Turk. J. Math. Comput. Sci.
8(2018) 49–64
© MatDer
http://dergipark.gov.tr/tjmcs
http://tjmcs.matder.org.tr

MATDER

A Truncated $\mathcal{V}$-Fractional Derivative in $\mathbb{R}^n$

J. Vanterler da C. Sousa [a], E. Capelas de Oliveira [a,*]

[a]*Department of Applied Mathematics, Imecc–UNICAMP, Sérgio Buarque de Holanda 651, 13083–859, Campinas SP, Brazil*

Received: 22-09-2017 • Accepted: 27-06-2018

Abstract. Using the six parameters truncated Mittag-Leffler function, we introduce a convenient truncated function to define the so-called truncated $\mathcal{V}$-fractional derivative type. In this sense, we propose the derivative of a vector valued function and define the $\mathcal{V}$-fractional Jacobian matrix whose properties allow us to say that: the multivariable truncated $\mathcal{V}$-fractional derivative type, as proposed here, generalizes the truncated $\mathcal{V}$-fractional derivative type and can be extended to obtain a truncated $\mathcal{V}$-fractional partial derivative type. As applications, we discuss and prove the order change associated with two indexes of two truncated $\mathcal{V}$-fractional partial derivative type and propose the truncated V-fractional Green theorem. Finally, we obtain the analytical solution of the $\mathcal{V}$-fractional heat equation and present a graphical analysis.

2010 AMS Classification: 26A33, 26B12, 26A24, 33EXX, 34AXX.

Keywords: Truncated $\mathcal{V}$-fractional derivative, multivariable truncated $\mathcal{V}$-fractional derivative, truncated $\mathcal{V}$-fractional partial derivative, truncated $\mathcal{V}$-fractional Jacobian matrix, truncated $\mathcal{V}$-fractional Green's theorem, $\mathcal{V}$-fractional heat equation.

1. Introduction

Recently, Sousa and Oliveira [10] introduced the truncated $\mathcal{V}$-fractional derivative in the domain $\mathbb{R}$, satisfying classical properties of the integer-order calculus, having as special property, to unify five other formulations of local fractional derivatives of which we mention the derivatives: conformable fractional, alternative fractional, truncated alternative fractional, M-fractional and truncated M-fractional [5, 6, 9, 11].

In 2015, Atangana et al. [1],performed a work approaching new properties of the conformable fractional derivative, being the domain of the functions considered in $\mathbb{R}^n$. In 2017, Gözütok and Gözütok [4] introduced the multivariable conformable fractional calculus, presenting interesting results found in $\mathbb{R}^n$. However, such a result is restricted only to the conformable fractional derivative. In this sense, we extend our definition of the truncated $\mathcal{V}$-fractional derivative to the $\mathbb{R}^n$ [10], since such a derivative formulation unifies the remaining five. We denote this new differential operator by ${}_{i}^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(z)$, $z \in \mathbb{R}^n$, to differentiate from the operator ${}_{i}^{\rho}\mathcal{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(z)$, $z \in \mathbb{R}$, where the parameter α, associated with the order of the derivative is such that $0 < \alpha < 1$, where $\gamma, \beta, \rho, \delta \in \mathbb{C}$ and $p, q > 0$ such that $Re(\gamma) > 0$, $Re(\beta) > 0$, $Re(\rho) > 0$, $Re(\delta) > 0$ and $Re(\gamma) + p \geq q$.

The article is organized as follows: in section 2, we present the truncated $\mathcal{V}$-fractional derivative by means of the truncated six parameters Mittag-Leffler function. Also, three theorems have been introduced that address linearity,

*Corresponding Author
Email addresses: ra160908@ime.unicamp.br (J. Vanterler da C. Sousa), capelas@ime.unicamp.br (E. Capelas de Oliveira)

product, divisibility, continuity, and the α-differentiable chain rule. In section 3, we introduce our main result, the multivariable truncated $\mathcal{V}$-fractional derivative as well as results that justifies its continuity and uniqueness. In this sense, we introduce the $\mathcal{V}$-fractional Jacobian matrix and introduce and prove two theorems dealing with: chain rule, linearity and the product of functions through the α-differentiable operator. In section 4, we present the concept of $\mathcal{V}$-fractional partial derivative and discuss two applications i.e., a theorem associated with the commutativity of two truncated $\mathcal{V}$-fractional derivatives and $\mathcal{V}$-fractional Green's theorem. Finally, we obtain the analytical solution of the $\mathcal{V}$-fractional heat equation and present a graphical analysis. Concluding remarks close the article.

2. Preliminaries

We will present the definition of the truncated $\mathcal{V}$-fractional derivative through the truncated six parameters Mittag-Leffler function and the gamma function. In this sense, we will present theorems that relate to the continuity and linearity, product, divisibility, as well as the chain rule.

Then, we begin with the definition of the six parameters truncated Mittag-Leffler function given by [10],

$$ {}_i\mathbb{E}^{\rho,\delta,q}_{\gamma,\beta,p}(z) = \sum_{k=0}^{i} \frac{(\rho)_{qk}}{(\delta)_{pk}} \frac{z^k}{\Gamma(\gamma k+\beta)}, \tag{2.1} $$

being $\gamma,\beta,\rho,\delta \in \mathbb{C}$ and $p,q > 0$ such that $Re(\gamma) > 0$, $Re(\beta) > 0$, $Re(\rho) > 0$, $Re(\delta) > 0$, $Re(\gamma) + p \geq q$ and $(\delta)_{pk}$, $(\rho)_{qk}$ given by

$$ (\rho)_{qk} = \frac{\Gamma(\rho+qk)}{\Gamma(\rho)}, \tag{2.2} $$

a generalization of the Pochhammer symbol and $\Gamma(\cdot)$ is the function gamma.

From Eq. (2.1), we introduce the following truncated function, denoted by ${}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(z)$, by means of

$$ {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(z) := \Gamma(\beta)\ {}_i\mathbb{E}^{\rho,\delta,q}_{\gamma,\beta,p}(z) = \Gamma(\beta) \sum_{k=0}^{i} \frac{(\rho)_{kq}}{(\delta)_{kp}} \frac{z^k}{\Gamma(\gamma k+\beta)}. \tag{2.3} $$

In order to simplify notation, in this work, if the truncated $\mathcal{V}$-fractional derivative of order α, according to Eq. (2.4) below, of a function f exists, we simply say that the f function is α-differentiable.

So, we start with the following definition, which is a generalization of the usual definition of a derivative presented as a particular limit.

Definition 2.1. Let $f : [0,\infty) \to \mathbb{R}$. For $0 < \alpha < 1$ the truncated $\mathcal{V}$-fractional derivative of f of order α, denoted by ${}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}(\cdot)$, is defined as

$$ {}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha} f(t) := \lim_{\epsilon\to 0} \frac{f\left(t\ {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\epsilon t^{-\alpha})\right) - f(t)}{\epsilon}, \tag{2.4} $$

for $\forall t > 0$, ${}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\cdot)$ is a truncated function as defined in Eq. (2.3) and being $\gamma,\beta,\rho,\delta \in \mathbb{C}$ and $p,q > 0$ such that $Re(\gamma) > 0$, $Re(\beta) > 0$, $Re(\rho) > 0$, $Re(\delta) > 0$, $Re(\gamma) + p \geq q$ and $(\delta)_{pk}$, $(\rho)_{qk}$ given by Eq. (2.2) [10].

