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Some Expressions for the Group Inverse of the Block Partitioned Matrices with an Invertible Subblock

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Abstract

Let K be a skew field and K^{mxm} be the set of all mxm matrices over K. The purpose of this paper is to give some necessary and sufficient conditions for the existence and some expressions of the group inverse of the block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K^{mxm}$ (A is square) under some conditions, where M be a square block matrix with an invertible subblock.

***Keywords*:** Skew field, Block partititoned matrix, Group Inverse of a matrix, Invertible subblock, Drazin inverse.

MSC Subject Clasifications: 15A09, 65F20

Tersinir Bir Altbloğa Sahip Blok Parçalanmiş Matrislerin Grup İnversleri İçin Bazi İfadeler

Öz

K bir kesir cismi ve K^{mxm} de bu kesir cismi üzerindeki mxm tipindeki tüm matrislerin kümesi olsun. Bu çalışmadaki amaç, M tersinir bir alt bloğa sahip blok parçalanmış bir kare matris olmak üzere, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K^{mxm}$ (A bir kare matris) matrisinin grup inversinin mevcut olması için gerek ve yeter şartları vermek ve bazı koşullar altında blok parçalanmış matrislerin grup inversi için ifadeler elde etmektir.

***Anahtar kelimeler*:** Kesir cismi, Blok parçalanmış matris, Bir matrisin grup inversi, Tersinir altblok, Drazin invers.

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1. Introduction

It is well known that the expressions for the Drazin(group) inverse of block matrices are very important not only in matrix theory, but also in singular differential and difference equations, probability statistical, numerical analysis, game theory, econometrics, control theory and so on (see references Campbell & Meyer 1991; Ben-Israel & Greville 2003; Campbell 1983; Wei & Diao 2005; Zang, Cao & Ge, 2014 and etc.). In 1991, Campbell and Meyer proposed an open problem to find an explicit representation for the Drazin inverse of any $2x2$ block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where the blocks A and D are supposed to be square matrices but their sizes need not be same (see Campbell & Meyer 1991.). Until now, this problem has not been solved completely. However, there are many literatures about the existence and the repretentation of the Drazin(group) inverse for the block matrix matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ under some conditions (see Chen & Hartwig 1996; Hartwig & Shoaf 1977; Meyer & Rose 1977; Li & Wei 2007; Cvetkovic –Ilic 2008;, González and E. Dopazo 2005; Bu, Zhao & Zheng 2008; Bu, Zhao & Zheng 2009; González, Dopazo & Robles 2006; Cao 2001; Cao & Tang 2006; Wei 1998; Hartwig, Li & Wei 2006.).

In 1996, Chen and Hartwig, a condition for the existence of group inverse of matrix M is given under the assumption that A and $I + CA^2B$ are both invertible over any field, however, the representation of the Drazin (group) inverse is not given. Specially, in papers, the existence and the representations of the group inverse for the following block matrices are researched.

(i) $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, (see Cao 2001.),

(ii) $M = \begin{bmatrix} A & A \\ C & 0 \end{bmatrix}$, $A = A^2$, (see Bu, Zhao & Zheng 2008.),

(iii) $M = \begin{bmatrix} A & A \\ C & 0 \end{bmatrix}$, $r(C) \geq r(A)$, (Bu, Zhao & Zheng 2009.),

(iv) $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ or $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, A is square, (Cambell 1983.),

(v) $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, $A = I_n$, $r((CB)^2) = r(B) = r(C)$, (Cambell 1983.),

(vi) $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$, A is square , (Cambell 1983.).

Suppose K is a skew field and K^{mxm} denote the set of all mxm matrices over K. For a matrix $A \in K^{mxm}$, the matrix $X \in K^{mxm}$ is said to be the Drazin inverse of A, if it holds

$$A^kXA = A^k, XAX = X, AX = XA, \tag{1.1}$$

for some integer k. Denote by $X = A^D$. Where k is the index of A, the smallest non-negative integer such that $r(A^k) = r(A^{k+1})$ and is denoted by $k = Ind(A)$. It is well-known that A^D exists and is unique (see Rao 2002.). If $Ind(A) \leq 1$, A^D is also called the group inverse of A and denoted by $A^\#$. Then the group inverse of A exists if and only if $r(A) = r(A^2)$ (see Rao 2002.). In particular, if $Ind(A) = 0$, then $A^D = A^\# = A^{-1}$. I, $\mathbf{0}$ and A^π denote the identity matrix, the zero matrix of suitable size and $I - AA^\#$, respevtively.

