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On the Duality of the Natural Transform with Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal, ARA Transforms

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Sawi Transform

Abstract: In this study, the duality between the Natural transform and the Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal and ARA transforms is systematically investigated. We chose these transforms because they are widely used in the literature. These transforms, many of which are recent developments in operational calculus, are examined in the context of their structural similarities and behavior when applied to a representative function. To illustrate this duality, we compute and graphically compare the transforms of a chosen function under each method. The analysis highlights not only the computational and theoretical similarities between the transforms, but also their respective efficiencies and distinguishing features with examples. This comparative approach provides new insights into the interconnected nature of integral transforms and their applicability in solving differential and integral equations across various domains of applied mathematics.

Naturel Dönüşümün Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal, ARA Dönüşümleriyle Dualitesi Üzerine

Anahtar Kelimeler
Integral dönüşüm,
Naturel dönüşümün dualitesi,
Elzaki dönüşümü,
Rishi dönüşümü,
ARA dönüşümü,
Sawi dönüşümü

Öz: Bu çalışmada, Naturel dönüşüm ile Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal ve ARA dönüşümleri arasındaki dualite sistematik olarak incelenmiştir. Bu dönüşümleri seçmemizin nedeni literatürde yaygın olarak kullanılmalarıdır. Birçoğu operasyonel hesaplamada son gelişmeler olan bu dönüşümler, yapısal benzerlikleri ve bir fonksiyona uygulandıklarında davranışları bağlamında incelenmiştir. Bu dualiteyi göstermek için, her yöntem altında seçilen bir fonksiyonun dönüşümlerini hesaplayıp grafiksel olarak karşılaştırdık. Analiz, yalnızca dönüşümler arasındaki hesaplamalı ve teorik benzerlikleri değil, aynı zamanda bunların verimliliklerini ve ayırt edici özelliklerini örnekler ile de vurgulamaktadır. Bu karşılaştırmalı yaklaşım, integral dönüşümlerin birbirine bağlı doğasına ve uygulamalı matematiğin çeşitli alanlarında diferansiyel ve integral denklemleri çözmede uygulanabilirliğine dair yeni bakış açıları sunmaktadır.

1. Introduction

In the realm of mathematical analysis, transforms play a pivotal role in the study of functional equations, and various applications within engineering, physics and mathematics. Among the numerous integral transforms utilized, the Natural transform stands out as a significant tool for solving various problems in these fields. In recent years, the study of duality principles within different transforms has gained attention due to its potential to link diverse mathematical structures, enhance solution strategies, and provide deeper insights into functional behavior. This duality is observed between Natural transform and some special transforms like Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal and ARA.

As described in contemporary literature, Natural transform is one of the well-known integral transforms, offering flexibility in treating various boundary conditions and operational equations. This makes it an important method for solving differential and integral equations with diverse applications in theoretical and applied mathematics. The duality of transforms in this context refers to the relationship between two distinct but interrelated transforms, which yield insights from two different perspectives when applied to a given function. These dualities often arise from their shared properties or structures, such as their kernels or operational forms. The dualities of Natural transform with Sumudu, Laplace, Fourier, Mellin, Aboodh, Formable, Abdallah Group, Polynomial Integral, Generalized Bivariate and Shehu transforms are available in the literature [1-8].

The Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal, and ARA transforms are some of the many integral transforms that have been studied for their unique properties and applications. Each of these transforms provides an alternative approach for handling boundary value problems, integral equations, and certain differential forms. These transforms are used in different applications. For instance, Sawi's transform, has been shown to be particularly effective in solving problems involving fractional calculus and operational equations [9]. Similarly, Anuj transform finds applications in the field of population dynamics [10]. On the other hand, Elzaki transform is employed for its ability to manage complex functional equations and is often used in engineering domains [11]. In the study by Turab et al., Rishi Transform was used to solve very high order fractional differential equations with constant coefficients [12]. Kumar et al. applied Mohand transform to mechanics and electrical circuit problems [13]. Aggarwal et al solved the second kind of linear Volterra integral problem using the Kamal transform [14]. Chu et al used ARA Transform in the solution of fractional order wave-like equations with variable coefficients [15]. Higazy et al determined the number of infected cells and the concentration of viral particles in plasma during HIV-1 infections using Shehu transform [16].

Recently, researchers have applied Natural transform in many areas. Some areas covered in the literature are given: Elbadri studied Natural transform to solve the Fractional Klein-Gordon equation [17]. Başkonuş et al. solved linear and nonlinear partial differential equations using Natural transform decomposition method [18]. Muti and Taflan presented a comparative study on electrical circuits using the Runge Kutta numerical method and Natural transform [19]. Nonhomogeneous fractional ordinary differential equations were solved using the Natural transform by Samuel and Gill [20]. Köklü studied that a solution to the first type of logarithmic kernel Volterra integral equations has been produced by Natural transform [21]. Alkan and Anaç obtained a numerical solution for the time-fractional Fornberg-Whitham equation using the fractional natural transform decomposition method [22].

