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A Decomposition Formula for Bivariate Hypergeometric-Trigonometric Series

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Abstract. A general identity is presented for bivariate hypergeometric-trigonometric series, which can be considered as a decomposition formula for the aforementioned series. Some special examples are also given in this sense.

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1. Introduction

Let

$$f^*(z) = \sum_{k=0}^{\infty} a_k^* z^k \tag{1.1}$$

be a convergent series in which are known real numbers. If

$$z = x + iy = r\, e^{i\theta} \ (i = \sqrt{-1})$$

is a complex variable with

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arg(x + iy)$$

then it can be verified from (1.1) that

$$\operatorname{Im}\left(z^\lambda f^*(z) + \bar{z}^\lambda f^*(\bar{z})\right) = \operatorname{Im}\left(\frac{z^\lambda f^*(z) - \bar{z}^\lambda f^*(\bar{z})}{i}\right) = 0\,, \quad \forall \lambda \in \mathbb{R}\,. \tag{1.2}$$

This means that $z^\lambda f^*(z) + \bar{z}^\lambda f^*(\bar{z})$ and $(z^\lambda f^*(z) - \bar{z}^\lambda f^*(\bar{z}))/i$ are always two real functions if (1.1) holds. Based on the results (1.2), we have recently introduced two bivariate series in [5] as

$$C_\alpha(f^*; r, \theta, m, n) = \sum_{k=0}^{\infty} a_{n\,k+m}^*\, r^k \cos(\alpha + k)\theta \tag{1.3}$$

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and

$$S_{\alpha}(f^*; r, \theta, m, n) = \sum_{k=0}^{\infty} a^*_{n\,k+m}\, r^k \sin(\alpha + k)\theta \tag{1.4}$$

where r, θ are real variables, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $m \in \{0, 1, ..., n-1\}$, and showed that they are convergent if the reduced series $\sum\limits_{k=0}^{\infty} a^*_{n\,k+m}\, r^k$ is convergent. Now, assume in (1.3) and (1.4) that

$$a^*_k = \frac{(a_1)_k (a_2)_k ... (a_p)_k}{(b_1)_k (b_2)_k ... (b_q)_k}$$

are hypergeometric terms where $(r)_k = \prod\limits_{j=0}^{k-1}(r+j)$ denotes the well-known Pochhammer symbol [1]. Then

$${}_pC_q\left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);\alpha \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k ... (a_p)_k}{(b_1)_k (b_2)_k ... (b_q)_k} \frac{r^k}{k!} \cos(\alpha + k)\theta, \tag{1.5}$$

and

$${}_pS_q\left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);\alpha \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k ... (a_p)_k}{(b_1)_k (b_2)_k ... (b_q)_k} \frac{r^k}{k!} \sin(\alpha + k)\theta, \tag{1.6}$$

are called bivariate hypergeometric-trigonometric series [6]. It is clear from (1.5) and (1.6) that

$$\begin{aligned} {}_pC_q\left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);\alpha \right) = \cos\alpha\theta\, {}_pC_q\left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);0 \right) \\ - \sin\alpha\theta\, {}_pS_q\left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);0 \right), \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} {}_pS_q\left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);\alpha \right) = \sin\alpha\theta\, {}_pC_q\left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);0 \right) \\ + \cos\alpha\theta\, {}_pS_q\left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);0 \right). \end{aligned} \tag{1.8}$$

Many ordinary hypergeometric series (when $\theta = 0$) and Fourier trigonometric series (when r is fixed and pre-assigned) can be represented in terms of the series (1.5) or (1.6). See also [2,6].

2. A General Identity for Bivariate Hypergeometric-Trigonometric Series

First, for any arbitrary series that we clearly have

$$\sum_{k=0}^{\infty} u_k = \sum_{j=0}^{\infty} u_{2j} + \sum_{j=0}^{\infty} u_{2j+1} = \sum_{j=0}^{\infty} u_{3j} + \sum_{j=0}^{\infty} u_{3j+1} + \sum_{j=0}^{\infty} u_{3j+2} = ... = \sum_{j=0}^{\infty} u_{mj} + \sum_{j=0}^{\infty} u_{mj+1} + ... + \sum_{j=0}^{\infty} u_{mj+m-1}, \tag{2.1}$$