In this paper, we consider the necessary and sufficient conditions of the existence of the group inverse for the block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K^{mxm}$, ($A$ is square), when A, B, C and D satisfy one of the conditions listed below:

(i) A is invertible, $S^\#$ exists, where $S = D - CA^{-1}B$;

(ii) D is invertible, $S^{\#}$ exists, where $S = A - BD^{-1}C$;

(iii) B or C is invertible.

2. Preliminaries

In this section, we present some important lemmas and investigate the expression of the group inverse. Let us began with a familiar lemma.

Lemma 2.1 [Rao, 2002] Let $A \in K^{mxm}$ be an arbitrary matrix. Then $A^{\#}$ exists if and only if $r(A) = r(A^2)$.

Lemma 2.2 [Rao, 2002] Let $A \in K^{mxm}$ be an arbitrary matrix. Then the following statements are equivalent:

(i) $A^{\#}$ exists

(ii) $A^2X = A$ for some $X \in K^{mxm}$. In this case, $A^{\#} = AX^2$.

(iii) $YA^2 = A$ for some $Y \in K^{mxm}$. In this case, $A^{\#} = X^2A$.

Lemma 2.3 [Li & Wei 2007] Let $A \in K^{mxm}$ be an arbitrary matrix. Then the matrix A has a group inverse if and only if there exists nonsingular matrices $P \in K^{mxm}$ and $A_1 \in K^{rxr}$ such that

$$A = P\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} \quad \text{and} \quad A^{\#} = P\begin{bmatrix} {A_1}^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1},$$

where $r(A) = r$.

Lemma 2.4 [Bu, Zhao & Zheng, 2008] Let $A = P\begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix}Q \in K^{mxm}$, where $P, Q \in K^{mxm}$ and $A_1 \in K^{rxr}$ are invertible and $r(A) = r$. Then

$$A^{(1)} = Q^{-1}\begin{bmatrix} X_1 & X_2 \\ X_3 & {A_1}^{-1} \end{bmatrix}P^{-1},$$

where $X_1 \in K^{rxr}$, $X_2 \in K^{rx(m-r)}$ and $X_3 \in K^{(m-r)xr}$ are arbitrary matrices.

Lemma 2.5 [Bu, Zhao & Zheng, 2008] Let $A \in K^{mxn}$ and $B \in K^{nxn}$. If $r(A) = r(BA)$ and $r(B) = r(AB)$, then the group inverses of both AB and BA exist.

Lemma 2.6 [Bu, Zhao & Zheng, 2009] Let $A, B \in K^{mxm}$. If $r(A) = r(BA) = r(B) = r(AB)$, then the following expressions hold:

(i) $(AB)^{\#}A = A(BA)^{\#}$,

(ii) $(BA)^{\#}B = B(AB)^{\#}$,
(iii) $B(AB)^{\pi} = 0$,
(iv) $(AB)^{\pi}A = 0$.

Lemma 2.7 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K^{mxm}$ be a matrix, where $A \in K^{nxn}$ is invertible. Let us denote $S = D - CA^{-1}B$. If $S^{\#}$ exists, $S^{\pi} = I_{m-n} - SS^{\#}$ and $R = A^2 + BS^{\pi}C$ is invertible, then

(i) $CAR^{-1} = CA^{-1} - DS^{\pi}CR^{-1}$,
(ii) $R^{-1}AB = A^{-1}B - R^{-1}BS^{\pi}D$.

Proof: It is easy to get

$$CA^{-1} = CA^{-1}I_n = CA^{-1}(\,A^2R^{-1} + BS^{\pi}CR^{-1}) = CAR^{-1} + (D - S)S^{\pi}CR^{-1},$$
$$CA^{-1} = CA^{-1}I_n = CA^{-1}(\,A^2R^{-1} + BS^{\pi}CR^{-1}) = CAR^{-1} + (D - S)S^{\pi}CR^{-1}.$$

Then, the conclusions hold.

Lemma 2.8 [Bu, Zhao & Zheng, 2008] Let $A \in K^{mxm}$ be a matrix. Then there exist invertible matrices $P, Q \in K^{mxm}$ such that

$$I_m - AA^{(1)} = P\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix}P^{-1} \quad \text{and} \quad I_m - A^{(1)}A = Q^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix}Q\,,$$

where $r(A) = r$ and $A^{(1)}$ is a $\{1\}$ −inverse of A, that is, $A^{(1)} \in K^{mxm}$ is a matrix satisfying the matrix equation $AA^{(1)}A = A$. Denote the set of all a $\{1\}$ −inverses of A by $A\{1\}$.