Apart from these transforms, there are also other transforms that contribute to the literature. For example, Kashuri Fundo and Formable transforms. Güngör studied first- and second-kind linear Volterra integral equations of convolution type are solved using the Formable transform [23]. Peker et al. studied that Kashuri Fundo is a useful strategy for resolving stable heat transfer issues, and the outcomes are contrasted with those of other approaches [24]. Also, Aggarwal et al. examined the dualities of the Kamal transform with some common transforms [25].

What is particularly intriguing in analyzing these transforms is the exploration of their duality with the Natural transform. The duality principle suggests a connection between the transforms in terms of their operational behavior, such as the Fourier-type relationships or the way they map certain function classes to others. This duality can potentially enhance the efficacy of these transforms when solving specific classes of problems, offering broader methods of solution and optimization.

Furthermore, the Rishi, Mohand, and Kamal transforms are linked with various analytical techniques in mathematics' theoretical and applied aspects. Their duality with the Natural transform enriches the methods available for solving problems related to partial differential equations and special functions. The study of these dualities, therefore, not only broadens the scope of integral transform techniques but also paves the way for more efficient and effective solutions in mathematical physics and engineering.

In this study, we will examine the duality between Natural transform and the transforms mentioned above. We will also highlight the characteristics of these dualities through example applications and differential equation applications. This research aims to provide information about the complex relationships between these transforms and their potential to solve real-world problems.

2. Materials and Methods

In this part, Natural, Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal, and ARA transforms are defined, and the dualities of the transforms are examined.

2.1. Definitions and Theorems

Khan and Khan defined Natural transform [26]. Belgacem and Silambarasan defined inverse Natural transform and studied some properties [27,1]. On the other hand, the Natural transform shares a dual relationship with the Laplace transform, enhancing its utility in solving integral and differential equations.

Throughout the article, Natural, Elzaki, Sawi, Mohand, and Kamal transforms are defined in the following domain $\mathfrak{D}$

$$\mathfrak{D} = \left\{ f(t) \,:\, \exists\, \delta, \lambda_1, \lambda_2 > 0, \qquad |f(t)| \leq \delta\, e^{\frac{|t|}{\lambda_j}} \; if\; t\epsilon(-1)^j \times [0,\infty) \right\}. \tag{1}$$

2.1.1. Definition: On the set of definitions, Natural transform was defined [26,1]:

$$N\{f(t)\} = R(p,s) = \int_0^\infty f(pt)e^{-st}dt. \tag{2}$$

The Natural transform retains many properties of Laplace transform while offering unique advantages, such as its straightforward application to problems involving initial conditions and boundary value problems.

2.1.2. Definition: The Anuj transform of a piecewise continuous exponential order function $f(t)$, $t \geq 0$ is given by

$$\mathcal{T}_{\mathcal{A}}\{f(t)\} = \mathcal{A}(p) = p^2 \int_0^\infty f(t)e^{-\left(\frac{1}{p}\right)t}dt, \qquad p > 0. \tag{3}$$

Here $\mathcal{T}_A$ denotes the Anuj transform operatör [28].

2.1.3. Definition: The Rishi transform of an exponential order piecewise continuous function, $f(t)$ defined in the interval $[0, \infty)$ is given by [29]:

$$\mathcal{T}_{\mathcal{R}}\{f(t)\} = \mathcal{R}(s,p) = \left(\frac{p}{s}\right)\int_0^\infty f(t)\, e^{-\left(\frac{s}{p}\right)t}dt, \qquad s > 0, \qquad p > 0. \tag{4}$$

2.1.4. Definition: The Elzaki transform of the function $f(t)$ defined on $\mathfrak{D}$ is denoted by $\mathcal{T}_E$ (.) and is defined by the following integral equation [30]:

$$\mathcal{T}_{\mathcal{E}}\{f(t)\} = \mathcal{E}(p) = p\int_0^\infty f(t)e^{\frac{-t}{p}}dt\;, \quad t \geq 0, \quad \lambda_1 \leq p \leq \lambda_2. \tag{5}$$

2.1.5. Definition: The Sawi transform of the function $f(t)$ defined on $\mathfrak{D}$ is denoted by $\mathcal{T}_S$ (.) and is defined by the following integral equation [31]:

$$\mathcal{T}_{\mathcal{S}}\{f(t)\} = \mathcal{S}(p) = \frac{1}{p^2}\int_0^\infty f(t)e^{\frac{-t}{p}}dt \qquad t \geq 0, \qquad \lambda_1 \leq p \leq \lambda_2. \tag{6}$$

2.1.6. Definition: The Mohand transform of the function $f(t)$ defined on $\mathfrak{D}$ is denoted by $\mathcal{T}_m$ (.) and is defined by the following integral equation

$$\mathcal{T}_m\{f(t)\} = m(p) = p^2\int_0^\infty f(t)e^{-pt}dt. \tag{7}$$

The variable p in this transform is used to factor the variable t in the argument of the function f [32].