where m is a natural number. By recalling the Pochhammer symbol and noting the series (1.5) and (1.6), if

$$u_k = \frac{(a_1)_k (a_2)_k ... (a_p)_k}{(b_1)_k (b_2)_k ... (b_q)_k (1)_k} r^k \left\{ \begin{array}{c} \cos(\alpha + k)\theta \\ \sin(\alpha + k)\theta \end{array} \right\},$$

is substituted in the last equality of (2.1), then we have

$$\begin{aligned} {}_p\left\{ \begin{array}{c} C \\ S \end{array} \right\}_q \left(\begin{array}{cccc} a_1, & a_2, & ... & a_p \\ b_1, & b_2, & ... & b_q \end{array} \middle| (r,\theta);\alpha \right) = \sum_{j=0}^{\infty} \frac{(a_1)_{mj}(a_2)_{mj}...(a_p)_{mj}}{(b_1)_{mj}(b_2)_{mj}...(b_q)_{mj}(1)_{mj}} r^{mj} \left\{ \begin{array}{c} \cos(\alpha + mj)\theta \\ \sin(\alpha + mj)\theta \end{array} \right\} \\ + \sum_{j=0}^{\infty} \frac{(a_1)_{mj+1}(a_2)_{mj+1}...(a_p)_{mj+1}}{(b_1)_{mj+1}(b_2)_{mj+1}...(b_q)_{mj+1}(1)_{mj+1}} r^{mj+1} \left\{ \begin{array}{c} \cos(\alpha + mj + 1)\theta \\ \sin(\alpha + mj + 1)\theta \end{array} \right\} + \cdots \\ + \sum_{j=0}^{\infty} \frac{(a_1)_{mj+m-1}...(a_p)_{mj+m-1}}{(b_1)_{mj+m-1}...(b_q)_{mj+m-1}(1)_{mj+m-1}} r^{mj+m-1} \left\{ \begin{array}{c} \cos(\alpha + mj + m - 1)\theta \\ \sin(\alpha + mj + m - 1)\theta \end{array} \right\}. \end{aligned} \tag{2.2}$$

On the other hand, since the two following identities hold

$$(a)_{mk} = m^{mk} \prod_{j=0}^{m-1} (\frac{a+j}{m})_k \,,$$

and

$$(a)_{mj+i} = (a+i)_{mj} (a)_i,$$

relation (2.2) can be re-written as

$$\begin{aligned} {}_p\left\{ \begin{array}{c} C \\ S \end{array} \right\}_q \left(\begin{array}{cccc} a_1, & a_2, & \ldots & a_p \\ b_1, & b_2, & \ldots & b_q \end{array} \middle| (r,\theta);\alpha \right) &= \sum_{j=0}^{\infty} \frac{\prod_{r=0}^{m-1} (\frac{a_1+r}{m})_j \ldots \prod_{r=0}^{m-1} (\frac{a_p+r}{m})_j}{\prod_{r=0}^{m-1} (\frac{b_1+r}{m})_j \ldots \prod_{r=0}^{m-1} (\frac{b_q+r}{m})_j \prod_{r=0}^{m-1} (\frac{1+r}{m})_j} \left(m^{(p-q-1)m} r^m\right)^j \left\{ \begin{array}{c} \cos(\frac{\alpha}{m} + j)(m\theta) \\ \sin(\frac{\alpha}{m} + j)(m\theta) \end{array} \right\} \\ &+ \frac{(a_1)_1 (a_2)_1 \ldots (a_p)_1}{(b_1)_1 (b_2)_1 \ldots (b_q)_1} \frac{r}{(1)_1} \sum_{j=0}^{\infty} \frac{\prod_{r=0}^{m-1} (\frac{a_1+1+r}{m})_j \ldots \prod_{r=0}^{m-1} (\frac{a_p+1+r}{m})_j}{\prod_{r=0}^{m-1} (\frac{b_1+1+r}{m})_j \ldots \prod_{r=0}^{m-1} (\frac{b_q+1+r}{m})_j \prod_{r=0}^{m-1} (\frac{2+r}{m})_j} \left(m^{(p-q-1)m} r^m\right)^j \left\{ \begin{array}{c} \cos(\frac{\alpha+1}{m} + j)(m\theta) \\ \sin(\frac{\alpha+1}{m} + j)(m\theta) \end{array} \right\} + \ldots \\ &+ \frac{(a_1)_{m-1} (a_2)_{m-1} \ldots (a_p)_{m-1}}{(b_1)_{m-1} (b_2)_{m-1} \ldots (b_q)_{m-1}} \frac{r^{m-1}}{(1)_{m-1}} \sum_{j=0}^{\infty} \frac{\prod_{r=0}^{m-1} (\frac{a_1+m-1+r}{m})_j \ldots \prod_{r=0}^{m-1} (\frac{a_p+m-1+r}{m})_j \left(m^{(p-q-1)m} r^m\right)^j}{\prod_{r=0}^{m-1} (\frac{b_1+m-1+r}{m})_j \ldots \prod_{r=0}^{m-1} (\frac{b_q+m-1+r}{m})_j \prod_{r=0}^{m-1} (\frac{m-1+r}{m})_j} \left\{ \begin{array}{c} \cos(\frac{\alpha+m-1}{m} + j)(m\theta) \\ \sin(\frac{\alpha+m-1}{m} + j)(m\theta) \end{array} \right\}, \end{aligned}$$

