective. Hence, by Proposition 2.16, we have the following corollary.

Corollary 2.18. *R is a strongly regular ring if and only if R is a GZI $MELT$ ring whose singular simple left modules are YJ-injective.*

Evidently, the class of GZI rings is closed under subrings and direct product.

Proposition 2.19. *Let R be a ring and Δ a multiplicatively closed subset of R consisting of central regular elements. Then R is a GZI ring if and only if $\Delta^{-1}R$ is a quasi-semicommutative ring.*

Proof. The sufficiency is clear.

Now let $\alpha\beta = 0$ with $\alpha = u^{-1}a \in N(\Delta^{-1}R), \beta = v^{-1}b \in \Delta^{-1}R, u, v \in \Delta$ and $a, b \in R$. Since Δ is contained in the center of R, we have $0 = \alpha\beta = u^{-1}av^{-1}b = (u^{-1}v^{-1})ab = (uv)^{-1}ab$, $a \in N(R)$ and $ab = 0$. Since R is a GZI ring, $aRba = 0$. Hence $\alpha(\Delta^{-1}R)\beta\alpha = (u^{-1})^2v^{-1}\Delta^{-1}aRba = 0$, this shows that $\Delta^{-1}R$ is a GZI ring. □

The ring of Laurent polynomials in x, coefficients in a ring R, consists of all formal sums $\Sigma_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denote it by $R[x; x^{-1}]$.

Corollary 2.20. *For a ring R, $R[x]$ is a GZI ring if and only if $R[x; x^{-1}]$ is a GZI ring.*

Proof. It suffices to establish necessity. Let $\Delta = \{1, x, x^2, \cdots, x^n, \cdots\}$. Then, clearly, Δ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = \Delta^{-1}R[x]$

and Δ is contained in the center of $R[x]$, it follows that $R[x;x^{-1}]$ is a GZI ring by Proposition 2.19. □

Proposition 2.21. *Let R be a GZI ring and $f(x) = a + bx, g(x) = c + dx \in R[x]$. If $f(x)g(x) = 0$, then $ac, ad, bc, bd \in N(R)$.*

Proof. Since $f(x)g(x) = 0$, one obtains

$$ac = 0 \tag{2.24}$$

$$ad + bc = 0 \tag{2.25}$$

$$bd = 0 \tag{2.26}$$

Multiply (2.25) on the left by c and on the right by b, it follows that

$$cadb + (cb)^2 = 0 \tag{2.27}$$

By (2.24), one has $ca \in N(R)$ and $(ca)(cadb) = 0$, this gives $(ca)R(cadbca) = 0$, so $cadb \in N(R)$. By (2.27), one has $(cb)^8 = 0$. Hence $bc \in N(R)$. Again by (2.25), we have $ad \in N(R)$. □

3. Some generalizations of GZI rings

Definition 3.1. A ring R is called *generalized GZI* if $ae = 0$ implies $aRea = 0$ for each $a \in N(R)$ and $e \in E(R)$.

Clearly, GZI rings are generalized GZI. Since *Abel* rings are generalized GZI and *Abel* rings need not be GZI by Example 2.6, one knows that generalized GZI rings need not be GZI.

Recall that a ring R is *quasi-normal* if $ae = 0$ implies $eaRe = 0$ for each $a \in N(R)$ and $e \in E(R)$. In [21, Theorem 2.1], it is shown that a ring R is quasi-normal if and only if $eR(1 - e)Re = 0$ for each $e \in E(R)$.

Let F be a field and $R = T_3(F)$. Then [21, P1858] implies that R is not quasi-normal. But by Proposition 2.3(3), R is GZI, so R is generalized GZI. Hence generalized GZI rings need not be quasi-normal. But quasi-normal rings are generalized GZI. (In fact, if $a \in N(R)$ and $e \in E(R)$, with $ae = 0$, then $area = a(1 - e)rea(1 - e)$ for each $r \in R$. Since R is quasi-normal, $(1 - e)ReR(1 - e) = 0$, this gives $area = 0$. Thus R is generalized GZI.)

Proposition 3.2. *Let R be a ring. If $T_2(R)$ is a generalized GZI ring, then R is quasi-normal.*

Proof. Let $e \in E(R)$ and $a, b \in R$, write $h = ea(1 - e)$. Then $h^2 = 0, eh = h$, so $A = \begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix} \in N(T_2(R))$ and $B = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \in E(T_2(R))$ with $AB = 0$.