2.1.7. Definition: The Kamal transform of the function $f(t)$ defined on $\mathfrak{D}$ is denoted by $\mathcal{T}_{\mathcal{K}}$ (.) and is defined by the following integral equation [33]:

$$\mathcal{T}_{\mathcal{K}}\{f(t)\} = \mathcal{K}(p) = \int_0^\infty f(t)e^{\frac{-t}{p}}dt. \tag{8}$$

2.1.8. Definition: The ARA integral transform of order n of the continuous function $f(t)$ on the interval $(0,\infty)$ is defined [34]:

$$\mathcal{T}_{\mathcal{AR}}\{f(t)\}(p)=\mathcal{AR}(n,p)=G(n,p)=p\int_0^\infty t^{n-1}\,e^{-pt}f(t)dt,\quad p>0. \tag{9}$$

2.2. Duality of Natural transform

In this section we introduce dualities between Natural transform and some useful integral transforms.

Theorem 2.2.1: Natural - Anuj Duality

Let $N\{f(t)\}$ and $\mathcal{T}_{\mathcal{A}}\{f(t)\}$ be Natural and Anuj transforms of the function $f(t)$, respectively. Then the following equations are satisfied.

$$N\{f(t)\}=\frac{s^2}{p^3}\mathcal{T}_{\mathcal{A}}\left(\frac{p}{s}\right) \tag{10}$$

$$\mathcal{T}_{\mathcal{A}}\{f(t)\}=p^2s\,R(s,\frac{s}{p}) \tag{11}$$

Proof: We know that the Natural transform is $N\{f(t)\}=R(p,s)=\int_0^\infty f(pt)e^{-st}dt$ and Anuj transform is $\mathcal{T}_{\mathcal{A}}\{f(t)\}=p^2\int_0^\infty f(t)e^{-\left(\frac{1}{p}\right)t}dt$. If we make $w=pt$, then we obtain $dw=pdt$. And by substituting $t=\frac{w}{p}$ in the equation (2), we get $R(p,s)=\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}\frac{dw}{p}$. If we rearrange the integral
$R(p,s)=\frac{1}{p}\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw=\frac{1}{p}\int_0^\infty f(w)e^{\left(-\frac{1}{\frac{p}{s}}\right)w}dw=\frac{s^2}{p^3}\left[\frac{p^2}{s^2}\int_0^\infty f(w)e^{\left(-\frac{1}{\frac{p}{s}}\right)w}dw\right]$, equation (10) is obtained.

On the contrary, we take $\mathcal{T}_{\mathcal{A}}\{f(t)\}=p^2\int_0^\infty f(t)e^{-\left(\frac{1}{p}\right)t}dt$. If we make $t=sw$, then we obtain $dt=sdw$. If these are substituted in equation (3), we get
$\mathcal{T}_{\mathcal{A}}\{f(t)\}=p^2\int_0^\infty f(t)e^{-\left(\frac{1}{p}\right)t}dt=p^2\int_0^\infty f(sw)e^{-\left(\frac{1}{p}\right)sw}sdw=p^2s\,R(s,\frac{s}{p})$. So equation (11) is held.

Theorem 2.2.2: Natural - Rishi Duality

Let $N\{f(t)\}=R(p,s)$ and $\mathcal{T}_{\mathcal{R}}\{f(t)\}=h(s,p)$. Then equations (12) and (13) are provided

$$N\{f(t)\}=\frac{s}{p^2}\mathcal{T}_{\mathcal{R}}(s,p) \tag{12}$$

$$\mathcal{T}_{\mathcal{R}}\{f(t)\}=pR\left(s,\frac{s^2}{p}\right). \tag{13}$$

Proof: If we make $w=pt$, then we obtain $dw=pdt$. And substituting $t=\frac{w}{p}$ in equation (2), we get
$R(p,s)=\frac{1}{p}\int_0^\infty f(w)e^{-\frac{s}{p}w}dw=\frac{s}{p^2}\left[\frac{p}{s}\int_0^\infty f(w)e^{-\frac{s}{p}w}dw\right]=\frac{s}{p^2}\mathcal{T}_{\mathcal{R}}(s,p)$.
If we make $t=sw$, then we obtain $dt=sdw$. And substituting $t=sw$ in the equation (4), we obtain
$\mathcal{T}_{\mathcal{R}}\{f(t)\}=\left(\frac{p}{s}\right)\int_0^\infty f(t)\,e^{-\left(\frac{s}{p}\right)t}dt=\left(\frac{p}{s}\right)\int_0^\infty f(sw)e^{-\left(\frac{s}{p}\right)sw}sdw=pR\left(s,\frac{s^2}{p}\right)$.