which eventually leads to the main theorem.

Theorem 2.1. *For any natural number m, the two series* (1.5) *and* (1.6) *satisfy the relation*

$$\begin{aligned} &{}_p\left\{ \begin{array}{c} C \\ S \end{array} \right\}_q \left(\begin{array}{cccc} a_1, & a_2, & \ldots & a_p \\ b_1, & b_2, & \ldots & b_q \end{array} \middle| (r,\theta);\alpha \right) \\ &= \sum_{k=0}^{m-1} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{r^k}{k!} {}_{(mp+1)}\left\{ \begin{array}{c} C \\ S \end{array} \right\}_{(mq+m)} \left(\begin{array}{cccc} \vec{A}_{1,k}, & \vec{A}_{2,k}, & \ldots & \vec{A}_{p,k},\ 1 \\ \vec{B}_{1,k}, & \vec{B}_{2,k}, & \ldots & \vec{B}_{q,k}, \vec{I}_{1,k} \end{array} \middle| (m^{(p-q-1)m} r^m, m\theta); \frac{\alpha+k}{m} \right), \end{aligned} \tag{2.3}$$

where

$$\vec{A}_{j,k} = (\frac{a_j+k}{m}, \frac{a_j+1+k}{m}, \ldots, \frac{a_j+m-1+k}{m}) \quad (j = 1, 2, \ldots, p),$$

$$\vec{B}_{j,k} = (\frac{b_j+k}{m}, \frac{b_j+1+k}{m}, \ldots, \frac{b_j+m-1+k}{m}) \quad (j = 1, 2, \ldots, q),$$

and

This theorem can be interpreted as a decomposition formula for many hypergeometric-trigonometric series of type (1.5) and (1.6).

Example 2.1. *Since we have [4]*

$${}_0C_0\left(\begin{array}{c} - \\ - \end{array} \middle| (r,\theta);0 \right) = \sum_{k=0}^{\infty} \frac{r^k}{k!} \cos k\theta = e^{r\cos\theta} \cos(r\sin\theta)\,,$$

and

$${}_0S_0\left(\begin{array}{c} - \\ - \end{array} \middle| (r,\theta);0 \right) = \sum_{k=0}^{\infty} \frac{r^k}{k!} \sin k\theta = e^{r\cos\theta} \sin(r\sin\theta)\,,$$

relations (1.7) *and* (1.8) *respectively yield*

$${}_0C_0\left(\begin{array}{c} - \\ - \end{array} \middle| (r,\theta);\alpha \right) = \sum_{k=0}^{\infty} \frac{r^k}{k!} \cos(k+\alpha)\theta = e^{r\cos\theta} \cos(\alpha\theta + r\sin\theta)\,,$$

and

$${}_0S_0\left(\begin{array}{c} - \\ - \end{array} \middle| (r,\theta);0 \right) = \sum_{k=0}^{\infty} \frac{r^k}{k!} \sin(k+\alpha)\theta = e^{r\cos\theta} \sin(\alpha\theta + r\sin\theta)\,,$$

which are valid for any $r \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $\theta \in [-\pi, \pi]$. Hence, by taking $m = 2$ in (2.3) we respectively obtain