Choose $C = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in T_2(R)$. Since $T_2(R)$ is a generalized GZI ring, $ACBA = 0$, that is $\begin{pmatrix} hbh & hbe \\ 0 & 0 \end{pmatrix} = 0$, this gives $ea(1-e)be = hbe = 0$ for each $a, b \in R$. Hence $eR(1-e)Re = 0$ for each $e \in E(R)$ and so R is quasi-normal. □

Theorem 3.3. *A ring R is Abel if and only if $T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is quasi-normal.*

Proof. First, we assume that R is *Abel* and $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in E(T_2(R))$. Then

$$a^2 = a \tag{3.1}$$

$$c^2 = c \tag{3.2}$$

$$b = ab + bc \tag{3.3}$$

Now for any $B = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, C = \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in T_2(R)$, one has $AB(1-A)CA =$

$$\begin{pmatrix} ax(1-a)ua & ax(1-a)ub + ax(1-a)vc - axbwc + ay(1-c)wc + bz(1-c)wc \\ 0 & cz(1-c)wc \end{pmatrix}$$

Since R is *Abel*, (3.1), (3.2) and (3.3) imply $a, c \in Z(R)$. Hence

$$ax(1-a)ua = ax(1-a)ub = ax(1-a)vc = 0 \tag{3.4}$$

$$cz(1-c)wc = ay(1-c)wc = bz(1-c)wc = 0 \tag{3.5}$$

By (3.3), one gets

$$axbwc = ax(ab + bc)wc = axabwc + axbcwc = axbwc + axbwc \tag{3.6}$$

this gives

$$axbwc = 0 \tag{3.7}$$

Thus $AB(1-A)CA = 0$ and so $T_2(R)$ is quasi-normal.

Conversely, assume that $T_2(R)$ is quasi-normal and $e \in E(R)$. Then $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \in E(T_2(R))$, so for each $x \in R$, one has

$$\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = 0$$

that is, $\begin{pmatrix} 0 & ex(1-e) \\ 0 & 0 \end{pmatrix} = 0$. Thus $ex(1-e) = 0$ for each $x \in R$, this implies R is *Abel*. □

Corollary 3.4. *If R is an Abel ring, then $T_2(R)$ is generalized GZI.*

If R is a quasi-normal ring, is $T_2(R)$ generalized GZI?

Lemma 3.5. *Let R be a generalized GZI ring and $a \in R$. If $a \in aRa$, then $a \in Ra^2$.*

Proof. Assume that $a = aba$ for some $b \in R$ and write $e = ba$. Then $a = ae$ and $e \in E(R)$. Let $h = a - ea$. Then $he = h$, $eh = 0$ and $h^2 = 0$. Since R is a generalized GZI ring and $h(1-e) = 0$, $hR(1-e)h = 0$, this gives $hbh = hb(1-e)h = 0$. Since $bh = e - bea$, $0 = hbh = h - hbea$, one has $h = hbea$, this leads to $a = h + ea = (hb+1)ea \in Ra^2$. □

Recall that a ring R is
n-regular if $a \in aRa$ for each $a \in N(R)$ ([19]);
Von Neumann regular if $a \in aRa$ for each $a \in R$;
strongly regular if $a \in a^2R \cap Ra^2$ for each $a \in R$;
π-regular if for each $a \in R$, there exists a positive integer n such that $a^n \in a^nRa^n$;
strongly π-regular if for each $a \in R$, there exists a positive integer n such that $a^n \in a^{n+1}R \cap Ra^{n+1}$;
left universally mininjective if $k \in kRk$ for each $k \in M_l(R) = \{k \in R | Rk$ is a minimal left ideal of $R\}$ ([14]);
strongly left DS if $k^2 \neq 0$ for each $k \in M_l(R)$ ([20]).

The following theorem generalizes [21, Theorem 2.4, Theorem 2.5 and Corollary 2.7].

Theorem 3.6. *Let R be a generalized GZI ring. Then*

(1) *R is directly finite;*
(2) *R is left min-abel;*
(3) *R is reduced if and only if R is n-regular;*
(4) *R is strongly regular if and only if R is von Neumann regular;*
(5) *R is strongly π-regular if and only if R is π-regular;*
(6) *R is strongly left DS if and only if R is left universally mininjective.*

Proof. (1) Let $a, b \in R$ with $ab = 1$. Then $a = aba$, this implies $a = ca^2$ for some $c \in R$ by Lemma 3.5. Hence $1 = ab = ca^2b = ca$ and $b = 1b = cab = c$, one gets $ba = ca = 1$, this shows that R is directly finite.