Theorem 2.2. [10] *If the function $f : [0,\infty) \to \mathbb{R}$ is α-differentiable for $t_0 > 0$, with $0 < \alpha \leq 1$, then f is continuous in t_0.*

Theorem 2.3. *Let $0 < \alpha \leq 1$, $a,b \in \mathbb{R}$, $\gamma,\beta,\rho,\delta \in \mathbb{C}$ and $p,q > 0$ such that $Re(\gamma) > 0$, $Re(\beta) > 0$, $Re(\rho) > 0$, $Re(\delta) > 0$, $Re(\gamma) + p \geq q$ and f,g α-differentiable, for $t > 0$. Then,*

(1) ${}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}(af+bg)(t) = a\,{}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}f(t) + b\,{}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}g(t)$

(2) ${}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}(f\cdot g)(t) = f(t)\,{}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}g(t) + g(t)\,{}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}f(t)$

(3) ${}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}\left(\frac{f}{g}\right)(t) = \dfrac{g(t)\,{}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}f(t) - f(t)\,{}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}g(t)}{[g(t)]^2}$

(4) ${}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}(c) = 0$, *where $f(t) = c$ is a constant.*

(5) *If f is differentiable, then* ${}_{i}^{\rho}\mathcal{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f(t)=\dfrac{t^{1-\alpha}\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\dfrac{df(t)}{dt}$.

(6) ${}_{i}^{\rho}\mathcal{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(t^a)=\dfrac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}at^{a-\alpha}$.

Proof. See [10]. □

Theorem 2.4. (Chain rule) *Assume $f,g:(0,\infty)\to\mathbb{R}$ be two α-differentiable functions where $0<\alpha\leq 1$. Let $\gamma,\beta,\rho,\delta\in\mathbb{C}$ and $p,q>0$ such that $Re(\gamma)>0$, $Re(\beta)>0$, $Re(\rho)>0$, $Re(\delta)>0$, $Re(\gamma)+p\geq q$ then $(f\circ g)$ is α-differentiable and for all $t>0$, we have*

$$ {}_{i}^{\rho}\mathcal{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(f\circ g)(t)=f'(g(t))\,{}_{i}^{\rho}\mathcal{V}_{\gamma,\beta,\alpha}^{\delta,p,q}g(t), $$

for f differentiable in $g(t)$.

Proof. See [10]. □

Definition 2.5. [10] ($\mathcal{V}$-fractional integral) Let $a\geq 0$ and $t\geq a$. Also, let f be a function defined on $(a,t]$ and $0<\alpha<1$. Then, the $\mathcal{V}$-fractional integral of f of order α is defined by

$$ {}_{a}^{\rho}\mathcal{I}_{\gamma,\beta,\alpha}^{\delta,p,q}f(t):=\frac{\Gamma(\gamma+\beta)(\delta)_p}{\Gamma(\beta)(\rho)_q}\int_a^t\frac{f(x)}{x^{1-\alpha}}dx, $$

with $\gamma,\beta,\rho,\delta\in\mathbb{C}$ and $p,q>0$ such that $Re(\gamma)>0$, $Re(\beta)>0$, $Re(\rho)>0$, $Re(\delta)>0$ and $Re(\gamma)+p\geq q$.

Theorem 2.6. *Let $a\geq 0$ and $t\geq a$. Also, let f be a function defined on $(a,t]$ and $0<\alpha<1$. Then, the $\mathcal{V}$-fractional integral of f of order α is unique.*

Proof. Consider the $\mathcal{V}$-fractional integral of f of order $0<\alpha\leq 1$ given by Eq. (25). Assume that L_1 and L_2 are $\mathcal{V}$-fractional integrals of f on $[a,b]$. We want to prove that $L_1=L_2$. Let $\varepsilon>0$. Then for each $j=1,2$, exist $\delta_j>0$ such that $\|P\|<\delta_j\Rightarrow\left|\sigma-L_j\right|<\frac{\varepsilon}{2}$, where P is a partition on $[a,b]$. Taking $\delta=\min\{\delta_1,\delta_2\}$. Fixed a partition P on the interval $[a,b]$ and suppose that $\|P\|<\delta$.

Note that $\delta\leq\delta_j$, for $j=1,2$. Then

$$ 0\leq|L_1-L_2|\leq|\sigma-L_1|+|\sigma-L_2|<\varepsilon $$

for all $\varepsilon>0$. Like this, we conclude that $|L_1-L_2|=0$, so $L_1=L_2$. Therefore, the $\mathcal{V}$-fractional integral is unique. □

Remark 2.7. In order to simplify notation, in this work, the $\mathcal{V}$-fractional integral of order α, will be denoted by

$$ \frac{\Gamma(\gamma+\beta)(\delta)_p}{\Gamma(\beta)(\rho)_q}\int_a^b\frac{f(t)}{t^{1-\alpha}}dt=\int_a^b f(t)\,d_\omega t $$

where, $d_\omega t=\dfrac{\Gamma(\gamma+\beta)(\delta)_p}{\Gamma(\beta)(\rho)_q}t^{\alpha-1}dt$.

3. $\mathcal{V}$-Fractional Derivative of A Vector Valued Function

In this section, we present our main result, the truncated $\mathcal{V}$-fractional derivative in $\mathbb{R}^n$ and check its continuity as well as the uniqueness of linear transformation. We present the definition of the truncated $\mathcal{V}$-fractional Jacobian matrix, the chain rule and the theorem that refers to linearity and product. We conclude the section discussing some examples.

Definition 3.1. Let f be a vector valued function with n real variables such that $f(x_1,x_2,...,x_n)=(f_1(x_1,x_2,...,x_n),$ $f_2(x_1,x_2,...,x_n),...,f_m(x_1,x_2,...,x_n))$. We say that f is α-differentiable at $a=(a_1,...,a_n)\in\mathbb{R}^n$ where each $a_i>0$, if there is a linear transformation $L:\mathbb{R}^n\to\mathbb{R}^m$ such that

$$ \lim_{\varepsilon\to 0}\frac{\left\|f\left(a_1\,{}_{i}H_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\,{}_{i}H_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_n a_n^{-\alpha}\right)\right)-f(a_1,...,a_n)-L(\varepsilon)\right\|}{\|\varepsilon\|}=0, $$

where $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$, $0 < \alpha \leq 1$, ${}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\cdot)$ is the truncated function and $\rho, \delta, \gamma, \beta \in \mathbb{C}$, $p, q > 0$ with, $Re(\rho) > 0$, $Re(\delta) > 0$, $Re(\gamma) > 0$, $Re(\beta) > 0$ and $Re(\gamma) + p \geq q$. The linear transformation is denoted by ${}^{\rho}_{i}\mathbb{V}^{\delta,p,q}_{\gamma,\beta,\alpha} f(a)$ and called the multivariable truncated $\mathcal{V}$-fractional derivative of f of order α at a.

Remark 3.2. Taking $m = n = 1$ in Definition 3.1, we have

$$L(\varepsilon) = f\left(a\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon a^{-\alpha})\right) - f(a) - r(\varepsilon). \tag{3.1}$$

Dividing by ε both sides of Eq. (3.1) and taking the limit $\varepsilon \to 0$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{L(\varepsilon)}{\varepsilon} &= \lim_{\varepsilon\to 0}\frac{f\left(a\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon a^{-\alpha})\right) - f(a) - r(\varepsilon)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{f\left(a\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon a^{-\alpha})\right) - f(a)}{\varepsilon}\\ &= {}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha} f(a),\end{aligned}$$

where $\lim_{\varepsilon\to 0}\frac{r(\varepsilon)}{\varepsilon} = 0$. Thus, we conclude that, Definition 3.1 is equivalent to Definition 2.1.

Theorem 3.3. *Let f be a vector valued function with n variables. If f is α-differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$ with $a_i > 0$, then there is a unique linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{\varepsilon\to 0}\frac{\left\|f\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right), ..., a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon_n a_n^{-\alpha})\right) - f(a_1, ..., a_n) - L(\varepsilon)\right\|}{\|\varepsilon\|} = 0,$$

with $0 < \alpha \leq 1$, ${}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\cdot)$ is the truncated function and $\rho, \delta, \gamma, \beta \in \mathbb{C}$, $p, q > 0$ such that, $Re(\rho) > 0$, $Re(\delta) > 0$, $Re(\gamma) > 0$, $Re(\beta) > 0$ and $Re(\gamma) + p \geq q$.