$$e^{r\cos\theta}\cos(\alpha\theta + r\sin\theta) = {}_0C_1\left(\begin{array}{c} - \\ 1/2 \end{array}\,\middle|\, (\frac{1}{4}r^2, 2\theta); \frac{\alpha}{2}\right) + r\,{}_0C_1\left(\begin{array}{c} - \\ 3/2 \end{array}\,\middle|\, (\frac{1}{4}r^2, 2\theta); \frac{\alpha+1}{2}\right),$$

and

$$e^{r\cos\theta}\sin(\alpha\theta + r\sin\theta) = {}_0S_1\left(\begin{array}{c} - \\ 1/2 \end{array}\,\middle|\, (\frac{1}{4}r^2, 2\theta); \frac{\alpha}{2}\right) + r\,{}_0S_1\left(\begin{array}{c} - \\ 3/2 \end{array}\,\middle|\, (\frac{1}{4}r^2, 2\theta); \frac{\alpha+1}{2}\right).$$

Similarly for $m = 3$ in (2.3) the decomposition formulae read as

$$\begin{aligned} e^{r\cos\theta}\cos(\alpha\theta + r\sin\theta) &= {}_0C_2\left(\begin{array}{c} - \\ 1/3,\ 2/3 \end{array}\,\middle|\, (\tfrac{1}{27}r^3, 3\theta); \tfrac{\alpha}{3}\right) \\ &+ r\,{}_0C_2\left(\begin{array}{c} - \\ 2/3,\ 4/3 \end{array}\,\middle|\, (\tfrac{1}{27}r^3, 3\theta); \tfrac{\alpha+1}{3}\right) + \tfrac{1}{2}r^2{}_0C_2\left(\begin{array}{c} - \\ 4/3,\ 5/3 \end{array}\,\middle|\, (\tfrac{1}{27}r^3, 3\theta); \tfrac{\alpha+2}{3}\right), \end{aligned}$$

and

$$\begin{aligned} e^{r\cos\theta}\sin(\alpha\theta + r\sin\theta) &= {}_0S_2\left(\begin{array}{c} - \\ 1/3,\ 2/3 \end{array}\,\middle|\, (\tfrac{1}{27}r^3, 3\theta); \tfrac{\alpha}{3}\right) \\ &+ r\,{}_0S_2\left(\begin{array}{c} - \\ 2/3,\ 4/3 \end{array}\,\middle|\, (\tfrac{1}{27}r^3, 3\theta); \tfrac{\alpha+1}{3}\right) + \tfrac{1}{2}r^2{}_0S_2\left(\begin{array}{c} - \\ 4/3,\ 5/3 \end{array}\,\middle|\, (\tfrac{1}{27}r^3, 3\theta); \tfrac{\alpha+2}{3}\right). \end{aligned}$$

Finally for $m = 4$ in (2.3) we have

$$\begin{aligned} e^{r\cos\theta}\cos(\alpha\theta + r\sin\theta) &= {}_0C_3\left(\begin{array}{c} - \\ 1/4,\ 2/4,\ 3/4 \end{array}\,\middle|\, (\tfrac{1}{256}r^4, 4\theta); \tfrac{\alpha}{4}\right) \\ &+ r\,{}_0C_3\left(\begin{array}{c} - \\ 2/4,\ 3/4,\ 5/4 \end{array}\,\middle|\, (\tfrac{1}{256}r^4, 4\theta); \tfrac{\alpha+1}{4}\right) + \tfrac{1}{2}r^2{}_0C_3\left(\begin{array}{c} - \\ 3/4,\ 5/4,\ 6/4 \end{array}\,\middle|\, (\tfrac{1}{256}r^4, 4\theta); \tfrac{\alpha+2}{4}\right) \\ &+ \tfrac{1}{6}r^3{}_0C_3\left(\begin{array}{c} - \\ 5/4,\ 6/4,\ 7/4 \end{array}\,\middle|\, (\tfrac{1}{256}r^4, 4\theta); \tfrac{\alpha+3}{4}\right), \end{aligned}$$