(2) Let $e \in ME_l(R)$ and $a \in R$. If $h = (1-e)ae \neq 0$, then $Rh = Re$. Clearly, $h \in hRh$, by Theorem 3.5, $h \in Rh^2 = 0$, which is a contradiction. Thus $(1-e)ae = 0$ for each $a \in R$, so R is left min-abel.

(3) Assume that R is n-regular and $a \in R$ with $a^2 = 0$. Then $a = aba$ for some $b \in R$. By Lemma 3.5, $a \in Ra^2 = 0$, that is $a = 0$, so R is reduced.

(4) It is an immediate result of (3).

(5) and (6) are direct results of Lemma 3.5. □

Theorem 3.7. *A ring R is a quasi-normal ring if and only if $V_2(R)$ is a generalized GZI ring.*

Proof. If R is a quasi-normal ring, then by [21, Theorem 2.9], $V_2(R)$ is quasi-normal, hence $V_2(R)$ is generalized GZI.

Conversely, assume that $V_2(R)$ is a generalized GZI ring and $e \in E(R)$ and $a \in R$. Write $h = ea(1-e)$ and $g = e + h$. Then $he = 0, eh = h$, $h^2 = 0$, $hg = 0$, $gh = h$, $g^2 = g$, $ge = e$ and $eg = g$. Clearly, $A = \begin{pmatrix} h & 1-e \\ 0 & h \end{pmatrix} \in N(V_2(R))$ and $E = \begin{pmatrix} e & e-g \\ 0 & e \end{pmatrix} \in E(V_2(R))$ with $AE = 0$. Since $V_2(R)$ is a generalized GZI ring, $A \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} EA = 0$ for each $x, y \in R$, that is

$$hxh = 0 \tag{3.8}$$

$$hyh + (1-e)xh = 0 \tag{3.9}$$

Insteading y for x, one gets

$$(1-e)xh = 0 \tag{3.10}$$

Hence $(1-e)xea(1-e) = 0$ for each $x, a \in R$, so R is quasi-normal. □

Let R be a ring and let $T(R,R) = \{(a,b) | a, b \in R\}$ with addition and multiplication are defined as follows: $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)(c,d) = (ac, ad+bc)$. Then $T(R,R)$ forms a ring. Clearly, $T(R,R) \cong V_2(R) \cong R[x]/(x^2)$.

Corollary 3.8. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-normal;*
(2) *$T(R,R)$ is generalized GZI;*
(3) *$R[x]/(x^2)$ is generalized GZI.*

A ring R is called *quasi-abel* if $ea(1-e)Rea(1-e) = 0$ for each $e \in E(R)$ and $a \in R$, and R is called *quasi-normal* if $eR(1-e)Re = 0$ for each $e \in E(R)$ (c.f. [21]). Clearly, quasi-normal rings are quasi-abel.

A ring R is called *idempotent semiprime* if for each $e \in E(R)$ and $a \in R$, $ea(1-e)Rea(1-e) = 0$ implies $ea(1-e) = 0$. Clearly, *Abel* rings and semiprime rings are idempotent semiprime.

Proposition 3.9. (1) *A ring R is an Abel ring if and only if R an idempotent semiprime quasi-abel ring.*
(2) *Generalized GZI rings are quasi-abel.*

Proof. (1) It is trivial.

(2) Let $e \in E(R)$ and $a \in R$. Write $h = ea(1-e)$. Then $he = 0$, $eh = h$ and $h^2 = 0$. Since R is generalized GZI, $hReh = 0$, that is $hRh = 0$. Hence, for each $a \in R$, one has $ea(1-e)Rea(1-e) = 0$, this implies R is quasi-abel. □

Since *Abel* rings are quasi-abel, by Example 2.6, one knows that quasi-abel rings need not be GZI.

Example 3.10. *Let F be a field and $R = T_3(F)$. By Proposition 2.3(3), R is GZI, so R is generalized GZI. By Proposition 3.9, R is quasi-abel. But by* [21, P1858], *R is not quasi-normal. Hence quasi-abel rings need not be quasi-normal.*

The following proposition illustrates that quasi-abel rings need not be generalized GZI.