Theorem 2.2.3: Natural - Elzaki Duality

Let $N\{f(t)\}$ and $\mathcal{T}_{\mathcal{E}}\{f(t)\}$ be the Natural and Elzaki transforms of the function $f(t)$, respectively. Then equations (14) and (15) are provided

$$N\{f(t)\}=\frac{s}{p^2}\mathcal{E}\left(\frac{p}{s}\right) \tag{14}$$

$$\mathcal{T}_{\mathcal{E}}\{f(t)\} = psR\left(s, \frac{s}{p}\right). \tag{15}$$

Proof: Substituting in equation (2) $w = pt \Rightarrow dw = pdt$, we have

$$R(p,s) = \frac{1}{p}\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw = \frac{s}{p^2}\left[\frac{p}{s}\int_0^\infty f(w)e^{\left(-\frac{w}{\frac{p}{s}}\right)}dw\right] = \frac{s}{p^2}\mathcal{E}\left(\frac{p}{s}\right).$$

If we make $t = sw$, then we obtain $dt = sdw$. And we substitute $t = sw$ in the equation (5). We get

$$\mathcal{T}_{\mathcal{E}}\{f(t)\} = p\int_0^\infty f(t)e^{\frac{-t}{p}}dt = p\int_0^\infty f(sw)e^{\frac{-sw}{p}}sdw = psR\left(s, \frac{s}{p}\right).$$

Theorem 2.2.4: Natural - Sawi Duality

Let $N\{f(t)\}$ and $\mathcal{T}_{\mathcal{A}}\{f(t)\}$ be the Natural and Sawi transforms of the function $f(t)$, respectively. Then equations (16) and (17) are provided

$$N\{f(t)\} = \frac{p}{s^2}\mathcal{S}\left(\frac{p}{s}\right) \tag{16}$$

$$\mathcal{T}_{\mathcal{S}}\{f(t)\} = \frac{s}{p^2}\ R\left(s, \frac{s}{p}\right). \tag{17}$$

Proof: Substituting in equation (2) $w = pt \Rightarrow dw = pdt$, we have

$$R(p,s) = \frac{1}{p}\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw = \frac{p}{s^2}\left[\frac{1}{\frac{p^2}{s^2}}\int_0^\infty f(w)e^{\left(-\frac{w}{\frac{p}{s}}\right)}dw\right] = \frac{p}{s^2}\mathcal{S}\left(\frac{p}{s}\right)$$

If we make $t = sw$, then we obtain $dt = sdw$. And we substitute $t = sw$ in the equation (6). We get

$$\mathcal{T}_{\mathcal{S}}\{f(t)\} = \frac{1}{p^2}\int_0^\infty f(t)e^{-\frac{t}{p}}dt = \frac{1}{p^2}\int_0^\infty f(sw)e^{-\frac{sw}{p}}sdw = \frac{s}{p^2}R\left(s, \frac{s}{p}\right).$$

Theorem 2.2.5: Natural - Mohand Duality

Let $N\{f(t)\}$ and $\mathcal{T}_m\{f(t)\}$ be the Natural and Mohand transforms of the function $f(t)$, respectively. Then equations (18) and (19) are provided

$$N\{f(t)\} = \frac{p}{s^2}m\left(\frac{s}{p}\right) \tag{18}$$

$$\mathcal{T}_m\{f(t)\} = p^2 s\ R(s, ps). \tag{19}$$

Proof: Substituting in equation (2) $w = pt \Rightarrow dw = pdt$, we have

$$R(p,s) = \frac{1}{u}\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw = \frac{p}{s^2}\left[\left(\frac{s}{p}\right)^2\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw\right] = \frac{p}{s^2}m\left(\frac{s}{p}\right).$$

If we make $t = sw$, then we obtain $dt = sdw$. And we substitute $t = sw$ in the equation (7). We get

$$\mathcal{T}_m\{f(t)\} = p^2\int_0^\infty f(t)e^{-pt}dt = p^2\int_0^\infty f(sw)e^{-psw}sdw = p^2 s\ \mathcal{R}(s, ps).$$

Theorem 2.2.6: Natural - Kamal Duality

Let $N\{f(t)\}$ and $\mathcal{T}_{\mathcal{K}}\{f(t)\}$ be the Natural and Kamal transforms of the function $f(t)$, respectively. Then equations (20) and (21) are provided

$$N\{f(t)\} = \frac{1}{p}\mathcal{K}\left(\frac{p}{s}\right) \tag{20}$$

$$\mathcal{T}_{\mathcal{K}}\{f(t)\} = sR\left(s, \frac{s}{p}\right). \tag{21}$$

Proof: Substituting in equation (2) $w = pt \Rightarrow dw = pdt$, we have

$R(p,s) = \frac{1}{p}\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw = \frac{1}{p}\left[\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw\right] = \frac{1}{p}\mathcal{K}\left(\frac{p}{s}\right).$

If we make $t = sw$, then we obtain $dt = sdw$. And we substitute $t = sw$ in the equation (8). We get
$\mathcal{T}_{\mathcal{K}}\{f(t)\} = \int_0^\infty f(t)e^{-\frac{t}{p}}dt = \int_0^\infty f(sw)e^{-\frac{sw}{p}}sdw = sR\left(s,\frac{S}{p}\right).$