Proof. Let $M : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\varepsilon\to 0}\frac{\left\|f\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right), ..., a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon_n a_n^{-\alpha})\right) - f(a_1, ..., a_n) - M(\varepsilon)\right\|}{\|\varepsilon\|} = 0.$$

Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\|L(\varepsilon) - M(\varepsilon)\|}{\|\varepsilon\|} \leq\; & \lim_{\varepsilon\to 0}\frac{\left\|L(\varepsilon) - f\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right), ..., a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon_n a_n^{-\alpha})\right) + f(a)\right\|}{\|\varepsilon\|}\\ &+\lim_{\varepsilon\to 0}\frac{\left\|f\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right), ..., a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(\varepsilon_n a_n^{-\alpha})\right) - f(a) - M(\varepsilon)\right\|}{\|\varepsilon\|} = 0,\end{aligned}$$

then

$$\lim_{\varepsilon\to 0}\frac{\|L(\varepsilon) - M(\varepsilon)\|}{\|\varepsilon\|} \leq 0.$$

If $x \in \mathbb{R}^n$, then $\varepsilon x \to 0$ as $\varepsilon \to 0$. Hence, for $x \neq 0$ we have

$$0 = \lim_{\varepsilon\to 0}\frac{\|L(\varepsilon x) - M(\varepsilon x)\|}{\|\varepsilon x\|} = \frac{\|L(x) - M(x)\|}{\|x\|}.$$

Therefore $L(x) = M(x)$. We conclude that, L is unique. □

Example 3.4. Let us consider the function f defined by $f(x, y) = sin(x)$ and the point $(a, b) \in \mathbb{R}^2$ such that $a, b > 0$, then ${}^{\rho}_{i}\mathbb{V}^{\delta,p,q}_{\gamma,\beta,\alpha} f(a, b) = L$ satisfies $L(x, y) = \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma + \beta)(\delta)_p} x a^{1-\alpha} cos(a)$.

To prove this, we note that

$$\begin{aligned}
&\lim_{(\varepsilon_1,\varepsilon_2)\to(0,0)} \frac{\left|f\left(a\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a^{-\alpha}\right), b\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_2 b^{-\alpha}\right)\right) - f(a,b) - L(\varepsilon_1,\varepsilon_2)\right|}{\|(\varepsilon_1,\varepsilon_2)\|} \\
= &\lim_{(\varepsilon_1,\varepsilon_2)\to(0,0)} \frac{\left|\sin\left(a\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a^{-\alpha}\right)\right) - \sin(a) - L(\varepsilon_1,\varepsilon_2)\right|}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}} \\
\leq &\lim_{\varepsilon_1\to 0} \frac{\left|\sin\left(a\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a^{-\alpha}\right)\right) - \sin(a) - \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\varepsilon_1 a^{1-\alpha}\cos(a)\right|}{|\varepsilon_1|} = 0
\end{aligned}$$

Example 3.5. Let us consider the function f defined by $f(x,y) = e^x$ and the point $(a,b) \in \mathbb{R}^2$ such that $a,b > 0$, then ${}^{\rho}_{i}\mathbb{V}^{\delta,p,q}_{\gamma,\beta,\alpha} f(a,b) = L$ satisfies $L(x,y) = \dfrac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} x a^{1-\alpha} e^a$.

To prove this, we note that

$$\begin{aligned}
&\lim_{(\varepsilon_1,\varepsilon_2)\to(0,0)} \frac{\left|f\left(a\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a^{-\alpha}\right), b\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_2 b^{-\alpha}\right)\right) - f(a,b) - L(\varepsilon_1,\varepsilon_2)\right|}{\|(\varepsilon_1,\varepsilon_2)\|} \\
= &\lim_{(\varepsilon_1,\varepsilon_2)\to(0,0)} \frac{\left|e^{a\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a^{-\alpha}\right)} - e^a - L(\varepsilon_1,\varepsilon_2)\right|}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}} \\
\leq &\left|\lim_{\varepsilon_1\to 0} \frac{e^{a\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a^{-\alpha}\right)} - e^a}{\varepsilon_1} - \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} a^{1-\alpha} e^a\right| = 0.
\end{aligned}$$

Definition 3.6. Consider the matrix of the linear transformation ${}^{\rho}_{i}\mathbb{V}^{\delta,p,q}_{\gamma,\beta,\alpha} f(a) : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the usual base of $\mathbb{R}^n$ and $\mathbb{R}^m$. This $m \times n$ matrix is called the truncated $\mathcal{V}$-fractional Jacobian matrix of f at a, and denoted by ${}^{\rho}J^{\delta,p,q}_{\gamma,\beta,\alpha} f(a)$, where $\rho,\delta,\gamma,\beta \in \mathbb{C}$, $p,q > 0$ with, $Re(\rho) > 0$, $Re(\delta) > 0$, $Re(\gamma) > 0$, $Re(\beta) > 0$ and $Re(\gamma) + p \geq q$.

Example 3.7. If $f(x,y) = \sin(x)$, then we have the matrix

$${}^{\rho}J^{\delta,p,q}_{\gamma,\beta,\alpha} f(a,b) = \begin{bmatrix} \dfrac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} a^{1-\alpha}\cos(a) & 0 \end{bmatrix}.$$

Theorem 3.8. *If a vector valued function f with n variables is α-differentiable at $a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$, with $a_i > 0$, then f is continuous at $a \in \mathbb{R}^n$.*

Proof. Note that,

$$\begin{aligned}
&\left\|f\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right), ..., a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f(a_1,...,a_n)\right\| \\
\leq &\frac{\left\|f\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right), ..., a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f(a_1,...,a_n) - L(\varepsilon)\right\| \|\varepsilon\|}{\|\varepsilon\|} \\
&+ \|L(\varepsilon)\|. \qquad (3.2)
\end{aligned}$$

Taking the limit $\varepsilon \to 0$ in both sides of the Eq. (3.2), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left\|f\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right), ..., a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f(a_1,...,a_n)\right\| \\
\leq &\lim_{\varepsilon\to 0}\frac{\left\|f\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right), ..., a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f(a_1,...,a_n) - L(\varepsilon)\right\|}{\|\varepsilon\|} \\
&\times\lim_{\varepsilon\to 0}\|\varepsilon\| + \lim_{\varepsilon\to 0}\|L(\varepsilon)\|.
\end{aligned}$$

Let $(u_1,...,u_n) = (\varepsilon_1 a_1^{-\alpha}, ..., \varepsilon_n a_n^{-\alpha})$, then $u \to 0$ as $\varepsilon \to 0$. Since

$$\lim_{\varepsilon\to 0}\left\|f\left(a\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(u)\right) - f(a)\right\| \leq 0,$$

we have,

$$\lim_{\varepsilon\to 0}\left\|f\left(a\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}(u)\right)-f(a)\right\|=0.$$

Hence, f is continuous at $a\in\mathbb{R}^n$. □

Theorem 3.9. (Chain rule) *Let $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$. If $f(x)=(f_1(x),...,f_m(x))$ is α-differentiable at $a=(a_1,...,a_n)\in\mathbb{R}^n$, with $a_i>0$ such that $\alpha\in(0,1]$, and $g(y)=(g_1(y),...,g_p(y))$ is α-differentiable at $f(a)\in\mathbb{R}^m$, with $f_i(a)>0$ such that $\alpha\in(0,1]$, then the composition $g\circ f$ is α-differentiable at a and*

$${}^{\rho}_{i}\mathbb{V}^{\delta,p,q}_{\gamma,\beta,\alpha}(g\circ f)(a)=g'(f(a))\,{}^{\rho}_{i}\mathbb{V}^{\delta,p,q}_{\gamma,\beta,\alpha}f(a),$$

for g differentiable in $f(a)$ and $\rho,\delta,\gamma,\beta\in\mathbb{C}$, $p,q>0$ such that, $Re(\rho)>0$, $Re(\delta)>0$, $Re(\gamma)>0$, $Re(\beta)>0$ and $Re(\gamma)+p\geq q$.