Proposition 3.11. *Let $R = \{\begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix} | a_1, a_2 \in \mathbb{Z}_2\}$ and $S = T_3(R)$. Then*

(1) *S is a quasi-abel ring;*
(2) *S is not a generalized GZI ring.*

Proof. (1) Clearly, R is commutative and

$$E(S) = \{\begin{pmatrix} \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix} & \begin{pmatrix} a_2 & a_3 \\ 0 & a_2 \end{pmatrix} & \begin{pmatrix} a_4 & a_5 \\ 0 & a_4 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix} & \begin{pmatrix} a_6 & a_7 \\ 0 & a_6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix} \end{pmatrix} | e_i^2 = e_i,$$

$a_2 = (e_1+e_2)a_2; a_3 = (e_1+e_2)a_3; a_6 = (e_2+e_3)a_6; a_7 = (e_2+e_3)a_7; a_4 = (e_1+e_3)a_4 + a_2a_6; a_5 = (e_1+e_3)a_5 + a_2a_7 + a_3a_6, e_i, a_i \in \mathbb{Z}_2\}$. Choose

$$E = \begin{pmatrix} \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix} & \begin{pmatrix} a_2 & a_3 \\ 0 & a_2 \end{pmatrix} & \begin{pmatrix} a_4 & a_5 \\ 0 & a_4 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix} & \begin{pmatrix} a_6 & a_7 \\ 0 & a_6 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix} \end{pmatrix} \in E(S) \text{ and}$$

$$B = \begin{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} & \begin{pmatrix} b_3 & b_4 \\ 0 & b_3 \end{pmatrix} & \begin{pmatrix} b_5 & b_6 \\ 0 & b_5 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} b_7 & b_8 \\ 0 & b_7 \end{pmatrix} & \begin{pmatrix} b_9 & b_{10} \\ 0 & b_9 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{11} \end{pmatrix} \end{pmatrix} \in S.$$

Case 1 : If $e_1 = e_2 = e_3 = 1$, then $a_i = 0$, $i = 2,3,4,5,6,7$ and $EB(1-E) = 0$.

Case 2 : if $e_1 = e_2 = 1$ and $e_3 = 0$, then $a_2 = a_3 = 0$ and $EB(1-E) =$

$$\begin{pmatrix} \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}-b_1a_4+b_5+a_4b_{11} & c_1\\ 0 & -b_1a_4+b_5+a_4b_{11}\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}b_9 & b_{10}+a_7b_{11}-b_7a_7\\ 0 & b_9\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} \end{pmatrix}$$

where $c_1 = -b_1a_5 - b_2a_4 - b_3a_7 - b_7a_7 + b_6 + a_4b_{12} + a_5b_{11}$.

Case 3 : If $e_1 = e_3 = 1$ and $e_2 = 0$, then $EB(1-E) =$

$$\begin{pmatrix} \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}-b_1a_2+b_3+a_2b_7 & c_4\\ 0 & -b_1a_2+b_3+a_2b_7\end{pmatrix} & \begin{pmatrix}c_2&c_3\\0&c_2\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} \end{pmatrix}$$

where $c_2 = -b_1a_4 - b_3a_6 - a_2b_7a_6, c_3 = -b_1a_5 - b_3a_7 - b_4a_6 - a_2b_7a_7 - a_2b_8a_6 - a_3b_7a_6$ and $c_4 = -b_1a_3 - b_2a_2 + a_4 + a_2b_8 + a_3b_7$.

Case 4 : If $e_1 = 0$ and $e_2 = e_3 = 1$, then $a_6 = a_7 = 0$ and $EB(1-E) = 0$.

Case 5 : If $e_1 = 1$ and $e_2 = e_3 = 0$, then $a_6 = a_7 = 0$ and $EB(1-E) =$

$$\begin{pmatrix} \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}-b_1a_2+b_3+a_2b_7 & c_5\\ 0 & -b_1a_2+b_3+a_2b_7\end{pmatrix} & \begin{pmatrix}c_6&c_7\\0&c_6\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} \end{pmatrix}$$

where $c_5 = -b_1a_3 - b_2a_2 + b_4 + a_2b_8 + a_3b_7, c_6 = -b_1a_4 + b_5 + a_2b_9 + a_4b_{11}$ and $c_7 = -b_1a_5 - b_2a_4 + b_6 + a_2b_{10} + a_3b_9 + a_4b_{12} + a_5b_{11}$.