and

$$\begin{aligned} e^{r\cos\theta}\sin(\alpha\theta + r\sin\theta) &= {}_0S_3\left(\begin{array}{c} - \\ 1/4,\ 2/4,\ 3/4 \end{array}\,\middle|\, (\tfrac{1}{256}r^4, 4\theta); \tfrac{\alpha}{4}\right) \\ &+ r\,{}_0S_3\left(\begin{array}{c} - \\ 2/4,\ 3/4,\ 5/4 \end{array}\,\middle|\, (\tfrac{1}{256}r^4, 4\theta); \tfrac{\alpha+1}{4}\right) + \tfrac{1}{2}r^2{}_0S_3\left(\begin{array}{c} - \\ 3/4,\ 5/4,\ 6/4 \end{array}\,\middle|\, (\tfrac{1}{256}r^4, 4\theta); \tfrac{\alpha+2}{4}\right) \\ &+ \tfrac{1}{6}r^3{}_0S_3\left(\begin{array}{c} - \\ 5/4,\ 6/4,\ 7/4 \end{array}\,\middle|\, (\tfrac{1}{256}r^4, 4\theta); \tfrac{\alpha+3}{4}\right). \end{aligned}$$

Example 2.2. *Since*

$${}_1C_0\left(\begin{array}{c} b \\ - \end{array}\,\middle|\, (r,\theta); 0\right) = \sum_{k=0}^{\infty}(b)_k\frac{r^k}{k!}\cos k\theta = (1 + r^2 - 2r\cos\theta)^{-\frac{b}{2}}\cos(b\ \arctan\frac{r\sin\theta}{r\cos\theta - 1}),$$

and

$${}_1S_0\left(\begin{array}{c} b \\ - \end{array}\,\middle|\, (r,\theta); 0\right) = \sum_{k=0}^{\infty}(b)_k\frac{r^k}{k!}\sin k\theta = (1 + r^2 - 2r\cos\theta)^{-\frac{b}{2}}\sin(b\ \arctan\frac{r\sin\theta}{r\cos\theta - 1}),$$

relations (1.7) and (1.8) respectively yield

$${}_1C_0\left(\begin{array}{c} b \\ - \end{array}\,\middle|\, (r,\theta); \alpha\right) = \sum_{k=0}^{\infty}(b)_k\frac{r^k}{k!}\cos(k+\alpha)\theta = (1 + r^2 - 2r\cos\theta)^{-\frac{b}{2}}\cos(\alpha\theta + b\ \arctan\frac{r\sin\theta}{r\cos\theta - 1}),$$

and

$${}_1S_0\left(\begin{array}{c} b \\ - \end{array}\,\middle|\, (r,\theta); \alpha\right) = \sum_{k=0}^{\infty}(b)_k\frac{r^k}{k!}\sin(k+\alpha)\theta = (1 + r^2 - 2r\cos\theta)^{-\frac{b}{2}}\sin(\alpha\theta + b\ \arctan\frac{r\sin\theta}{r\cos\theta - 1}),$$

which are valid for any $|r| < 1$, $\alpha \in \mathbb{R}$ *and* $\theta \in [-\pi, \pi]$. *Hence, by taking* $m = 2$ *in* (2.3) *we respectively obtain*

$$(1 + r^2 - 2r\cos\theta)^{-\frac{b}{2}} \cos(\alpha\theta + b\ \arctan \tfrac{r\sin\theta}{r\cos\theta - 1}) = {}_2C_1\left(\begin{matrix} b/2,\ \ (b+1)/2 \\ 1/2 \end{matrix}\ \middle|\ (r^2, 2\theta);\ \tfrac{\alpha}{2}\right)$$
$$+br\,{}_2C_1\left(\begin{matrix} (b+1)/2,\ \ (b+2)/2 \\ 3/2 \end{matrix}\ \middle|\ (r^2, 2\theta);\ \tfrac{\alpha+1}{2}\right),$$

and

$$(1 + r^2 - 2r\cos\theta)^{-\frac{b}{2}} \sin(\alpha\theta + b\ \arctan \tfrac{r\sin\theta}{r\cos\theta - 1}) = {}_2S_1\left(\begin{matrix} b/2,\ \ (b+1)/2 \\ 1/2 \end{matrix}\ \middle|\ (r^2, 2\theta);\ \tfrac{\alpha}{2}\right)$$
$$+br\,{}_2S_1\left(\begin{matrix} (b+1)/2,\ \ (b+2)/2 \\ 3/2 \end{matrix}\ \middle|\ (r^2, 2\theta);\ \tfrac{\alpha+1}{2}\right).$$