Theorem 2.2.7: Natural - ARA Duality

Let $N\{f(t)\}$ and $\mathcal{T}_{\mathcal{AR}}\{f(t)\}$ be the Natural and ARA transforms of the function $f(t)$, respectively. Firstly for $n = 1$; equation (22) is obtained

$$N\{f(t)\} = \frac{1}{s}G_1\left(\frac{s}{p}\right), \quad G_1(p) = psR(s,ps) \tag{22}$$

Secondly; in general equation (23) is provided

$$N\{t^{n-1}f(t)\} = \frac{1}{s}G\left(n,\frac{s}{p}\right). \tag{23}$$

Proof: For $n = 1$; we get equation (9) as $G_1[f(t)](p) = p\int_0^\infty t^0\, e^{-pt}f(t)dt$. We arrange equation (2) as:
$R(p,s) = \frac{1}{p}\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw = \frac{1}{s}\left[\frac{s}{p}\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw\right].$
We get $N\{f(t)\} = \frac{1}{s}G_1\left[\left(\frac{s}{p}\right)\right].$
For $G_1(p) = psR(s,ps)$ we get $G_1(p) = p\int_0^\infty e^{-pt}f(t)dt$ and substitute $t = sw$ then
$G_1(p) = ps\int_0^\infty e^{-psw}f(sw)dw = psR(s,ps)$ is found.
Secondly, let us consider equations (9) and (2). If $w = pt$ is written in equation (2),
$R(p,s) = \int_0^\infty f(pt)e^{-st}dt = \frac{1}{p}\int_0^\infty f(w)e^{\left(-\frac{s}{p}\right)w}dw$ is obtained. Let us compute the Natural transform of the function $t^{n-1}f(t)$ in this context. Then
$N\{t^{n-1}f(t)\} = \int_0^\infty (pt)^{n-1}f(pt)e^{-st}dt = \int_0^\infty p^{n-1}t^{n-1}f(pt)e^{-st}dt$
Substituting $w = pt \Rightarrow dw = pdt$ and $t = \frac{w}{p}$ in the above equation

$$\begin{aligned} N\{t^{n-1}f(t)\} &= \int_0^\infty p^{n-1}\left(\frac{w}{p}\right)^{n-1}f(w)e^{\left(-\frac{s}{p}\right)w}\frac{dw}{p} \\ &= \frac{1}{p}\int_0^\infty w^{n-1}f(w)e^{-\frac{s}{p}w}dw = \frac{1}{s}\left[\frac{s}{p}\int_0^\infty w^{n-1}f(w)e^{-\frac{s}{p}w}dw\right] \\ &= \frac{1}{s}G_n[f(t)]\left(\frac{s}{p}\right) = \frac{1}{s}G\left(n,\frac{s}{p}\right). \end{aligned}$$

Then equation (23) is obtained.

3. Application of dualities

In this section, we have selected three examples. In Example 1, we applied dualities to seven commonly used functions consisting of polynomial, exponential, and trigonometric functions. These are presented in Table 1. In Example 2, the graphs of the function $f(t) = t\,e^{2t} + \cos(3t)$ are compared with the Natural transform and other transforms studied above. In Table 2, Natural, Anuj, Elzaki, Sawi, Mohand, Kamal, Rishi, ARA transforms of the function $f(t) = t\,e^{2t} + \cos(3t)$ are given. In Example 3, the undamped spring-mass system problem selected in the field of applied mathematics is examined, solved using all the transforms in the study and dualities are obtained.

3.1. Example 1. The following table provides examples of the Natural transform and dualities of commonly used functions (polynomial, exponential and trigonometric functions). The dualities of other functions not included in this study are computed similarly utilizing these dualities.