Case 6 : If $e_2 = 1$ and $e_1 = e_3 = 0$, then $a_4 = a_2a_6$, $a_5 = a_2a_7 + a_3a_6$ and $EB(1-E) =$

$$\begin{pmatrix} \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}-b_7a_4+a_2b_9+a_4b_{11} & c_8\\ 0 & -b_7a_4+a_2b_9+a_4b_{11}\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}-b_7a_6+b_9+a_6b_{11} & -b_7a_7-b_8a_6+b_{10}+a_7b_{11}\\ 0 & -b_7a_6+b_9+a_6b_{11}\end{pmatrix} \\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} \end{pmatrix}$$

where $c_8 = -b_7a_5 - b_8a_4 + a_2b_{10} + a_3b_9 + a_4b_{12} + a_5b_{11}$.

Case 7 : If $e_3 = 1$ and $e_1 = e_2 = 0$, then $a_2 = a_3 = 0$ and $EB(1-E) = 0$.

Case 8 : If $e_1 = e_2 = e_3 = 0$, then $EB(1-E) = 0$.

In any case, one can easy to see that $EB(1-E)SEB(1-E) = 0$, hence S is quasi-abel.

(2) Choose $A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in N(S)$ and

$E = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in E(S)$ and

$C = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in S$. Then $AE = 0$ and

$ACEA = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0$. Hence S is not generalized GZI. □

4. Some properties of quasi-abel rings

Let R be a ring and $e \in E(R)$. Then $(1-e)Re = (1-e)N(R)e$, this implies the following proposition.

Proposition 4.1. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-abel;*
(2) *$ea(1-e)N(R)ea(1-e) = 0$ for each $e \in E(R)$ and $a \in R$;*
(3) *$ea(1-e)Rea(1-e) = 0$ for each $e \in E(R)$ and $a \in N(R)$;*
(4) *$ea(1-e)N(R)ea(1-e) = 0$ for each $e \in E(R)$ and $a \in N(R)$.*

Proposition 4.2. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-abel;*
(2) *$ae = 0$ implies $eaRea = 0$ for each $e \in E(R)$ and $a \in R$;*
(3) *$ea = 0$ implies $aeRae = 0$ for each $e \in E(R)$ and $a \in R$.*

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (3) Let $ea = 0$. Then $(ae)(1-e) = 0$, by (2), $(1-e)(ae)R(1-e)(ae) = 0$, that is $aeRae = 0$.

(3) $\Rightarrow$ (1) Let $a \in R$ and $e \in E(R)$. Then $(1-e)(ea) = 0$, by (3), $(ea)(1-e)R(ea)(1-e) = 0$. Thus R is quasi-abel. □

It is well known that a ring R is *Abel* if and only if $ab = 0$ implies $aE(R)b = 0$ for each $a, b \in R$.

Proposition 4.3. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-abel;*
(2) $ae = 0$ *implies* $eaE(R)ea = 0$ *for each* $e \in E(R)$ *and* $a \in R$*;*
(3) $ea = 0$ *implies* $aeE(R)ae = 0$ *for each* $e \in E(R)$ *and* $a \in R$*.*

Proof. By Proposition 4.2, (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial.

(2) $\Rightarrow$ (1) Let $ae = 0$. For any $r \in R$, write $g = e+(1-e)re$. Then $eg = e, ge = g$ and $g^2 = g$. By (2), $eagea = 0$. But $eagea = earea$, this gives $earea = 0$ for each $r \in R$. Thus $eaRea = 0$, by Proposition 4.2, R is quasi-abel.

Similarly, we can show (3) $\Rightarrow$ (1). □

Similarly, we can give the following characterization of quasi-normal rings.