Proof. Taking $L={}^{\rho}_{i}\mathbb{V}^{\delta,p,q}_{\gamma,\beta,\alpha}f(t)$ and $M=Dg(f(a))$, where D is the derivative operator of integer order, we define,

$$\begin{aligned}&\varphi\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1a_1^{-\alpha}\right),...,a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_na_n^{-\alpha}\right)\right)\\ =\;&f\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1a_1^{-\alpha}\right),...,a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_na_n^{-\alpha}\right)\right)-f(a)-L(\varepsilon),\end{aligned}$$

$$\begin{aligned}&\psi\left(f_1(a)\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(k_1f_1(a)^{-\alpha}\right),...,f_n(a)\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(k_nf_n(a)^{-\alpha}\right)\right)\\ =\;&g\left(f_1(a)\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(k_1f_1(a)^{-\alpha}\right),...,f_n(a)\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(k_nf_n(a)^{-\alpha}\right)\right)-g(f(a))-M(k)\end{aligned}$$

and

$$\begin{aligned}&\rho\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1a_1^{-\alpha}\right),...,a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_na_n^{-\alpha}\right)\right)=g\circ f\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1a_1^{-\alpha}\right),...,a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_na_n^{-\alpha}\right)\right)\\ &-g\circ f(a)-M\circ L(\varepsilon).\end{aligned}\tag{3.3}$$

Hence, taking $\varepsilon\to 0$ and $k\to 0$ in both sides of Eq.(3.3) and Eq.(3.3), we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\left\|\varphi\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1a_1^{-\alpha}\right),...,a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_na_n^{-\alpha}\right)\right)\right\|}{\|\varepsilon\|}\\ =\;&\lim\frac{\left\|f\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1a_1^{-\alpha}\right),...,a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_na_n^{-\alpha}\right)\right)-f(a)-L(\varepsilon)\right\|}{\|\varepsilon\|}=0\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&\lim_{k\to 0}\frac{\left\|\psi\left(f_1(a)\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(k_1f_1(a)^{-\alpha}\right),...,f_n(a)\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(k_nf_n(a)^{-\alpha}\right)\right)\right\|}{\|k\|}\\ =\;&\lim_{k\to 0}\frac{\left\|g\left(f_1(a)\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(k_1f_1(a)^{-\alpha}\right),...,f_n(a)\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(k_nf_n(a)^{-\alpha}\right)\right)-g(f(a))-M(k)\right\|}{\|k\|}=0.\end{aligned}\tag{3.5}$$

On the other hand, taking $\varepsilon\to 0$ and $k\to 0$ on both sides of Eq.(3.3), we will show that

$$\lim_{\varepsilon\to 0}\frac{\left\|\rho\left(a_1\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1a_1^{-\alpha}\right),...,a_n\,{}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_na_n^{-\alpha}\right)\right)\right\|}{\|\varepsilon\|}=0.$$

Now, let

$$\begin{aligned}
& \rho\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) \\
= \;& g\left(f\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right)\right) - g\left(f(a)\right) - M\circ L(\varepsilon) \\
= \;& g\left(\begin{array}{c} f_1\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right),... \\ ...,f_m\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) \end{array}\right) - g\left(f(a)\right) \\
& -M\left(\begin{array}{c} f\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) \\ -f(a) - \varphi\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) \end{array}\right) \\
= \;& \left[g\left(\begin{array}{c} f_1\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right),... \\ ...,f_m\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) \end{array}\right) - g\left(f(a)\right)\right. \\
& \left. -M\left(\begin{array}{c} f_1\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f_1(a),... \\ ...,f_m\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f_m(a) \end{array}\right)\right] \\
& +M\left[\varphi\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right)\right].
\end{aligned}$$

If we put $u_j = f_j\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f_j(a)$, with $j = 1,2,...,m$, then we have $f_j\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) = u_j + f_j(a)$, and $u \to 0$ as $\varepsilon \to 0$. Hence, using Eq. (3.3), we have

$$\begin{aligned}
\rho\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) &= \left[g\left(\begin{array}{c} f_1\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right),... \\ ...,f_m\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right) \end{array}\right) - g\left(f(a)\right) - M(u)\right] \\
& \quad +M\left[\varphi\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right)\right] \\
&= \psi\left(f_1(a)\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(u_1 f(a)^{-\alpha}\right),...,f_m(a)\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(u_m f_m(a)^{-\alpha}\right)\right) \\
& \quad +M\left[\varphi\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right)\right].
\end{aligned}$$

Thus we will show,

$$\lim_{u\to 0}\frac{\left\|\psi\left(f_1(a)\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(u_1 f(a)^{-\alpha}\right),...,f_m(a)\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(u_m f_m(a)^{-\alpha}\right)\right)\right\|}{\|u\|} = 0 \tag{3.6}$$

and

$$\lim_{\varepsilon\to 0}\frac{\left\|M\left(\varphi\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right)\right)\right\|}{\|\varepsilon\|} = 0. \tag{3.7}$$

For Eq. (3.6), it is obvious from of Eq. (3.5). Now, for Eq. (3.7), we have

$$\begin{aligned}
\left\|M\left(\varphi\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right)\right)\right\| &\leq \|M\|\left\|\left(\varphi\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right)\right)\right\| \\
&\leq K\left\|\left(\varphi\left(a_1\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\, {}_iH^{\rho,\delta,q}_{\gamma,\beta,p}\left(\varepsilon_n a_n^{-\alpha}\right)\right)\right)\right\|, \qquad (3.8)
\end{aligned}$$

such that $K > 0$. Taking the limit $\varepsilon \to 0$ on both sides of Eq. (3.8) and using Eq. (3.4), we get Eq. (3.6). Hence, we conclude the proof. □

Corollary 3.10. *For $m = n = p = 1$, the* Theorem 3.9 *states that*

$${}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha}\left(g\circ f\right)(a) = g'\left(f(a)\right)\, {}^{\rho}_{i}\mathcal{V}^{\delta,p,q}_{\gamma,\beta,\alpha} f(a).$$

Corollary 3.10 says that Theorem 3.9 generalizes Theorem (2.3).

Corollary 3.11. *Consider all the conditions of* Theorem 3.9 *satisfied. Then*

$$\begin{aligned} & {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(g\circ f)(a) \\ = \ & g'(f(a))\begin{pmatrix} \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}f_1(a)^{1-\alpha} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}f_2(a)^{1-\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}f_n(a)^{1-\alpha} \end{pmatrix} \end{aligned}$$

where, $\begin{pmatrix} \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}f_1(a)^{1-\alpha} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}f_2(a)^{1-\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}f_n(a)^{1-\alpha} \end{pmatrix}$, *is the matrix corresponding to the linear transformation* ${}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f(a)$.

Corollary 3.12. *Consider all the conditions of* Theorem 3.9 *satisfied. For $f(a) = a$,* Corollary 3.11, *says that*

$$\begin{aligned} {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}g(a) &= g'(a)\begin{pmatrix} \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}a_1^{1-\alpha} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}a_2^{1-\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}a_n^{1-\alpha} \end{pmatrix} \\ &= g'(a)\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}L_{\alpha}^{1-\alpha}. \end{aligned}$$

Remark 3.13. The Corollary 3.12 generalizes part 5 of the Theorem 2.3.

Theorem 3.14. *Let f be a vector valued function with n variables such that $f(x_1,...,x_n) = (f_1(x_1,...,x_n),...,f_n(x_1,...,x_n))$. Then f is α-differentiable function at $a = (a_1,...,a_n) \in \mathbb{R}^n$, with $a_i > 0$ if, and only if, each f_i is,*

$${}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f(a) = \left({}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f_1(a),...,{}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f_m(a)\right),$$

where $\alpha \in (0,1]$ and $\rho,\delta,\gamma,\beta \in \mathbb{C}$, $p,q > 0$ with, $Re(\rho) > 0$, $Re(\delta) > 0$, $Re(\gamma) > 0$, $Re(\beta) > 0$ and $Re(\gamma) + p \geq q$.