Lemma 2.9 [Bu, Zhao & Zheng, 2008] Let $A, B, C \in K^{mxm}$, where B is invertible, $Z = (I_m - CC^{(1)})B^{-1}A(I_n - C^{(1)}C)$ and $\Delta = \left(I_m - C^{(1)}C\right)Z^{(1)}\left(I_m - CC^{(1)}\right)$, for any $C^{(1)} \in C\{1\}$ and $Z^{(1)} \in Z\{1\}$. In this case, if $r(Z) = m - r(C)$, then the following expressions satisfy:

(i) $(I_m - \Delta B^{-1}A)C^{(1)}C = I_m - \Delta B^{-1}A$
(ii) $CC^{(1)}(I_m - BA^{-1}\Delta) = I_m - BA^{-1}\Delta$,
(iii) $(I_m - \Delta B^{-1}A)\Delta = \Delta(I_m - B^{-1}A\Delta) = 0$,
(iv) $(I_m - \Delta B^{-1}A)^2 = I_m - \Delta B^{-1}A$,
(v) $(I_m - B^{-1}A\Delta)^2 = I_m - B^{-1}A\Delta$.

Proof: Suppose that $r(C) = r$. Then, by Lemma 2.7, we see that there exist invertible matrices $P, Q \in K^{mxm}$, such that

$$I_m - CC^{(1)} = P\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^{-1} \text{ and } I_m - C^{(1)}C = Q^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} Q ,$$

and so we have

$$CC^{(1)} = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \text{ and } C^{(1)}C = Q^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q.$$

Suppose $B^{-1}A = P\begin{bmatrix} X & Y \\ U & V \end{bmatrix} Q$, where $U \in K^{(m-r)xr}$, $V \in K^{(m-r)x(m-r)}$, $Y \in K^{rx(m-r)}$, $X \in K^{rxr}$. Therefore, we have

$$Z = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P^{-1} P\begin{bmatrix} X & Y \\ U & V \end{bmatrix} QQ^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} Q = P\begin{bmatrix} 0 & 0 \\ 0 & V \end{bmatrix} Q$$

since $r(Z) = m - r(C)$. So we get that V is invertible. From this, applying Lemma 2.4, we can write

$$Z^{(1)} = Q^{-1}\begin{bmatrix} X_1 & X_2 \\ X_3 & V \end{bmatrix} P^{-1},$$

where $X_1 \in K^{rxr}$, $X_2 \in K^{rx(m-r)}$ and $X_3 \in K^{(m-r)xr}$ are arbitrary. Consequently, we have

$$\Delta = Q^{-1}\begin{bmatrix} 0 & 0 \\ 0 & V^{-1} \end{bmatrix} P^{-1},$$

$$I_m - B^{-1}A\Delta = P\begin{bmatrix} I_r & -YV^{-1} \\ 0 & 0 \end{bmatrix} P^{-1},$$

$$I_m - \Delta B^{-1}A = Q^{-1}\begin{bmatrix} I_r & 0 \\ -V^{-1}U & 0 \end{bmatrix} Q.$$

According to the above results, we can easily prove that (i)-(v) hold.

3. Main Results

Theorem 3.1 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K^{mxm}$, where $A \in K^{nxn}$ is invertible, $S = D - CA^{-1}B$. If the group inverse $S^{\#}$ exists, then

(i) $M^{\#}$ exists iff $R = A^2 + BS^{\pi}C$ is invertible and $S^{\pi} = I_{m-n} - SS^{\#}$,

(ii) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}, \quad (3.1)$$

where

$$X = AR^{-1}(A + BS^{\#}C)R^{-1}A,$$

$Y = AR^{-1}(A + BS^{\#}C)R^{-1}BS^{\pi} - AR^{-1}BS^{\#}$,

$Z = S^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}A - S^{\#}CR^{-1}A$,

$W = S^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}BS^{\pi} - S^{\#}CR^{-1}BS^{\pi} - S^{\pi}CR^{-1}BS^{\#} + S^{\#}$.

Proof: (i) Since A is invertible, we get

$$r(M) = r\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = r\left(\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}\right) = r\left(\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}\right),$$

where $S = D - CA^{-1}B$, so $r(M) = r(A) + r(S)$. On the other hand, as $S^{\#}$ exists, we can write

$$\begin{aligned} r(M^2) &= r\left(\begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A^2 + BC & AB + BD \\ CA + DC - CA^{-1}(A^2 + BC) & CB + D^2 - CA^{-1}(AB + BD) \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A^2 + BC & AB + BD \\ SC & SD \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A^2 + BC & AB + BD - (A^2 + BC)A^{-1}B \\ SC & SD - SCA^{-1}B \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A^2 + BC & BS \\ SC & S^2 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A^2 + BC - BSS^{\#}C & BS \\ SC - S^2S^{\#}C & S^2 \end{bmatrix}\right) \end{aligned}$$

and

$$r\left(\begin{bmatrix} R & RS \\ 0 & S^2 \end{bmatrix}\right) = r\left(\begin{bmatrix} R & BS^{\#}S^2 \\ 0 & S^2 \end{bmatrix}\right) = r\left(\begin{bmatrix} R & 0 \\ 0 & S^2 \end{bmatrix}\right).$$