Proposition 4.4. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-normal;*
(2) $ae = 0$ *implies* $eaE(R)e = 0$ *for each* $e \in E(R)$ *and* $a \in R$*;*
(3) $ea = 0$ *implies* $eE(R)ae = 0$ *for each* $e \in E(R)$ *and* $a \in R$*.*

Proposition 4.5. *Let R be a quasi-abel ring and* $e \in E(R)$*. Then*

(1) *For every maximal left ideal M of R, either* $e \in M$ *or* $1-e \in M$*.*
(2) *For each* $a \in R$*,* $Ra + R(ae-1) = R$*.*
(3) *For every maximal left ideal M of R,* $Me \subseteq M$*.*
(4) *If* $ReR = R$*, then* $e = 1$*.*

Proof. (1) If $e \notin M$, then $M+Re = R$. Let $1 = m+ae$ for some $a \in R$ and $m \in M$. Since R is quasi-abel, $(1-e)aeR(1-e)ae = 0$, this gives $(1-e)ae \in J(R) \subseteq M$, so $1-e = (1-e)m + (1-e)ae \in M$.

(2) If $Ra + R(ae-1) \neq R$, then there exists a maximal left ideal M such that $Ra+R(ae-1) \subseteq M$. Since $ae-1 \in M$, $e \notin M$, by (1), $1-e \in M$, so $a(1-e) \in M$. Since $a \in M$, $ae \in M$, so $1 = ae-(ae-1) \in m$, which is a contradiction. Thus $Ra + R(ae-1) = R$.

(3) If $Me \nsubseteq M$, then $M + Me = R$. Let $1 = m + ae$ for some $a, m \in M$. By (2), $R = Ra + R(ae-1) = Ra + Rm \subseteq M$, which is a contradiction. Thus $Me \subseteq M$.

(4) Let $1 = \Sigma_{i=1}^{n} a_i e b_i$. Since $eb_i(1-e)Reb_i(1-e) = 0$, $eb_i(1-e) \in J(R)$, this gives $1-e = \Sigma_{i=1}^{n} a_i e b_1(1-e) \in J(R)$. Thus $e = 1$. □

A ring R is called *left pp* if for any $a \in R, {}_R Ra$ is a projective module.

Corollary 4.6. *Let R be a quasi-abel ring. If R is left pp, then $al(a) \subseteq J(R)$ for each $a \in R$.*

Proof. Let $a \in R$. Since R is a left *pp* ring, ${}_R Ra$ is projective. Thus there exists $e \in E(R)$ such that $l(a) = l(e)$ and $ea = a$. Since R is a quasi-abel ring and $(1-e)ar = 0$ for each $r \in R$, by Proposition 4.2, $ar(1-e)Rar(1-e) = 0$, this gives $ar(1-e) \in J(R)$ for each $r \in R$. Thus $aR(1-e) \subseteq J(R)$, which implies $al(a) = aR(1-e) \subseteq J(R)$. □

Corollary 4.7. *Let R be a quasi-abel ring. If $x, z \in R$ are such that $x+z \in zxE(R)$, then $xR = zR$.*

Proof. Let $x + z = zxe$ for some $e \in E(R)$. Then $x = z(xe-1)$. Since R is a quasi-abel ring, $R = Rx + R(xe-1)$ by Proposition 4.5, this implies $R = R(xe-1)$. Let $1 = u(xe-1)$ for some $u \in R$. Write $g = (xe-1)u$. Then $g^2 = g$ and $xe - 1 = g(xe-1)$. By Proposition 4.2, $(xe-1)(1-g)R(xe-1)(1-g) = 0$, this gives $(xe-1)(1-g)R(1-g) = 0$ because $R = R(xe-1)$, and so $R(1-g) = R(1-g)R(1-g) = R(xe-1)(1-g)R(1-g) = 0$, one obtains $g = 1$, that is $(xe-1)u = 1$. Hence $xe-1$ is invertible, this leads to $xR = z(xe-1)R = zR$. □

Following [13], an element a of a ring R is called clean if a is a sum of a unit and an idempotent of R, and a is said to be exchange if there exists $e \in E(R)$ such that $e \in aR$ and $1-e \in (1-a)R$. A ring R is called clean if every element of R is clean, and R is said to be exchange if every element of R is exchange. According to [13], clean rings are always exchange, but the converse is not true unless R satisfies one of the following conditions (1) R is a left quasi-duo ring [24]; (2) R is an Abelian ring [25]; (3) R is a quasi-normal ring [21]; (4) R is a weakly normal ring [20].