Proof. If each f_i is α-differentiable at a and $L = \left({}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f_1(a),...,{}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f_m(a)\right)$, then

$$\begin{aligned} & f\left(a_1\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f(a) - L(\varepsilon) \\ = \ & \left[f_1\left(a_1\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f_1(a) - {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f_1(a)(\varepsilon),...\right. \\ & \left. ...,f_m\left(a_1\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f_m(a) - {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f_m(a)(\varepsilon)\right]. \end{aligned} \tag{3.9}$$

Taking the limit $\varepsilon \to 0$ on both sides of Eq. (3.9), we have

$$\begin{aligned} & \lim_{\varepsilon\to 0}\frac{\left\|f\left(a_1\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_1 a_1^{-\alpha}\right),...,a_n\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_n a_n^{-\alpha}\right)\right) - f(a) - L(\varepsilon)\right\|}{\|\varepsilon\|} \\ = \ & \lim_{\varepsilon\to 0}\frac{\left\|\sum_{j=1}^{n} f_j\left(a_j\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_j a_j^{-\alpha}\right)\right) - f_j(a) - {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f_j(a)(\varepsilon)\right\|}{\|\varepsilon\|} \\ \leq \ & \lim_{\varepsilon\to 0}\sum_{j=1}^{n}\frac{\left\|f_j\left(a_j\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_j a_j^{-\alpha}\right)\right) - f_j(a) - {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}f_j(a)(\varepsilon)\right\|}{\|\varepsilon\|} = 0, \end{aligned}$$

which is the result. □

Theorem 3.15. *Let* $0 < \alpha \leq 1$, $\lambda,\mu \in \mathbb{R}$, $\gamma,\beta,\rho,\delta \in \mathbb{C}$ *and* $p,q > 0$ *such that* $Re(\gamma) > 0$, $Re(\beta) > 0$, $Re(\rho) > 0$, $Re(\delta) > 0$, $Re(\gamma) + p \geq q$ *and* f,g *α-differentiable at* $a = (a_1,...,a_n) \in \mathbb{R}^n$, *with* $a_i > 0$. *Then,*

(1) ${}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(\lambda f + \mu g)(a) = \lambda_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a) + \mu_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a)$.

(2) ${}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(f \cdot g)(a) = f(a)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a) + g(a)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)$.

Proof. 1. Let $A = a_1\, {}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_1 a_1^{-\alpha}\right) + \cdots + a_n\, {}_iH_{\gamma,\beta,p}^{\rho,\delta,q}(\varepsilon_1 a_n^{-\alpha})$, then we have,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\left\|(\lambda f + \mu g)(A) - (\lambda f + \mu g)(a) - \left(\lambda\, {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a) + \mu\, {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a)\right)(\varepsilon)\right\|}{\|\varepsilon\|}\\
= &\lim_{\varepsilon\to 0}\frac{\left\|\lambda f(A) - \lambda f(a) - \lambda_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon) + \mu g(A) - \mu g(a) - \mu\, {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a)(\varepsilon)\right\|}{\|\varepsilon\|}\\
\leq &\lim_{\varepsilon\to 0}\frac{\left\|\lambda f(A) - \lambda f(a) - \lambda_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon)\right\|}{\|\varepsilon\|} + \lim_{\varepsilon\to 0}\frac{\left\|\mu g(A) - \mu g(a) - \mu\, {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a)(\varepsilon)\right\|}{\|\varepsilon\|}\\
= &\lambda\lim_{\varepsilon\to 0}\frac{\left\|f(A) - f(a) - {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon)\right\|}{\|\varepsilon\|} + \mu\lim_{\varepsilon\to 0}\frac{\left\|g(A) - g(a) - {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a)(\varepsilon)\right\|}{\|\varepsilon\|} = 0.
\end{aligned}$$

So, the proof is complete.

2. Let $A = a_1\, {}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon_1 a_1^{-\alpha}\right) + \cdots + a_n\, {}_iH_{\gamma,\beta,p}^{\rho,\delta,q}(\varepsilon_1 a_n^{-\alpha})$, then we have,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\left\|(f\cdot g)(A) - (f\cdot g)(a) - \left(f(a)\, {}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a) + g(a)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)\right)(\varepsilon)\right\|}{\|\varepsilon\|}\\
= &\lim_{\varepsilon\to 0}\frac{\left\|\begin{array}{c} f(A)\,g(A) - f(a)\,g(A) - g(A)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon) + f(a)\,g(A) - f(a)\,g(a)\\ -f(a)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a)(\varepsilon) + g(A)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon) - g(a)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon)\end{array}\right\|}{\|\varepsilon\|}\\
\leq &\lim_{\varepsilon\to 0}\frac{\left\|f(A)\,g(A) - f(a)\,g(A) - g(A)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon)\right\|}{\|\varepsilon\|} +\\
&\lim_{\varepsilon\to 0}\frac{\left\|f(a)\,g(A) - f(a)\,g(a) - f(a)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} g(a)(\varepsilon)\right\|}{\|\varepsilon\|} +\\
&\lim_{\varepsilon\to 0}\frac{\left\|g(A)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon) - g(a)_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon)\right\|}{\|\varepsilon\|}\\
= &\lim_{\varepsilon\to 0}\left\|{}_i^{\rho}\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)(\varepsilon)\right\|\frac{\|g(A) - g(a)\|}{\|\varepsilon\|}\\
\leq &K\lim_{\varepsilon\to 0}\|(\varepsilon)\|\frac{\|g(A) - g(a)\|}{\|\varepsilon\|} = 0,
\end{aligned}$$

with $K > 0$. So, the proof is complete. □

4. Truncated $\mathcal{V}$-Fractional Partial Derivatives and Applications

In this section, we introduce the truncated $\mathcal{V}$-fractional partial derivative and discuss applications: the theorem associated with the commutativity property of two truncated $\mathcal{V}$-fractional partial derivatives, the truncated $\mathcal{V}$-fractional Green's theorem and analytical solution of the $\mathcal{V}$-fractional heat equation and present a graphical analysis.

Definition 4.1. Let f be a real valued function with n variables and $a = (a_1,...,a_n) \in \mathbb{R}^n$ be a point whose i^{th} component is positive. Then, the limit

$$\lim_{\varepsilon\to 0}\frac{f\left(a_1,...,a_j\, {}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon a_j^{-\alpha}\right),...,a_n\right) - f(a_1,...,a_n)}{\varepsilon},$$

if it exists, is denoted by $\frac{\partial^\alpha}{\partial x^\alpha} f(a) := \frac{\partial^\alpha}{\partial x^\alpha} f(x)\,|_{x=a}$, and called the i^{th} truncated $\mathcal{V}$-fractional partial derivative of f of order $\alpha \in (0,1]$ at a.

Theorem 4.2. *Let f be a vector valued function with n variables. If f is α-differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, with $a_j > 0$, then $\frac{\partial^\alpha}{\partial x_p^\alpha} f_j(a)$ of order $\alpha \in (0,1]$ exists for $1 \le j \le m$, $1 \le p \le n$ and the Jacobian of f at a is the $m \times n$ matrix $\left(\frac{\partial^\alpha}{\partial x_p^\alpha} f_j(a)\right)$.*

Proof. Let $f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$. Suppose first that $m = 1$, so that $f(x_1, ..., x_n) \in \mathbb{R}^n$. Define $h : \mathbb{R} \to \mathbb{R}^n$ by $h(y) = (a_1, ..., y, ..., a_n)$ with y in the place of p^{th}. Then $\frac{\partial^\alpha}{\partial x_p^\alpha} f_j(a) = {}_i^\rho\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(f \circ h)\left(a_p\right)$. Hence, by Corollary (3.11), we have

$$\begin{aligned}
{}_i^\rho\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q}(f \circ h)\left(a_p\right) &= f'\left(h\left(a_p\right)\right)\begin{pmatrix} \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} h_1\left(a_p\right)^{1-\alpha} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} h_2\left(a_p\right)^{1-\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} h_n\left(a_p\right)^{1-\alpha} \end{pmatrix} \\
&= f'(a)\begin{pmatrix} \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} a_1^{1-\alpha} & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} a_j^{1-\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p} a_n^{1-\alpha} \end{pmatrix} \\
&= {}_i^\rho\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a).
\end{aligned}$$

Since $(f \circ h)\left(a_p\right)$ has a single entry $\frac{\partial^\alpha}{\partial x_p^\alpha} f_j(a)$, this shows that $\frac{\partial^\alpha}{\partial x_p^\alpha} f_j(a)$ exists and is the p^{th} entry of the $1 \times n$ matrix ${}_i^\rho\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)$. The theorem now follows for arbitrary m since, by Theorem (3.14), each f_j, is α-differentiable and the p^{th} row of ${}_i^\rho\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f(a)$ is ${}_i^\rho\mathbb{V}_{\gamma,\beta,\alpha}^{\delta,p,q} f_j(a)$. □

For the next result, we use the Clairaut-Schwarz theorem integer order [12], and realize an application of the truncated $\mathcal{V}$-fractional partial derivative.