Therefore, we get $r(M^2) = r(R) + r(S^2)$. Then it follows from Lemma 2.1 and existence of $S^{\#}$ that $r(S) = r(S^2)$ and so $r(M^2) = r(R) + r(S)$. From this, the matrix $M^{\#}$ exists if and only if $r(A) = (R)$, i.e., $M^{\#}$ exists if and only if R is invertible.

(ii) Let $N = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$. Then we prove that $N = M^{\#}$. In this case, we observe that

$$MN = \begin{bmatrix} AX + BZ & AY + BW \\ CX + DZ & CY + DW \end{bmatrix}$$

and

$$NM = \begin{bmatrix} XA + YC & XB + YD \\ ZA + WC & ZB + WD \end{bmatrix}.$$

Therefore, we can prove that

$$\begin{aligned} AX + BZ &= A^2R^{-1}(A + BS^{\#}C)R^{-1}A + BS^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}A - BS^{\pi}CR^{-1}A \\ &= (A^2R^{-1} + BS^{\pi}CR^{-1})(A + BS^{\#}C)R^{-1}A - BS^{\pi}CR^{-1}A \end{aligned}$$

$$= I_m(A + BS^\#C)R^{-1}A - BS^\pi CR^{-1}A$$
$$= AR^{-1}A,$$
$$AY + BW = (A^2R^{-1} + BS^\pi CR^{-1})(A + BS^\#C)R^{-1}BS^\pi + BS^\#$$
$$-(A^2R^{-1} + BS^\pi CR^{-1})BS^\# - BS^\#CR^{-1}BS^\pi$$
$$= (A + BS^\#C)R^{-1}BS^\pi + BS^\# - BS^\# - BS^\#CR^{-1}BS^\pi$$
$$= AR^{-1}BS^\pi,$$

according to Lemma 2.6(i), we can write

$$CX + DZ = CAR^{-1}(A + BS^\#C)R^{-1}A + DS^\pi CR^{-1}(A + BS^\#C)R^{-1}A - DS^\pi CR^{-1}A$$
$$= (CA^{-1} - DS^\pi CR^{-1})(A + BS^\#C)R^{-1}A - DS^\pi CR^{-1}$$
$$+DS^\pi CR^{-1}(A + BS^\#C)R^{-1}A$$
$$= CA^{-1}(A + BS^\#C)R^{-1}A - DS^\pi CR^{-1}A$$
$$= S^\pi CR^{-1}A,$$
$$CY + DW = CAR^{-1}(A + BS^\#C)R^{-1}BS^\pi - CAR^{-1}BS^\# - DS^\pi CR^{-1}BS^\# + DS^\#$$
$$+ DS^\pi CR^{-1}(A + BS^\#C)R^{-1}BS^\pi - DS^\#CR^{-1}BS^\pi$$
$$= CA^{-1}(A + BS^\#C)R^{-1}BS^\pi - CA^{-1}BS^\# - DS^\#CR^{-1}BS^\pi + DS^\#$$
$$= S^\pi CR^{-1}BS^\pi + SS^\#,$$
$$XA + YC = AR^{-1}(A + BS^\#C)R^{-1}A^2 + AR^{-1}(A + BS^\#C)R^{-1}BS^\pi C - AR^{-1}BS^\#C$$
$$= AR^{-1}(A + BS^\#C)(R^{-1}A^2 + R^{-1}BS^\pi C) - AR^{-1}BS^\#C$$
$$= AR^{-1}(A + BS^\#C)I_m - AR^{-1}BS^\#C$$
$$= AR^{-1}A,$$