Theorem 4.8. *Let R be a quasi-abel ring and $a \in R$. Then*

(1) *If a is exchange, then a is clean.*
(2) *If R is an exchange ring, then R is clean.*
(3) *If a^n is clean for some $n \geq 1$, then a is clean.*
(4) *If a^2 is clean, then a and $-a$ are clean.*

Proof. (1) Let $e \in E(R)$ such that $e \in aR$ and $1-e \in (1-a)R$. Write $e = ab$ and $1-e = (1-a)c$ for some $b = be, c = c(1-e) \in R$. Then $(a-(1-e))(b-c) = ab - ac - (1-e)b + (1-e)c = ab + (1-a)c - (1-e)b - ec = 1 - (1-e)b - ec$. Since R is a quasi-abel ring, $(1-e)bR(1-e)b = (1-e)beR(1-e)be = 0$, this gives $(1-e)b \in J(R)$. Similarly, $ec \in J(R)$. Hence $v = 1-(1-e)b-ec$ is a unit of R, so $(a-(1-e))(b-c)u = 1$ where $u = v^{-1}$. Let $g = (b-c)u(a-(1-e))$. Then $g^2 = g$ and $g(b-c)u = (b-c)u$. Since R is quasi-abel, $g(b-c)u(1-g)Rg(b-c)u(1-g) = 0$, this implies $g(b-c)u(1-g)(a-(1-e))g(b-c)u(1-g) = 0$, that is, $(b-c)u(1-g)(a-(1-e))(b-c)u(1-g) = 0$. Since $(a-(1-e))(b-c)u = 1$, $(b-c)u(1-g) = 0$, this leads to $1-g=(a-(1-e))(b-c)u(1-g)=0$, so $(b-c)u(a-(1-e))=g=1$, one obtains $a-(1-e)$ is an unit of R. Hence a is a clean element.

(2) It is an immediate result of (1).

(3) Since a^n is clean, there exist $u \in U(R)$ and $f \in E(R)$ such that $a^n = u+f$. Let $e = u(1-f)u^{-1}$. Then $(a^n - e)u = (u+f)u - u(1-f) = a^n(a^n-1) \in aR$, so $e = a^n + (a^n - a^{2n})u^{-1} \in aR$ and $1-e \in (1-a)R$, this implies a is exchange, by (1), a is clean.

(4) Since $a^2 = (-1a)^2$ is clean, by (3), a and $-a$ are clean. □

Corollary 4.9. *Let R be a quasi-abel ring and idempotent can be lifted modulo $J(R)$. If $a \in R$ is clean and $e \in E(R)$. Then*

(1) *ae is clean.*

(2) *If $-a$ is also clean, then $a+e$ is clean.*

Proof. Since a is clean, $\bar{a}$ is clean in $\bar{R} = R/J(R)$. Since R is a quasi-abel ring and idempotent can be lifted modulo $J(R)$, $\bar{R}$ is *Abel*, this illustrates that $\bar{e}$ is a central idempotent in $\bar{R}$. Since a is clean in R, there exist $u \in U(R)$ and $f \in E(R)$ such that $a = u+f$. Let $v \in R$ such that $uv = vu = 1$. Then, in $\bar{R}$, $\bar{a}\bar{e} = (\bar{u}\bar{e}+\bar{e}-\bar{1})+(\bar{f}\bar{e}+\bar{1}-\bar{e})$. Clearly, $(\bar{u}\bar{e}+\bar{e}-\bar{1})(\bar{v}\bar{e}+\bar{e}-\bar{1}) = (\bar{v}\bar{e}+\bar{e}-\bar{1})(\bar{u}\bar{e}+\bar{e}-\bar{1}) = \bar{1}$ and $(\bar{f}\bar{e}+\bar{1}-\bar{e})^2 = \bar{f}\bar{e}+\bar{1}-\bar{e}$, so $\bar{a}\bar{e}$ is clean in $\bar{R}$. Since idempotent can be lifted modulo $J(R)$, there exists $g \in E(R)$ such that $\bar{g} = \bar{f}\bar{e}+\bar{1}-\bar{e}$. Let $w \in R$ such that $\bar{w} = \bar{u}\bar{e}+\bar{e}-\bar{1}$. Then $w \in U(R)$ and $ae-w-g \in J(R)$. Let $ae-w-g = x \in J(R)$. Then $ae = g + w(1+w^{-1}x)$. Since $w(1+w^{-1}x) \in U(R)$, ae is clean in R.