Theorem 4.3. *Assume that $f(t,s)$ is a function for which $\partial_t^\alpha(\partial_s^\kappa f(t,s))$ is of order $\alpha \in (0,1]$ and $\partial_s^\kappa(\partial_t^\alpha f(t,s))$ is of order $\kappa \in (0,1]$ exist and are continuous over the domain $D \subset \mathbb{R}^2$, then*

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa} f(t,s)\right) = \frac{\partial^\kappa}{\partial t^\kappa}\left(\frac{\partial^\alpha}{\partial t^\alpha} f(t,s)\right).$$

Proof. By means of the Definition (2.1), truncated $\mathcal{V}$-fractional derivative at the s variable, we have

$$\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa} f(t,s)\right) &= \frac{\partial^\alpha}{\partial t^\alpha}\left(\lim_{\varepsilon\to 0}\frac{f\left(t, s\,{}_iH_{\gamma,\beta,p}^{\rho,\delta,q}\left(\varepsilon s^{-\kappa}\right)\right) - f(t,s)}{\varepsilon}\right) \\
&= \frac{\partial^\alpha}{\partial t^\alpha}\left(\lim_{\varepsilon\to 0}\frac{f\left(t, s + \frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\varepsilon s^{1-\kappa} + O\left(\varepsilon^2\right)\right) - f(t,s)}{\varepsilon}\right).
\end{aligned}$$

Introducing the following change of variable $h=\varepsilon s^{1-\kappa}\left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}+O(\varepsilon)\right)$ implies $\varepsilon=\dfrac{h}{s^{1-\kappa}\left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}+O(\varepsilon)\right)}$, we get

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa}f(t,s)\right)=\frac{\partial^\alpha}{\partial t^\alpha}\left(\lim_{\varepsilon\to 0}\frac{\dfrac{f(t,s+h)-f(t,s)}{hs^{\kappa-1}}}{\dfrac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}+O(\varepsilon)}\right).$$

Since f is differentiable in s-direction, we obtain

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa}f(t,s)\right)=s^{1-\kappa}\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial}{\partial s}f(t,s)\right).$$

Again, by the definition of the truncated $\mathcal{V}$-fractional derivative we have

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa}f(t,s)\right)=s^{1-\kappa}\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\left\{\lim_{\varepsilon\to 0}\frac{\frac{\partial}{\partial s}f\left(t\,{}_iH^{\rho,\delta,q}_{\gamma,\delta,p}(\varepsilon t^{-\alpha}),s\right)-\frac{\partial}{\partial s}f(t,s)}{\varepsilon}\right\}.$$

In analogy to the expression, after making a similar change of variable, we have

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa}f(t,s)\right)=\frac{\Gamma(\beta)(\rho)_q\,s^{1-\kappa}t^{1-\alpha}}{\Gamma(\gamma+\beta)(\delta)_p}\lim_{k\to 0}\frac{\frac{\partial}{\partial s}f(t+k,s)-\frac{\partial}{\partial s}f(t,s)}{k}.$$

Since f is differentiable in t-direction, we obtain

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa}f(t,s)\right)=\left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\right)^2 s^{1-\kappa}t^{1-\alpha}\frac{\partial^2}{\partial t\partial s}f(t,s). \tag{4.1}$$

Being f a continuous function and using the Clairaut-Schwarz theorem for partial derivative, it follows that

$$\frac{\partial^2}{\partial t\partial s}f(t,s)=\frac{\partial^2}{\partial s\partial t}f(t,s).$$

Therefore the Eq. (4.1), becomes

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa}f(t,s)\right) &= \left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\right)^2 s^{1-\kappa}t^{1-\alpha}\frac{\partial^2}{\partial s\partial t}f(t,s)\\ &= \left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\right)^2 s^{1-\kappa}t^{1-\alpha}\lim_{h\to 0}\frac{\frac{\partial}{\partial t}f(t,s+h)-\frac{\partial}{\partial t}f(t,s)}{h}.\end{aligned} \tag{4.2}$$

Thus, taking $h=\varepsilon s^{1-\kappa}\left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}+O(\varepsilon)\right)$ and later $k=\varepsilon t^{1-\alpha}\left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}+O(\varepsilon)\right)$ in the Eq. (4.2), we arrive at

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial t^\kappa}f(t,s)\right)=\frac{\partial^\kappa}{\partial t^\kappa}\left(\lim_{h\to 0}\frac{\frac{\partial}{\partial t}f(t,s+h)-\frac{\partial}{\partial t}f(t,s)}{h}\right)=\frac{\partial^\kappa}{\partial t^\kappa}\left(\frac{\partial^\alpha}{\partial t^\alpha}f(t,s)\right),$$

which completes the proof. □

We define the $\mathcal{V}$-fractional vector at the point a, given by

$$\triangledown_\alpha f(a)=\left(\frac{\partial^\alpha}{\partial t^\alpha}f(a),\frac{\partial^\kappa}{\partial s^\kappa}f(a)\right).$$

The next example, is a direct application of the Theorem 4.3.

Example 4.4. Consider $f(t,s) = e^{a(t+s)}$ with $a \in \mathbb{R}$ satisfying the conditions of Theorem 4.3, then we have

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial s^\kappa}f(t,s)\right) &= \frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{s^{1-\kappa}\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\frac{\partial}{\partial s}e^{a(s+t)}\right)\\ &= a\frac{s^{1-\kappa}\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\frac{\partial^\alpha}{\partial t^\alpha}e^{a(s+t)}\\ &= a^2 s^{1-\kappa}t^{1-\alpha}\left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\right)^2 e^{a(t+s)} \qquad (4.3)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^\kappa}{\partial s^\kappa}\left(\frac{\partial^\alpha}{\partial t^\alpha}f(t,s)\right) &= \frac{\partial^\kappa}{\partial s^\kappa}\left(\frac{t^{1-\alpha}\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\frac{\partial}{\partial t}e^{a(s+t)}\right)\\ &= a\frac{t^{1-\alpha}\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\frac{\partial^\kappa}{\partial s^\kappa}e^{a(s+t)}\\ &= a^2 s^{1-\kappa}t^{1-\alpha}\left(\frac{\Gamma(\beta)(\rho)_q}{\Gamma(\gamma+\beta)(\delta)_p}\right)^2 e^{a(t+s)}. \qquad (4.4)\end{aligned}$$

Thus, by Eq. (4.3) and Eq. (4.4) we conclude that

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\kappa}{\partial s^\kappa}f(t,s)\right) = \frac{\partial^\kappa}{\partial s^\kappa}\left(\frac{\partial^\alpha}{\partial t^\alpha}f(t,s)\right).$$

Theorem 4.5. (truncated $\mathcal{V}$-fractional Green theorem) *Let C be a simple positively oriented, piecewise smooth and close curve in $\mathbb{R}^2$, say for instance the $x-y$ plane, furthermore assume D in the interior of C. If $f(x,y)$ and $g(x,y)$ are two functions having continuous partial truncated $\mathcal{V}$-fractional derivative on D then*

$$\int\int_D\left(\frac{\partial^\alpha}{\partial x^\alpha}g - \frac{\partial^\alpha}{\partial y^\alpha}f\right)d_\omega S = \int_C \frac{\partial^{\alpha-1}}{\partial y^{\alpha-1}}f d_\omega x - \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}g d_\omega y,$$

where $d_\omega S = \left(\frac{\Gamma(\gamma+\beta)(\delta)_p}{\Gamma(\beta)(\rho)_q}\right)^2 x^{\alpha-1}y^{\alpha-1}dxdy$, *with $d_\omega x$ and $d_\omega y$, given by* Remark 2.7.