and, from Lemma 2.6(ii), we can write

$$XB + YD = AR^{-1}(A + BS^\#C)R^{-1}AB + AR^{-1}(A + BS^\#C)R^{-1}BS^\pi D - AR^{-1}BS^\#D$$
$$= AR^{-1}(A + BS^\#C)A^{-1}B - AR^{-1}BS^\#D$$
$$= AR^{-1}BS^\pi,$$
$$ZA + WC = S^\pi CR^{-1}(A + BS^\#C)(R^{-1}A^2 + R^{-1}BS^\pi C) - S^\pi CR^{-1}BS^\#C$$
$$+S^\#C - S^\#C(R^{-1}A^2 + A^{-1}BS^\pi C)$$
$$= S^\pi CR^{-1}(A + BS^\#C) - S^\pi CR^{-1}BS^\#C + S^\#C - S^\#C$$
$$= S^\pi CR^{-1}A,$$
$$ZB + WD = S^\pi CR^{-1}(A + BS^\#C)R^{-1}AB - S^\#CR^{-1}AB + S^\pi CR^{-1}(A + BS^\#C)R^{-1}BS^\pi D$$
$$-S^\#CR^{-1}BS^\pi D - S^\pi CR^{-1}BS^\pi D + S^\#D$$
$$= S^\pi CR^{-1}(A + BS^\#C)A^{-1}B - S^\#CA^{-1}B - S^\pi CR^{-1}BS^\#D + S^\#D$$
$$= S^\pi CR^{-1}BS^\pi + SS^\#.$$

Therefore, we have

$$MN = NM = \begin{bmatrix} AR^{-1}A & AR^{-1}BS^{\pi} \\ S^{\pi}CR^{-1}A & S^{\pi}CR^{-1}BS^{\pi} + SS^{\#} \end{bmatrix}.$$

From this, we get

$$MNM = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} AR^{-1}A & AR^{-1}BS^{\pi} \\ S^{\pi}CR^{-1}A & S^{\pi}CR^{-1}BS^{\pi} + SS^{\#} \end{bmatrix}$$

and

$$NMN = \begin{bmatrix} AR^{-1}A & AR^{-1}BS^{\pi} \\ S^{\pi}CR^{-1}A & S^{\pi}CR^{-1}BS^{\pi} + SS^{\#} \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix},$$

according to Lemma 2.6, we can prove that $MNM = M$ and $NMN = N$. Thus $N = M^{\#}$.

Corollary 3.1 Let $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in K^{mxm}$, where $A \in K^{nxn}$ is invertible and $S = -CA^{-1}B$. If the group inverse $S^{\#}$ exists, then

(i) $M^{\#}$ exists if and only if $R = A^2 + BS^{\#}C$ is invertible and $S^{\pi} = I_{m-n} - SS^{\#}$,

(ii) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}, \tag{3.2}$$

where

$X = AR^{-1}(A + BS^{\#}C)R^{-1}A,$

$Y = AR^{-1}(A + BS^{\#}C)R^{-1}BS^{\pi} - AR^{-1}BS^{\#},$

$Z = S^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}A - S^{\#}CR^{-1}A,$

$W = S^{\pi}CR^{-1}(A + BS^{\#}C)R^{-1}BS^{\pi} - S^{\#}CR^{-1}BS^{\pi} - S^{\pi}CR^{-1}BS^{\#} + S^{\#}.$

Proof: We can obtain the conclusions directly by applying Theorem 3.1 while $D = 0$.

Corollary 3.2 Let $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \mathbb{Z}^{mxm}$, where $A \in K^{nxn}$. If $AB = B$, $CA = C$, then

(i) $M^{\#}$ exists if and only if $(CB)^{\#}$ and $A^{\#}$ exist;

(ii) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} A^{\#} - 2B(CB)^{\pi}C - B(CB)^{\#}C & B(CB)^{\#} + B(CB)^{\pi} \\ (CB)^{\#}C + (CB)^{\pi}C & -(CB)^{\#} \end{bmatrix}. \tag{3.3}$$

Proof: We can obtain the conclusions directly by applying Theorem 3.1 and Reference [5].

Example 3.1 Let $\mathbb{Z}$ be the integer ring and let $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ be a matrix over $\mathbb{Z}/6\mathbb{Z}$, where

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 3 & 3 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}.$$

It is easy to verify that $AB = B,\ \ CA = C$ and $A^2 \neq A$. Furthermore, $(CB)^{\#}$ and $A^{\#}$ exist. By computating,

$$A^{\#} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (CB)^{\#} = CB = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix}.$$

Thus, by Corollary 3.2 $M^{\#}$ exist and

$$M^{\#} = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & -2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 3 & 3 \end{bmatrix}.$$

Theorem 3.2 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K^{mxm}$, where $D \in K^{nxn}$ is invertible and $S = A - BD^{-1}C$. If the group inverse $S^{\#}$ exists, then

(i) $M^{\#}$ exists iff $R = D^2 + CS^{\pi}B$ is invertible and $S^{\pi} = I_{m-n} - SS^{\#}$,