Table 1. The Natural transform of the functions $g(t)$ and dualities

$\boldsymbol{g(t)}$	$\mathbf{1}$	$\boldsymbol{t}$	$\boldsymbol{\frac{t^{n-1}e^{at}}{(n-1)!}}$, $\boldsymbol{n} = \mathbf{1, 2, \ldots}$	$\boldsymbol{\frac{\sin at}{a}}$	$\boldsymbol{\cos at}$	$\boldsymbol{\frac{e^{bt}\sin at}{a}}$	$\boldsymbol{e^{bt}\cos at}$
$\boldsymbol{N\{g(t)\} = R(p,s)}$	$\frac{1}{s}$	$\frac{p}{s^2}$	$\frac{p^{n-1}}{(s-ap)^n}$	$\frac{p}{s^2+a^2p^2}$	$\frac{s}{s^2+a^2p^2}$	$\frac{p}{(s-bp)^2+a^2p^2}$	$\frac{s-bp}{(s-bp)^2+a^2p^2}$
$\boldsymbol{\mathcal{T}_{\mathcal{A}}\{g(t)\} = p^2 s\, R(s, \frac{s}{p})}$	p^3	p^4	$\frac{p^{n+2}}{(1-ap)^n}$	$\frac{p^4}{1+a^2p^2}$	$\frac{p^3}{1+a^2p^2}$	$\frac{p^4}{(1-bp)^2+a^2p^2}$	$\frac{p^3(1-bp)}{(1-bp)^2+a^2p^2}$
$\boldsymbol{\mathcal{T}_{\mathcal{R}}\{g(t)\} = pR\left(s, \frac{s^2}{p}\right)}$	$\frac{p^2}{s^2}$	$\frac{p^3}{s^3}$	$\frac{p^{n+1}}{s(s-ap)^n}$	$\frac{p^3}{s(s^2+a^2p^2)}$	$\frac{p^2}{s^2+a^2p^2}$	$\frac{p^3}{s[(s-bp)^2+a^2p^2]}$	$\frac{(s-bp)p^2}{s[(s-bp)^2+a^2p^2]}$
$\boldsymbol{\mathcal{T}_{\mathcal{E}}\{g(t)\} = psR\left(s, \frac{s}{p}\right)}$	p^2	p^3	$\frac{p^{n+1}}{(1-ap)^n}$	$\frac{p^3}{1+a^2p^2}$	$\frac{p^2}{1+a^2p^2}$	$\frac{p^3}{(1-bp)^2+a^2p^2}$	$\frac{p^2(1-bp)}{(1-bp)^2+a^2p^2}$
$\boldsymbol{\mathcal{T}_{\mathcal{S}}\{g(t)\} = \frac{s}{p^2}\ R\left(s, \frac{s}{p}\right)}$	$\frac{1}{p}$	1	$\frac{p^{n-2}}{(1-ap)^n}$	$\frac{1}{1+a^2p^2}$	$\frac{1}{p(1+a^2p^2)}$	$\frac{1}{(1-bp)^2+a^2p^2}$	$\frac{(1-bp)}{p[(1-bp)^2+a^2p^2]}$
$\boldsymbol{\mathcal{T}_{m}\{g(t)\} = p^2 s\, R(s, ps)}$	p	1	$\frac{p^2}{(p-a)^n}$	$\frac{p^2}{p^2+a^2}$	$\frac{p^3}{p^2+a^2}$	$\frac{p^2}{(p-b)^2+a^2}$	$\frac{p^2(p-b)}{(p-b)^2+a^2}$
$\boldsymbol{\mathcal{T}_{\mathcal{K}}\{g(t)\} = s\mathcal{R}\left(s, \frac{s}{p}\right)}$	p	p^2	$\frac{p^n}{(1-ap)^n}$	$\frac{p^2}{1+a^2p^2}$	$\frac{p}{1+a^2p^2}$	$\frac{p^2}{(1-bp)^2+a^2p^2}$	$\frac{p(1-bp)}{(1-bp)^2+a^2p^2}$
$\boldsymbol{G_1(p) = psR(s, ps)}$	1	$\frac{1}{p}$	$\frac{p}{(p-a)^n}$	$\frac{p}{p^2+a^2}$	$\frac{p^2}{p^2+a^2}$	$\frac{p}{(p-b)^2+a^2}$	$\frac{p(p-b)}{(p-b)^2+a^2}$

3.2. Example 2. We considered the function $f(t) = t\, e^{2t} + \cos(3t)$. It is given in Figure 1. The Natural, Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal and ARA transforms are applied to this function, and a graph is plotted for each transform. These are presented comparatively in Figure 2. The purpose of selecting this function is that it contains both exponential and trigonometric functions as well as a polynomial product.

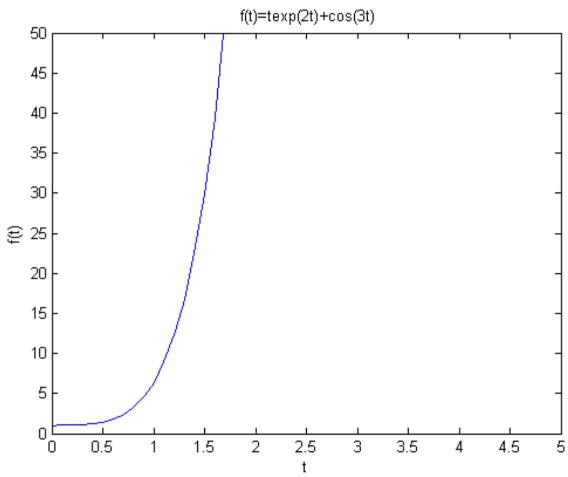

Figure 1. $f(t) = t\, e^{2t} + \cos(3t)$ for $t \in [0,5]$

Table 2: Natural, Anuj, Elzaki, Sawi, Mohand, Kamal, Rishi, ARA transforms of the function $f(t) = t\,e^{2t} + \cos(3t)$