Proof. In fact, note that

$$\int\int_D\left(\frac{\partial^\alpha}{\partial x^\alpha}g - \frac{\partial^\alpha}{\partial y^\alpha}f\right)d_\omega S = \int\int_D\left[\frac{\partial}{\partial x}\left(\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}g\right) - \frac{\partial}{\partial y}\left(\frac{\partial^{\alpha-1}}{\partial y^{\alpha-1}}f\right)\right]d_\omega S. \qquad (4.5)$$

Applying the classical version of the Green's theorem [7],

$$\int\int_D\left(\frac{\partial Q}{\partial x} - \frac{\partial}{\partial y}P\right)dS = \int_C (Pdx + Qdy)$$

into Eq. (4.5), we conclude that

$$\int\int_D\left(\frac{\partial^\alpha}{\partial x^\alpha}g - \frac{\partial^\alpha}{\partial y^\alpha}f\right)d_\omega S = \int_C \frac{\partial^{\alpha-1}}{\partial y^{\alpha-1}}f d_\omega x + \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}g d_\omega y.$$

□

The following application by means of the heat equation will be discussed in $\mathbb{R}$. However, it can be extended to $\mathbb{R}^n$. Using a $\mathcal{V}$-fractional derivative type, we propose a $\mathcal{V}$-fractional heat equation given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k\frac{\partial^2 u(x,t)}{\partial x^2},\ 0 < x < L,\ t > 0, \qquad (4.6)$$

where $0 < \alpha < 1$ and with the initial condition and boundary conditions given by

$$\begin{aligned}u(0,t) &= 0, t \geq 0, \qquad (4.7)\\ u(L,t) &= 0, t \geq 0,\\ u(x,0) &= f(x), 0 \leq x \leq L.\end{aligned}$$

We start, considering the so-called $\mathcal{V}$-fractional linear differential equation with constant coefficients

$$\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} \pm \mu^{2}v(x,t) = 0, \tag{4.8}$$

where μ^2 is a positive constant.

Using the item 5 in Theorem 2.3, the Eq. (4.6) can be written as follows

$$\frac{t^{1-\alpha}\Gamma(\beta)(\rho)_{q}}{\Gamma(\beta+\gamma)(\delta)_{p}}\frac{dv(x,t)}{dt} \pm \mu^{2}v(x,t) = 0,$$

whose solution is given by

$$v(t) = c\exp\left(\pm\frac{\Gamma(\beta+\gamma)(\delta)_{p}}{\Gamma(\beta)(\rho)_{q}}\frac{\mu^{2}t^{\alpha}}{\alpha}\right), \tag{4.9}$$

with $0<\alpha<1$ and $\beta,\gamma,\rho,\delta,p,q>0$.

Now, we will use separation of variables method to obtain the solution of the $\mathcal{V}$-fractional heat equation. Then, considering $u(x,t) = P(x)Q(t)$ and replacing in Eq. (4.6), we get

$$\frac{d^{\alpha}}{dt^{\alpha}}Q(t)P(x) = k\frac{d^{2}}{dx^{2}}P(x)Q(t)$$

which implies

$$\frac{1}{kQ(t)}\frac{d^{\alpha}}{dt^{\alpha}}Q(t) = \frac{1}{P(x)}\frac{d^{2}}{dx^{2}}P(x) = \xi, \tag{4.10}$$

where ξ is a constant.

From Eq. (4.10), we obtain a system of differential equations, given by

$$\frac{d^{\alpha}}{dt^{\alpha}}Q(t) - k\xi Q(t) = 0$$

and

$$\frac{d^{2}}{dx^{2}}P(x) - \xi P(x) = 0. \tag{4.11}$$

First, let's find the solution of Eq. (4.11). For this, we must study three cases, that is, $\xi=0$, $\xi=-\mu^2$ and $\xi=\mu^2$, with $\mu>0$

The Case 1, i.e. $\xi=0$ and the Case 3, i.e. $\xi=\mu^2$, we do not present the calculations, since it is a trivial solution.

Case 2: $\xi=-\mu^2$.

Substituting $\xi=-\mu^2$ into Eq. (4.11), we get

$$\frac{d^{2}}{dx^{2}}P(x) + \mu^{2}P(x) = 0,$$

whose solution is given by $P(x) = c_2\sin(\mu x)+c_1\cos(\mu x)$, with c_1 and c_2 arbitrary constant. Using the initial conditions Eq. (4.7), we obtain $c_1=0$ and $0=c_2\sin(\mu x)$ which implies that $\mu=\frac{n\pi}{L}$, with $n=1,2,....$ Then, we obtain

$$P_{n}(x) = a_{n}\sin\left(\frac{n\pi x}{L}\right) \text{ and } \mu = \frac{n\pi}{L}.$$

Therefore, the solution of Eq. (4.11) is given by

$$P_{n}(x) = a_{n}\sin\left(\frac{n\pi x}{L}\right) \text{ and} \mu = \frac{n\pi}{L}. \tag{4.12}$$

Using the Eq. (4.8) and Eq. (4.9), we have

$$Q_{n}(t) = b_{n}\exp\left(-\frac{\Gamma(\beta+\gamma)(\delta)_{p}}{\Gamma(\beta)(\rho)_{q}}\left(\frac{n\pi}{L}\right)^{2}\frac{k}{\alpha}t^{\alpha}\right), \tag{4.13}$$

where b_n are constant coefficients.

So, using the Eq. (4.12) and Eq. (4.13), the partial solutions of Eq. (4.6), is given by

$${}_{\alpha}u_{\beta,\gamma,p}^{\delta,\rho,q}(x,t) = \sum_{n=1}^{\infty}c_{n}\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{\Gamma(\beta+\gamma)(\delta)_{p}}{\Gamma(\beta)(\rho)_{q}}\left(\frac{n\pi}{L}\right)^{2}\frac{k}{\alpha}t^{\alpha}\right).$$

Using Eq. (4.7), we get

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

which provides c_n through

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

So, we conclude that the solution of $\mathcal{V}$-fractional heat equation Eq. (4.6), satisfying the conditions Eq. (4.7), is given by

$$ {}_{\alpha}u^{\delta,\rho,q}_{\beta,\gamma,p}(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{\Gamma(\beta+\gamma)(\delta)_p}{\Gamma(\beta)(\rho)_q}\left(\frac{n\pi}{L}\right)^2 \frac{k}{\alpha}t^{\alpha}\right)\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right). \tag{4.14}$$

Choosing $p = q = \gamma = \delta = \rho = \beta = 1$ in the Eq. (4.14), we have

$$ {}_{\alpha}u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\left(\frac{n\pi}{L}\right)^2 \frac{k}{\alpha}t^{\alpha}\right)\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right), \tag{4.15}$$

the solution of Eq. (4.8), in this sense of the conformable fractional derivative. (Note that, taking the limit $i \to 1$ in the Eq. (2.4)). We have the parameter α free.

Choosing $p = q = \gamma = \delta = \rho = 1$ in the Eq. (4.14), we get

$$ {}_{\alpha}u_{\beta}(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\Gamma(\beta+1)\left(\frac{n\pi}{L}\right)^2 \frac{k}{\alpha}t^{\alpha}\right)\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right), \tag{4.16}$$

the solution of Eq. (4.8), in this sense of the M-fractional derivative. (Note that, taking the limit $i \to \infty$ in the Eq. (2.4). We have the parameter α and β free.