(ii) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}, \tag{3.4}$$

where

$$X = S^{\pi}BR^{-1}(D + CS^{\#}B)R^{-1}CS^{\pi} - S^{\#}BR^{-1}CS^{\pi} - S^{\pi}BR^{-1}CS^{\#} + S^{\#},$$
$$Y = S^{\pi}BR^{-1}(D + CS^{\#}B)R^{-1}D - S^{\#}BR^{-1}D,$$
$$Z = DR^{-1}(D + CS^{\#}B)R^{-1}CS^{\pi} - DR^{-1}CS^{\#},$$
$$W = DR^{-1}(D + CS^{\#}B)R^{-1}D.$$

Proof: (i) The matrix M can be written as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ I_{m-n} & 0 \end{bmatrix} \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{bmatrix} 0 & I_n \\ I_{m-n} & 0 \end{bmatrix}^{-1} = PNP^{-1},$$

where

$$P = \begin{bmatrix} 0 & I_n \\ I_{m-n} & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} D & C \\ B & A \end{bmatrix}.$$

So $M^{\#}$ exists if and only if $N^{\#}$ exists. Since $D \in K^{nxn}$ is invertible, then using Theorem 3.1(i), we know that $N^{\#}$ exists if and only if $R = D^2 + CS^{\pi}B$ is invertible. Hence, we prowe that (i) holds.

(ii) If $M^{\#}$ exists, then $M^{\#} = (PNP^{-1})^{\#} = PN^{\#}P^{-1}$. Using Theorem 3.1(ii), we obtain the expression of $M^{\#}$ as in (3.3).

Theorem 3.2. Let $M=\begin{bmatrix} A & B\\ C & 0\end{bmatrix}\in K^{mxm}$ be a matrix, where $A,B,C\in K^{nxn}$ and B is invertible. Then

(i) $M^{\#}$ exists if and only if $r(Z)=n-r(C)$, where $Z=(I_n-CC^{(1)})B^{-1}A(I_n-C^{(1)}C)$,

(ii) If $M^{\#}$ exists, then

$$M^{\#}=\begin{bmatrix} \Delta B^{-1} & X\\ (I_n-BA^{-1}\Delta)B^{-1} & Y\end{bmatrix}, \tag{3.5}$$

where

$$\Delta=(I_n-C^{(1)}C)Z^{(1)}(I_n-CC^{(1)}),$$
$$X=\Delta B^{-1}\Delta+(I_n-BA^{-1}\Delta)C^{(1)}(I_n-B^{-1}A\Delta)$$
$$Y=(I_n-BA^{-1}\Delta)B^{-1}\Delta-B^{-1}A(I_n-\Delta B^{-1}A)C^{(1)}(I_n-B^{-1}A\Delta),$$

for any $C^{(1)}\in C\{1\}$ and $Z^{(1)}\in Z\{1\}$.

Proof. (i) We known that

$$r(M)=r\begin{bmatrix} A & B\\ C & 0\end{bmatrix}=r\begin{bmatrix} 0 & B\\ C & 0\end{bmatrix}=r\begin{bmatrix} B & 0\\ 0 & C\end{bmatrix}r(B)+r(C)=m+r(C)$$

and

$$r(M^2)=r\begin{bmatrix} A^2+BC & AB\\ 0 & CB\end{bmatrix}=r\begin{bmatrix} BC & AB\\ 0 & CB\end{bmatrix}$$

$$=r\begin{bmatrix} C & B^{-1} & 0\\ 0 & C & 0\\ 0 & 0 & 0\end{bmatrix}=r\begin{bmatrix} C & B^{-1}A & 0\\ 0 & C & 0\\ 0 & (I_n-CC^{(1)})B^{-1}A & 0\end{bmatrix}$$

$$=r\begin{bmatrix} C & B^{-1}A-CC^{(1)}B^{-1}A & 0\\ 0 & C & 0\\ 0 & (I_n-CC^{(1)})B^{-1}A & 0\end{bmatrix}$$

$$=r\begin{bmatrix} C & 0 & 0\\ 0 & C & 0\\ 0 & (I_n-CC^{(1)})B^{-1}A & 0\end{bmatrix}$$

$$=r\begin{bmatrix} C & 0 & 0\\ 0 & C & 0\\ 0 & (I_n-CC^{(1)})B^{-1}A & (I_n-CC^{(1)})B^{-1}A(I_n-C^{(1)}C)\end{bmatrix}$$

$$=r\begin{bmatrix} C & 0 & 0\\ 0 & C & 0\\ 0 & (I_n-CC^{(1)})B^{-1}A & (I_n-CC^{(1)})B^{-1}A(I_n-C^{(1)}C)\end{bmatrix}$$

$$= r\begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & (I_n - CC^{(1)})B^{-1}A(I_n - C^{(1)}C) \end{bmatrix}$$

$$= 2r(C) + r(Z).$$

It follows from Lemma 2.1 that $M^{\#}$ exist if and only if $r(Z) = n - r(C)$.