	Transform of $f(t) = t\,e^{2t} + \cos(3t)$		**Transform of $f(t) = t\,e^{2t} + \cos(3t)$**
Natural Transform	$N\{f(t)\} = \dfrac{p}{s-2p} + \dfrac{s}{s^2+9p^2}$	Sawi Transform	$\mathcal{T}_{\mathcal{S}}\{f(t)\} = \dfrac{1}{(1-2p)^2} + \dfrac{1}{p(1+9p^2)}$
Anuj Transform	$\mathcal{T}_{\mathcal{A}}\{f(t)\} = \dfrac{p^4}{(1-2p)^2} + \dfrac{p^3}{1+9p^2}$	Mohand Transform	$\mathcal{T}_{m}\{f(t)\} = \dfrac{p^2}{(p-2)^2} + \dfrac{p^3}{p^2+9}$
Rishi Transform	$\mathcal{T}_{\mathcal{R}}\{f(t)\} = \dfrac{p^3}{(2p-s)^2} + \dfrac{p^2}{s^2+9p^2}$	Kamal Transform	$\mathcal{T}_{\mathcal{K}}\{f(t)\} = \left(\dfrac{p}{1-2p}\right)^2 + \dfrac{p}{1+9p^2}$
Elzaki Transform	$\mathcal{T}_{\mathcal{E}}\{f(t)\} = \dfrac{p^3}{(1-2p)^2} + \dfrac{p^2}{1+9p^2}$	ARA Transform	$\mathcal{T}_{\mathcal{AR}}\{f(t)\} = \dfrac{p}{(2-p)^2} + \dfrac{p^2}{p^2+9}$

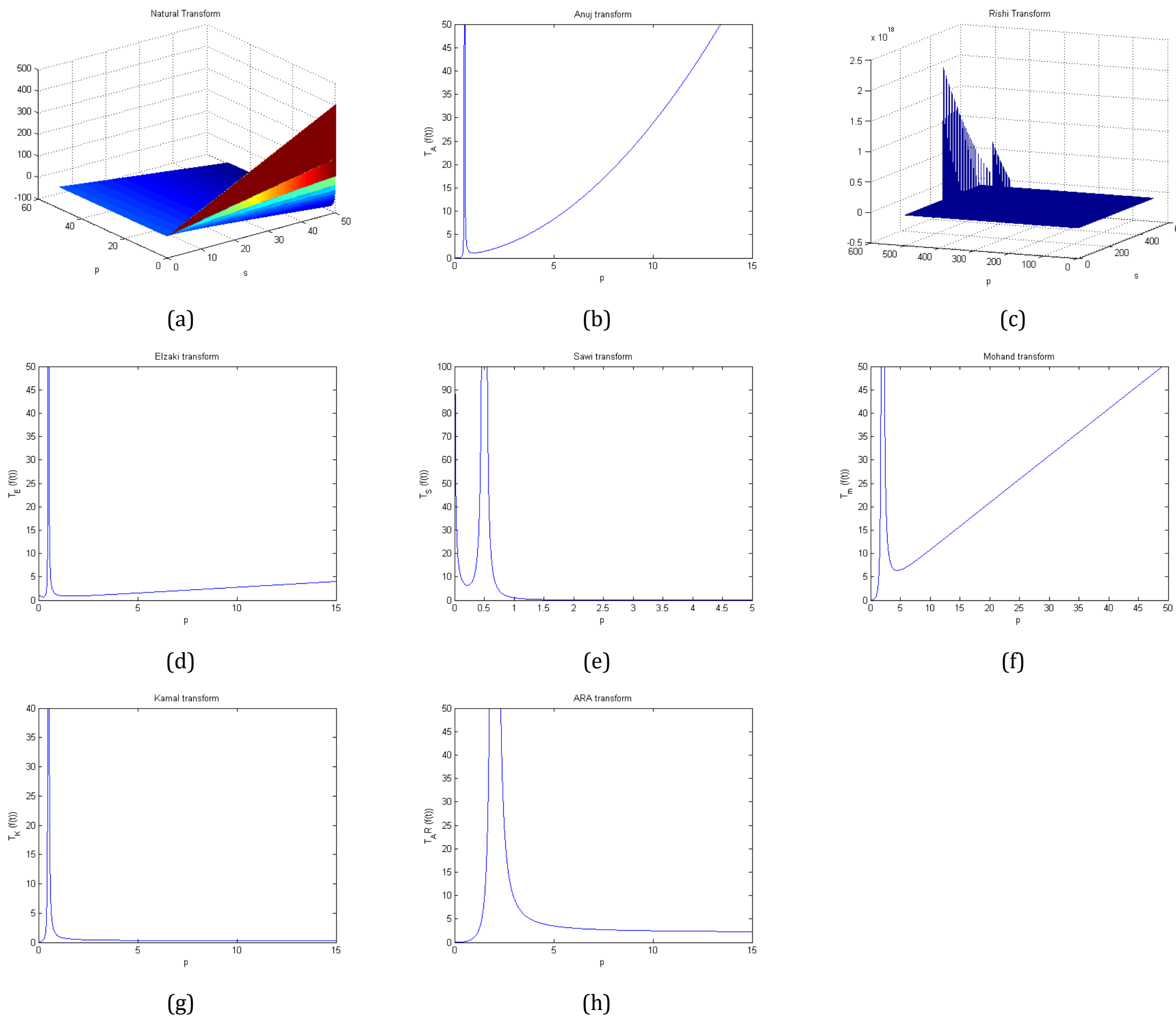

Figure 2. a. Natural, **b.** Anuj, **c.** Rishi, **d.** Elzaki, **e.** Sawi, **f.** Mohand, **g.** Kamal, **h.** ARA transform of the function $f(t) = t\,e^{2t} + \cos(3t)$

As seen in Table 2 and Figure 2, the Anuj, Elzaki, Sawi, and Kamal transforms exhibit similar characteristics at the same point as the function at $p = 0.5$. The Mohand and ARA transforms provide the same feature at $p = 2$. For the Natural and Rishi transforms (3-dimensional transforms), the similarity of the graphs is observed for $s - 2p = 0$.