Similarly for $m = 3$ *in* (2.3) *the decomposition formulae read as*

$$(1 + r^2 - 2r\cos\theta)^{-\frac{b}{2}} \cos(\alpha\theta + b\ \arctan \tfrac{r\sin\theta}{r\cos\theta - 1}) = {}_3C_2\left(\begin{matrix} b/3,\ \ (b+1)/3,\ \ (b+2)/3 \\ 1/3,\ \ 2/3 \end{matrix}\middle|(r^3, 3\theta);\ \tfrac{\alpha}{3}\right)$$
$$+br{}_3C_2\left(\begin{matrix} (b+1)/3,\ \ (b+2)/3,\ \ (b+3)/3 \\ 2/3,\ \ 4/3 \end{matrix}\middle|(r^3, 3\theta);\ \tfrac{\alpha+1}{3}\right)$$
$$+b(b+1)\tfrac{1}{2}r^2{}_3C_2\left(\begin{matrix} (b+2)/3,\ \ (b+3)/3,\ \ (b+4)/3 \\ 4/3,\ \ 5/3 \end{matrix}\middle|(r^3, 3\theta);\ \tfrac{\alpha+2}{3}\right),$$

and

$$(1 + r^2 - 2r\cos\theta)^{-\frac{b}{2}} \sin(\alpha\theta + b\ \arctan \tfrac{r\sin\theta}{r\cos\theta - 1}) = {}_3S_2\left(\begin{matrix} b/3,\ \ (b+1)/3,\ \ (b+2)/3 \\ 1/3,\ \ 2/3 \end{matrix}\middle|(r^3, 3\theta);\ \tfrac{\alpha}{3}\right)$$
$$+br{}_3S_2\left(\begin{matrix} (b+1)/3,\ \ (b+2)/3,\ \ (b+3)/3 \\ 2/3,\ \ 4/3 \end{matrix}\middle|(r^3, 3\theta);\ \tfrac{\alpha+1}{3}\right)$$
$$+b(b+1)\tfrac{1}{2}r^2{}_3S_2\left(\begin{matrix} (b+2)/3,\ \ (b+3)/3,\ \ (b+4)/3 \\ 4/3,\ \ 5/3 \end{matrix}\middle|(r^3, 3\theta);\ \tfrac{\alpha+2}{3}\right).$$

Example 2.3. *Since we have* [4]

$${}_2C_1\left(\begin{matrix} 1,\ \ 1 \\ 2 \end{matrix}\ \middle|\ (r, \theta); 0\right) = \sum_{k=0}^{\infty} \tfrac{r^k}{k+1} \cos k\theta = \tfrac{1}{r}\left(-\tfrac{\cos\theta}{2}\ \ln\left(1 + r^2 - 2r\cos\theta\right) + \sin\theta \arctan\left(\tfrac{r\sin\theta}{1 - r\cos\theta}\right)\right),$$

and

$${}_2S_1\left(\begin{matrix} 1,\ \ 1 \\ 2 \end{matrix}\ \middle|\ (r, \theta); 0\right) = \sum_{k=0}^{\infty} \tfrac{r^k}{k+1} \sin k\theta = \tfrac{1}{r}\left(\cos\theta \arctan\left(\tfrac{r\sin\theta}{1 - r\cos\theta}\right) + \tfrac{\sin\theta}{2}\ \ln\left(1 + r^2 - 2r\cos\theta\right)\right),$$

relations (1.7) *and* (1.8) *respectively yield*

$${}_2C_1\left(\begin{matrix} 1,\ \ 1 \\ 2 \end{matrix}\ \middle|\ (r, \theta);\ \alpha\right) = \sum_{k=0}^{\infty} \tfrac{r^k}{k+1} \cos(\alpha + k)\theta = -\tfrac{\sin(\alpha-1)\theta}{r} \arctan\left(\tfrac{r\sin\theta}{1 - r\cos\theta}\right) - \tfrac{\cos(\alpha-1)\theta}{2r} \ln\left(1 + r^2 - 2r\cos\theta\right), \tag{2.4}$$