Next, we will present some plots by choosing values for the parameters $\alpha, \beta, \gamma, \delta, \rho, p, q, k, t$ and L, to see the behavior of the solution presented in Eq (4.14) and recover the Eq. (4.15) and Eq. (4.16). The graphics were plotted using MATLAB 7:10 software (R2010a). For the elaboration of the following plots, we choose the function $f(x) = 50x(1-x)$.

FIGURE 1. Analytical solution of the $\mathcal{V}$-fractional heat equation Eq. (4.14). We consider the values $t = 50$, $L = 1$, $k = 0.01$ and $\alpha = 0.2$.

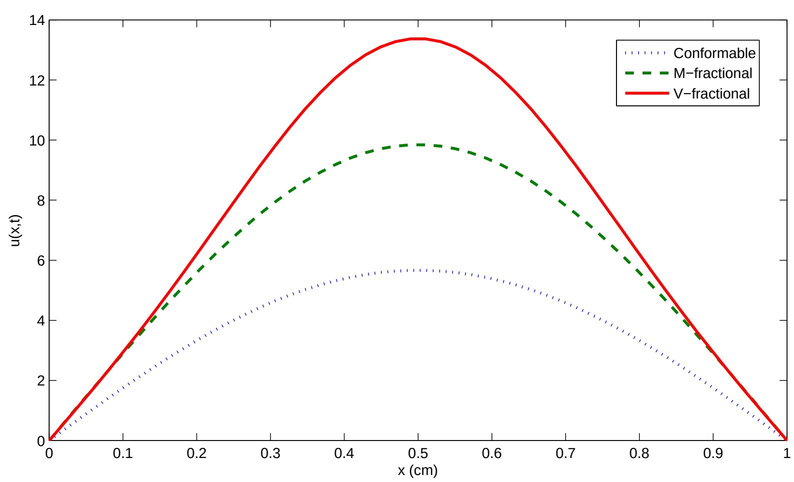


Figure 2. Analytical solution of the $\mathcal{V}$-fractional heat equation Eq. (4.14). We consider the values $t = 50$, $L = 1$, $k = 0.01$ and $\alpha = 0.5$.

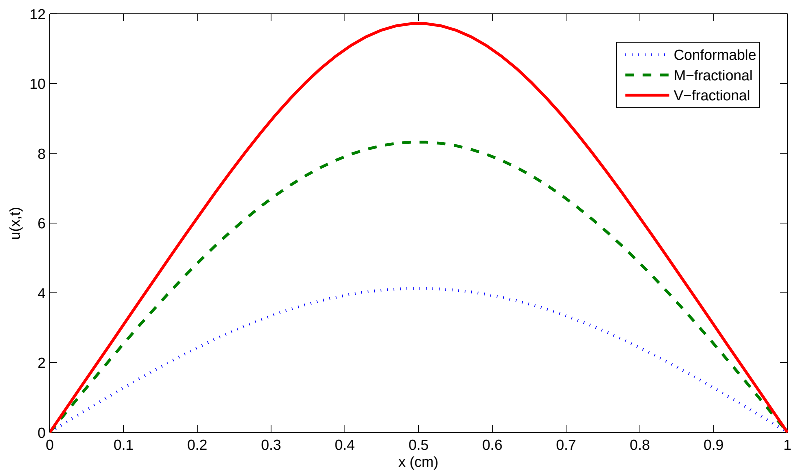
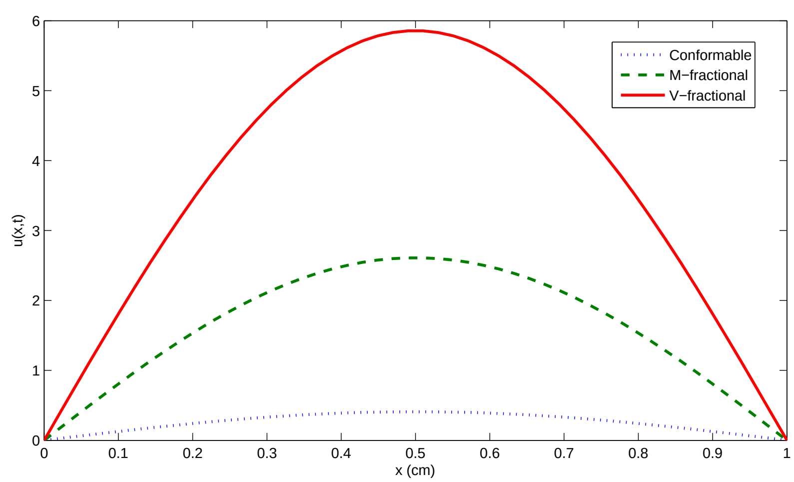


Figure 3. Analytical solution of the $\mathcal{V}$-fractional heat equation Eq. (4.14). We consider the values $t = 50$, $L = 1$, $k = 0.01$ and $\alpha = 0.9$.



5. Concluding Remarks

After a brief introduction to the truncated six-parameters Mittag-Leffler function and the truncated $\mathcal{V}$-fractional derivative with domain of function in $\mathbb{R}$ and the validity of some important results, we have introduced the multivariable truncated $\mathcal{V}$-fractional derivative, that is, with domain of the function in $\mathbb{R}^n$. In this sense, we discussed and proved classical theorems such as: the chain rule, the commutativity of the exponent of two truncated $\mathcal{V}$-fractional derivatives and Green's theorem.

We concluded that: a variety of new fractional derivatives of said local have been recently introduced, all them satisfy the requirements of the integer-order derivative, and have been employed to deal more effectively with real problems and their physical properties [2, 3, 7, 8]. The dynamics of systems over time, becomes more complex and more precise mathematical tools are needed to solve certain theoretical and practical problems. In this theoretical and applicable sense, we extended the idea of truncated $\mathcal{V}$-fractional derivative of a variable, so it is possible to work with differential equations with several variables consequently make comparisons with the results obtained by means of other fractional derivatives. Studies in direction will be published in a forthcoming paper.

Acknowledgment

We are grateful to the anonymous referees for the suggestions that improved the manuscript.

References

[1] Atangana, A., Baleanu, D., Alsaedi, A., *New properties of conformable derivative*, Open Mathematics, **13(1)**(2015), 1–10. 1
[2] Baleanu, D., Machado, J. A. T., Luo, A. C., Fractional Dynamics and Control, Springer, New York, 2011. 5
[3] Diethelm, K., Fractional differential equations. Theory and numerical treatment. Scriptum, Institute of Computational Mathematics, Technical University of Braunschweig, 2003. 5
[4] Gözütok, N. Y., Gözütok, U., *Multi-variable conformable fractional calculus*, Filomat, **32(1)**(2018), 45–53. 1
[5] Katugampola, U. N., *A new fractional derivative with classical properties*, arXiv preprint arXiv:1410.6535. 1
[6] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., *A new definition of fractional derivative*, J. Comput. and Appl. Math., **264**(2014), 65–70. 1
[7] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006. 4, 5
[8] Monje, C. A., Chen, Y., Vinagre, B. M., Xue, D., Feliu-Batlle, V., Fractional-order systems and controls: fundamentals and applications, Springer-Verlag, London, 2010. 5
[9] Vanterler da C. Sousa, J., Capelas de Oliveira, E., *M-fractional derivative with classical properties*, arXiv:1704.08186 [math.CA], (2017). 1
[10] Vanterler da C. Sousa, J., Capelas de Oliveira, E., *Mittag-Leffler functions and the truncated $\mathcal{V}$-fractional derivative*, Mediterr. J. Math., **14(6)**(2017) 244. 1, 2, 2.1, 2.2, 2, 2, 2.5
[11] Vanterler da C. Sousa, J., Capelas de Oliveira, E., *A new truncated M-fractional derivative unifying some fractional derivatives with classical properties*, Inter. J. Anal. and Appl., **16(1)**(2018), 83–96. 1
[12] Stewart, J., Calculus, Cengage Learning, Boston, 2015. 4