3.3. Example 3. Let us consider the following undamped spring-mass system with the initial values given below [35].

$$y'' + 2y = 0,\ y(0) = 0,\quad y'(0) = 2. \tag{24}$$

The analytical solution of this equation is $y = \sqrt{2}sin(\sqrt{2}\,x)$.
Let's solve equation (24) with the Natural transform. So, let's take the Natural transform of both sides of equation (24).
$N\{y''\} + 2N\{y\} = N\{0\}$ then we get $\frac{s^2}{p^2}R(p,s) - \frac{s}{p^2}y(0) - \frac{y'(0)}{p} + 2R(p,s) = 0$
Substituting the initial values and regularising the equation, we obtain

$$R(p,s) = \frac{2p}{s^2 + 2p^2}. \tag{25}$$

Let's solve equation (24) by similar procedures with Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal, and ARA transforms:
Solution with Anuj transform is:

$$\mathcal{A}(p) = \frac{2p^4}{2p^2 + 1}. \tag{26}$$

Solution with Rishi transform is:

$$\mathcal{R}(s,p) = \frac{2p^3}{s(s^2 + 2p^2)} \tag{27}$$

Solution with Elzaki transform is:

$$\mathcal{E}(p) = \frac{2p^3}{2p^2 + 1}. \tag{28}$$

Solution with Sawi transform is:

$$\mathcal{S}(p) = \frac{2}{2p^2 + 1}. \tag{29}$$

Solution with Mohand transform is:

$$m(p) = \frac{2p^2}{p^2 + 2}. \tag{30}$$

Solution with the Kamal transform is:

$$\mathcal{K}(p) = \frac{2p^2}{1 + 2p^2}. \tag{31}$$

Solution with the ARA transform is:
If we apply the ARA transform for $n = 1$, we get

$$G_1(s) = \frac{2s}{s^2 + 2}. \tag{32}$$

For the solution of equation (24), equations (10)-(23) are used for the relevant transforms, respectively, and thus the Natural-Anuj, Natural-Rishi, Natural-Elzaki, Natural-Sawi, Natural-Mohand, Natural-Kamal, and Natural-ARA transform dualities are calculated. Let's now illustrate Natural-Anuj duality:

Natural-Anuj duality are $R(p,s) = \frac{s^2}{p^3}\mathcal{A}\left(\frac{p}{s}\right)$, $\quad \mathcal{T}_{\mathcal{A}}\{f(t)\} = p^2 s\, R(s, \frac{s}{p})$.

Firstly, $\frac{s^2}{p^3}\mathcal{A}\left(\frac{p}{s}\right)=\frac{s^2}{p^3}\left(\frac{2\left(\frac{p}{s}\right)^4}{2\left(\frac{p}{s}\right)^2+1}\right)=\frac{2p}{2p^2+s^2}=R(p,s)$ is obtained.

Secondly, $p^2s\,R\left(s,\frac{s}{p}\right)=p^2s\;\frac{2s}{\left(\frac{s}{p}\right)^2+2s^2}=\frac{2p^4}{1+2p^2}=\mathcal{A}(p)$ is obtained. Equations (25) and (26) are found.

The Natural-Rishi, Natural-Elzaki, Natural-Sawi, Natural-Mohand, Natural-Kamal, and Natural-ARA transform dualities are obtained using similar mathematical operations.

4. Discussion and Conclusion

The comparative analysis conducted in this study demonstrates a notable duality between the Natural transform and the Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal and ARA transforms. By applying each transform to some well-defined functions and visualizing the results, we observed the conceptual closeness of these operators, their common kernel structure, and similar asymptotic behavior. Despite differences in form or origin, many of these transforms yield comparable results under specific parametric settings, suggesting a deeper theoretical relationship. It is aimed to contribute to the literature by using the dualities of Anuj, Rishi, Elzaki, Sawi, Mohand, Kamal and ARA transforms, whose dualities are found by Natural transform, instead of taking their integrals.

The Natural transform's role as a unifying framework indicates that future research should focus on mapping these relationships more comprehensively, potentially leading to the development of meta-theoretical approaches that can predict and exploit transform dualities across broader mathematical domains. These findings not only validate the utility of the Natural transform as a unifying framework but also provide a foundation for extending duality-based analysis to broader classes of functional transforms and their applications in engineering, physics, and fractional calculus. For example, these transforms can be explored for fractional-order respiratory system dynamics in biomedical modelling [36], hybrid modeling with optimal control [37], etc.

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