and

$${}_2S_1\left(\begin{matrix} 1,\ \ 1 \\ 2 \end{matrix}\ \middle|\ (r, \theta);\ \alpha\right) = \sum_{k=0}^{\infty} \tfrac{r^k}{k+1} \sin(\alpha + k)\theta = \tfrac{\cos(\alpha-1)\theta}{r} \arctan\left(\tfrac{r\sin\theta}{1 - r\cos\theta}\right) - \tfrac{\sin(\alpha-1)\theta}{2r} \ln\left(1 + r^2 - 2r\cos\theta\right), \tag{2.5}$$

which are valid for any $|r| < 1$, $\alpha \in \mathbb{R}$ *and* $\theta \in [-\pi, \pi]$. *Hence, by taking* $m = 2$ *in* (2.3) *we respectively obtain*

$$-\frac{\sin(\alpha-1)\theta}{r} \arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right) - \frac{\cos(\alpha-1)\theta}{2r} \ln\left(1 + r^2 - 2r\cos\theta\right) = {}_2C_1\left(\begin{array}{c} 1/2,\ 1 \\ 3/2 \end{array} \middle| (r^2, 2\theta);\ \frac{\alpha}{2}\right) + \frac{1}{2}r\,{}_2C_1\left(\begin{array}{c} 1,\ 1 \\ 2 \end{array} \middle| (r^2, 2\theta);\ \frac{\alpha+1}{2}\right),$$

and

$$\frac{\cos(\alpha-1)\theta}{r} \arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right) - \frac{\sin(\alpha-1)\theta}{2r} \ln\left(1 + r^2 - 2r\cos\theta\right) = {}_2S_1\left(\begin{array}{c} 1/2,\ 1 \\ 3/2 \end{array} \middle| (r^2, 2\theta);\ \frac{\alpha}{2}\right) + \frac{1}{2}r\,{}_2S_1\left(\begin{array}{c} 1,\ 1 \\ 2 \end{array} \middle| (r^2, 2\theta);\ \frac{\alpha+1}{2}\right).$$

Finally, it may be valuable to point out that it was first Euler [3] who obtained the following well known sum

$$\frac{\pi - \theta}{2} = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k}, \qquad 0 < \theta < 2\pi a.$$

Now, Euler's sum can be represented as

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k} = \sum_{j=0}^{\infty} \frac{\sin(j+1)\theta}{j+1} = \sum_{j=0}^{\infty} \frac{(1)_j(1)_j}{(2)_j} \frac{\sin(j+1)\theta}{j!} = {}_2S_1\left(\begin{array}{c} 1,\ 1 \\ 2 \end{array} \middle| (1, \theta);\ 1\right),$$

which according to (2.5) *is simplified as*

$${}_2S_1\left(\begin{array}{c} 1,\ 1 \\ 2 \end{array} \middle| (1, \theta);\ 1\right) = \arctan\left(\frac{\sin\theta}{1-\cos\theta}\right) = \arctan\left(\cot(\theta/2)\right) = \frac{\pi - \theta}{2}.$$

Similarly, relation (2.4) *yields*

$${}_2C_1\left(\begin{array}{c} 1,\ 1 \\ 2 \end{array} \middle| (1, \theta);\ 1\right) = \sum_{k=1}^{\infty} \frac{\cos k\theta}{k} = -\frac{1}{2}\ln\left(2 - 2\cos\theta\right).$$

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References

[1] Andrews, G.E., Askey, R., Roy, R., Special Functions, Cambridge: Cambridge University Press, 2000. 1

[2] Askey, R., Haimo, D.T., *Similarities between Fourier and power series*, Amer. Math. Monthly, **103**(1996), 297–304. 1

[3] Gottlieb, D., Shu, C.W., *On the Gibbs phenomenon and its resolution*, SIAM REV., **39**(1997), 644–668. 2.3

[4] Gradshteyn, I.S., Ryzhik, I.M., Table of Integrals, Series, and Products, Seventh Edition, Academic Press, Elsevier Inc, 2007. 2.1, 2.3

[5] Masjed-Jamei, M., Koepf, W., *Symbolic computation of some power-trigonometric series*, Journal of Symbolic Computation, **80**(2017), 273–284. 1

[6] Takemura, K., *Heun's equation, generalized hypergeometric function and exceptional Jacobi polynomial*, J. Phys. A: Math. Theor., **45**(2012), 085211. 1, 1