(ii) We denote the right side of the identity (3.5). by direct computation, we have

$$ME = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \text{ and } EM = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix},$$

where

$$A_1 = A\Delta B^{-1} + B(I_n - B^{-1}A\Delta)B^{-1} = I_n,$$

$$A_2 = A(I_n - \Delta B^{-1}A)C^{(1)}(I_n - B^{-1}A\Delta)B^{-1} + A\Delta B^{-1}\Delta$$
$$+B(I_n - B^{-1}A\Delta)B^{-1}\Delta - A(I_n - \Delta B^{-1}A)C^{(1)}(I_n - B^{-1}A\Delta) = \Delta,$$

$$A_3 = C\Delta B^{-1} = 0,$$

$$A_4 = C\Delta B^{-1} + C(I_n - \Delta B^{-1}A)C^{(1)}(I_n - B^{-1}A\Delta)$$
$$= CC^{(1)}(I_n - B^{-1}A\Delta).$$

It follows from Lemma 2.9(i) that $A_2 = I_n - B^{-1}A\Delta$;

$$B_1 = \Delta B^{-1}A + \Delta B^{-1}\Delta C + (I_n - \Delta B^{-1}A)C^{(1)}(I_n - B^{-1}A\Delta)C$$
$$= \Delta B^{-1}A + I_n - \Delta B^{-1}A = I_n,$$

$$B_2 = \Delta B^{-1}B = \Delta,$$

$$B_3 = (I_n - B^{-1}A\Delta)B^{-1}A + (I_n - B^{-1}A\Delta)B^{-1}\Delta C$$
$$-B^{-1}A(I_n - \Delta B^{-1}A)C^{(1)}(I_n - B^{-1}A\Delta)$$
$$= (I_n - B^{-1}A\Delta)B^{-1}A - B^{-1}A(I_n - \Delta B^{-1}A)C^{(1)}C.$$

By using Lemma 2.9 (i), we can get

$$B_3 = 0,\ B_4 = (I_n - B^{-1}A\Delta)B^{-1}B = I_n - B^{-1}A\Delta.$$

Therefore, we have

$$ME = EM = \begin{bmatrix} I_n & 0 \\ 0 & I_n - B^{-1}A\Delta \end{bmatrix}.$$

Furthermore, from Lemma 2.9 (ii), we can write

$$MEM = \begin{bmatrix} I_n & 0 \\ 0 & I_n - B^{-1}A\Delta \end{bmatrix}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A + \Delta C & B \\ (I_n - B^{-1}A\Delta)C & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = M,$$

$$EME=\begin{bmatrix}\Delta B^{-1} & X\\ (I_n-B^{-1}A\Delta)B^{-1} & Y\end{bmatrix}\begin{bmatrix}I_n & 0\\ 0 & I_n-B^{-1}A\Delta\end{bmatrix}$$

$$=\begin{bmatrix}\Delta B^{-1} & \Delta B^{-1}\Delta+(I_n-\Delta B^{-1}A)C^{(1)}(I_n-B^{-1}A\Delta)\\ (I_n-B^{-1}A\Delta)B^{-1} & (I_n-B^{-1}A\Delta)B^{-1}\Delta-B^{-1}A(I_n-\Delta B^{-1}A)C^{(1)}(I_n-B^{-1}A\Delta)\end{bmatrix}$$

$$=E.$$

As a result, we obtain $ME=EM, MEM=M$ and $EME=E$. Thus $E=M^{\#}$.

Corollary 3.3 Let $M=\begin{bmatrix}A & B\\ C & 0\end{bmatrix}\in K^{mxm}$ where $A,B,C\in K^{nxn}$ and C is invertible. Then

(i) $M^{\#}$ exists if and only if $r(Z)=n-r(B)$, where $Z=(I_n-BB^{(1)})AC^{-1}(I_n-B^{(1)}B)$,

(ii) If $M^{\#}$ exists, then

$$M^{\#}=\begin{bmatrix}C^{-1}\Delta & C^{-1}(I_n-\Delta AC^{-1})\\ X & Y\end{bmatrix}, \tag{3.6}$$

where

$\Delta=(I_n-B^{(1)}B)Z^{(1)}(I_n-BB^{(1)})$,

$X=\Delta C^{-1}\Delta+(I_n-\Delta AC^{-1})B^{(1)}(I_n-AC^{-1}\Delta)$

$Y=\Delta C^{-1}(I_n-\Delta AC^{-1})-(I_n-\Delta AC^{-1})B^{(1)}(I_n-AC^{-1}\Delta)AC^{-1}$,

for any $B^{(1)}\in B\{1\}$ and $Z^{(1)}\in Z\{1\}$.