(2) Since $-a$ is clean in R, $1+a$ is clean in R. Hence $\bar{a}$ and $\bar{1}+\bar{a}$ are all clean in $\bar{R} = R/J(R)$. Let $\bar{a} = \bar{u}+\bar{f}$ and $\bar{1}+\bar{a} = \bar{v}+\bar{g}$ where $u, v \in U(R)$ and $f, g \in E(R)$. Clearly, $\bar{a}+\bar{e} = \bar{a}(\bar{1}-\bar{e})+(\bar{1}+\bar{a})\bar{e}$, so $\bar{a}+\bar{e} = \bar{v}\bar{e}+\bar{u}(\bar{1}-\bar{e})+\bar{g}\bar{e}+\bar{f}(\bar{1}-\bar{e})$. Clearly, $(\bar{v}\bar{e}+\bar{u}(\bar{1}-\bar{e}))(\bar{v}^{-1}\bar{e}+\bar{u}^{-1}(\bar{1}-\bar{e})) = \bar{1}$ and $\bar{g}\bar{e}+\bar{f}(\bar{1}-\bar{e}) \in E(\bar{R})$. Therefore, $\bar{a}+\bar{e}$ is clean in $\bar{R}$, similar to (1), we obtain $a+e$ is clean in R. □

In [7], it is showed that if R is a unit regular ring, then every element of R is a sum of two units. A ring R is called an $(S,2)$-*ring* ([8]), if every element of R is a sum of two units of R. In [1], it is proved that if R is an *Abel* π-regular ring, then R is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.

Theorem 4.10. *Let R be a quasi-abel π-regular ring. Then R is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.*

Proof. Since R is a quasi-abel π-regular ring, $R/J(R)$ is π-regular ring. Since R is an exchange ring, idempotent can be lifted modulo $J(R)$, this implies $R/J(R)$ is an *Abel* ring. By [1], $R/J(R)$ is an $(S,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R/J(R)$. By [21, Lemma 4.3], we are done. □

In light of Theorem 4.10, we have the following corollaries:

Corollary 4.11. *Let R be a quasi-abel π-regular ring such that $2 = 1+1 \in U(R)$. Then R is an $(S, 2)$-ring.*

Corollary 4.12. *Let R be a quasi-abel π-regular ring. Then R is an $(S, 2)$-ring if and only if for some $d \in U(R)$, $1 + d \in U(R)$.*

Recall that a ring R is said to *have stable range 1* ([16]), if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. Clearly, R has stable range 1 if and only if $R/J(R)$ has stable range 1. In [25, Theorem 6], it is showed that exchange rings with all idempotents central have stable range 1.

Theorem 4.13. *Quasi-abel exchange rings have stable range 1.*

Proof. Let R be a quasi-abel exchange ring. Then $R/J(R)$ is exchange with all idempotents central, so, by [25, Theorem 6], $R/J(R)$ has stable range 1. Therefore R has stable range 1. □

In [23], a ring R is said to satisfy the *unit* 1*-stable condition* if for any $a, b, c \in R$ with $ab + c = 1$, there exists $u \in U(R)$ such that $au + c \in U(R)$. It is easy to prove that R satisfies the unit 1-stable condition if and only if $R/J(R)$ satisfies the unit 1-stable condition.

Theorem 4.14. *Let R be a quasi-abel exchange ring, then the following conditions are equivalent:*

(1) *R is an $(S, 2)$-ring.*
(2) *R satisfies the unit 1-stable condition.*
(3) *Every factor ring R_1 of R is an $(S, 2)$-ring.*
(4) *$\mathbb{Z}_2$ is not a homomorphic image of R.*

A ring R is called *left topologically boolean*, or a *left tb-ring* ([5]) for short, if for every pair of distinct maximal left ideals of R there is an idempotent in exactly one of them.

Theorem 4.15. *Let R be a quasi-abel clean ring. Then R is a left tb-ring.*

Proof. Suppose that M and N are distinct maximal left ideals of R. Let $a \in M\backslash N$. Then $Ra + N = R$ and $1 - xa \in N$ for some $x \in R$. Clearly, $xa \in M\backslash N$. Since R is clean, there exist an idempotent $e \in E(R)$ and a unit u in R such that $xa = e + u$. If $e \in M$, then $u = xa - e \in M$ from which it follows that $R = M$, a contradiction. Thus $e \notin M$. If $e \notin N$, then $1 - e \in N$ by Proposition 4.5, this gives $u = (1 - e) + (xa - 1) \in N$. It follows that $N = R$ which is also not possible. We thus have that e is an idempotent belonging to N only. □

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

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