Proof: (i) Now we consider that

$$M=\begin{bmatrix}A & B\\ C & 0\end{bmatrix}=\begin{bmatrix}AC^{-1} & I_n\\ I_n & 0\end{bmatrix}\begin{bmatrix}CAC^{-1} & C\\ B & 0\end{bmatrix}\begin{bmatrix}AC^{-1} & I_n\\ I_n & 0\end{bmatrix}^{-1}=PNP^{-1},$$

where $P=\begin{bmatrix}AC^{-1} & I_n\\ I_n & 0\end{bmatrix}$ and $N=\begin{bmatrix}CAC^{-1} & C\\ B & 0\end{bmatrix}$. Then we known that $M^{\#}$ exists if and only if $N^{\#}$ exists. Since C is invertible, then by applying Theorem 3.2 (i), we get $N^{\#}$ exists if and only if $r(Z)=n-r(B)$, where $Z=(I_n-BB^{(1)})AC^{-1}(I_n-B^{(1)}B)$. Hence (i) is proved.

(ii) Since the proof is similar to Theorem 3.2(ii), it is easy to check that the result holds.

Corollary 3.4 Let $M=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\in K^{mxm}$, where $A,B,C,D\in K^{nxn}$ and B is invertible. Then

(i) $M^{\#}$ exists iff $r(Z)=n-r(S)$, where $Z=\left(I_n-SS^{(1)}\right)W\left(I_n-S^{(1)}S\right)$, $W=B^{-1}A+DB^{-1}$ and $S=C-DB^{-1}A$.

(ii) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} \Delta B^{-1} - EDB^{-1} & E \\ (I_n - B^{-1}A\Delta)B^{-1} - FDB^{-1} & F \end{bmatrix}, \tag{3.7}$$

where

$$\Delta = (I_n - S^{(1)}S)Z^{(1)}(I_n - SS^{(1)}),$$

$$E = \Delta B^{-1}\Delta + (I_n - \Delta W)S^{(1)}(I_n - W\Delta),$$

$$F = (I_n - B^{-1}\Delta A)B^{-1}\Delta - B^{-1}A(I_n - \Delta W)S^{(1)}(I_n - W\Delta),$$

for any $S^{(1)} \in S\{1\}$ and $Z^{(1)} \in Z\{1\}$.

Proof: (i) Notice that

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ DB^{-1} & I_n \end{bmatrix} \begin{bmatrix} A + BDB^{-1} & B \\ C - DB^{-1}A & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ DB^{-1} & I_n \end{bmatrix}^{-1} = PNP^{-1},$$

where $P = \begin{bmatrix} I_n & 0 \\ DB^{-1} & I_n \end{bmatrix}$ and $N = \begin{bmatrix} A + BDB^{-1} & B \\ C - DB^{-1}A & 0 \end{bmatrix}$. So $M^{\#}$ exists if and only if $N^{\#}$ exists. Since B is invertible, then by applying Theorem 3.2(i), we know that $N^{\#}$ exists if and only if $r(Z) = n - r(S)$, where $Z = (I_n - SS^{(1)})W(I_n - S^{(1)}S)$ and $W = B^{-1}A + DB^{-1}$. Therefore, we prove that (i) is holds.

(ii) Since the proof is similar to Theorem 3.2(ii), it is easy to check that the result holds.

Corollary 3.5 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in K^{mxm}$ be a matrix, where $A, B, C, D \in K^{nxn}$ and C is invertible. Then

(i) $M^{\#}$ exists if and only if $r(Z) = n - r(S)$, where $Z = \left(I_n - SS^{(1)}\right)V\left(I_n - S^{(1)}S\right)$, $V = B^{-1}A + DB^{-1}$ and $S = C - DB^{-1}A$.

(ii) If $M^{\#}$ exists, then

$$M^{\#} = \begin{bmatrix} C^{-1}\Delta - C^{-1}DE & C^{-1}(I_n - \Delta AC^{-1}) - C^{-1}\Delta F \\ E & F \end{bmatrix}, \tag{3.8}$$

where

$$\Delta = (I_n - S^{(1)}S)Z^{(1)}(I_n - SS^{(1)}),$$

$$E = \Delta C^{-1}\Delta + (I_n - \Delta V)S^{(1)}(I_n - V\Delta),$$

$$F = \Delta C^{-1}(I_n - \Delta AC^{-1}) - (I_n - V\Delta)S^{(1)}(I_n - V\Delta)AC^{-1},$$

for any $S^{(1)} \in S\{1\}$ and $Z^{(1)} \in Z\{1\}$.

Proof: The proof is similar to Corollary 3.4.

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