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The $\Delta-$convergence of proximal point iteration based on the variational inequality in CAT(0) Spaces

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Abstract

In this paper, we build the inexact proximal point algorithm of Mann-type and Ishikawa-type iteration based on the variational inequality in Hadamard spaces and prove $\Delta-$convergence to a fixed point of the nonexpansive mapping.

Keywords: Mann-type, Ishikawa-type, variational inequality, nonexpansive mapping, Hadamard spaces, inexact, $\Delta-$convergence.
2010 MSC: 54H25, 47H10.

1. Introduction

Let (X, d) be a metric space[1]. A geodesic path joining $x \in X$ to $y \in X$(or, more briefly, a geodesic from x to y) is a map f from a closed interval $[0, l] \subset R$ to X such that $f(0) = x$, $f(l) = y$ and $d(f(t), f(t')) = |t-t'|$ for all $t, t' \in [0, l]$. In particular, f is an isometry and $d(x, y) = l$. The image α of f is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic is denoted $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic space (X, d) is a CAT(0) space if it satisfies the following CN-inequality for $x, z_0, z_1, z_2 \in X$ such that $d(z_0, z_1) = d(z_0, z_2) = \frac{1}{2}d(z_1, z_2)$:

$$d^2(x, z_0) \leqslant \frac{1}{2}d^2(x, z_1) + \frac{1}{2}d^2(x, z_2) - \frac{1}{4}d^2(z_1, z_2).$$

A complete CAT(0) space is called a Hadamard space.

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Berg and Nikolaev[3] introduced the concept of quasi-linearization in CAT(0) space X. They denoted a vector by $\overrightarrow{ab}$ for $(a,b)\in X\times X$ and defined the quasi-linearization map $\langle\cdot,\cdot\rangle:(X\times X)\times(X\times X)\to R$ as follow:

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\frac{1}{2}[d^2(a,d)+d^2(b,c)-d^2(a,c)-d^2(b,d)],$$

for $a,b,c,d\in X$. We can verify $\langle\overrightarrow{ab},\overrightarrow{ab}\rangle=d^2(a,b)$, $\langle\overrightarrow{ba},\overrightarrow{cd}\rangle=-\langle\overrightarrow{ab},\overrightarrow{cd}\rangle$, and $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\langle\overrightarrow{ae},\overrightarrow{cd}\rangle+\langle\overrightarrow{eb},\overrightarrow{cd}\rangle$ for all $a,b,c,d,e\in X$. For a space X, it satisfies the Cauchy-Schwarz inequality if

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle\leqslant d(a,b)d(c,d)$$

for all $a,b,c,d\in X$.It is known[3] that a geodesically connected metric space X is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Let (X,d) be a complete CAT(0) space. K is convex and closed in X and $T:K\to X$ is a nonexpansive mapping. We formulate the variational inequality problem in a Hadamard space X associated with the nonexpansive mapping T as follows[1]: Find $x\in K$, such that

$$\left\langle\overrightarrow{Txx},\overrightarrow{xz}\right\rangle\geqslant 0,\quad\forall z\in K. \tag{1.1}$$

Hadi Khatibzadeh and Sajad Ranjbar[2] have introduced the existence of solutions for the variational inequality problem and approximate a solution of (1.1) by the inexact proximal point algorithm. By the Lemma 2.8, the solution of the auxiliary problem (2.1) is unique but we may not be able to obtain the sequence (x_n) exactly. Therefore, they suppose that in each step a computational error occurs. Then, they proved the $\Delta-$convergence of the inexact proximal point algorithm

$$\begin{cases} x_0\in K,(\lambda_n)\subset(0,\infty),(\varepsilon_n)\subset[0,\infty)\\ \left\langle\overrightarrow{Tx_nx_n}-\frac{1}{\lambda_n}\overrightarrow{x_nx_{n-1}},\overrightarrow{x_nz}\right\rangle\geqslant-\varepsilon_n,\quad\forall z\in X,\end{cases} \tag{1.2}$$

with certain conditions on the parameter sequence λ_n and the error sequence ε_n to a solution of the variational inequality problem (1.1), which is also a fixed point of the nonexpansive mapping T. Meanwhile, they also proved the strong convergence of the Halpern-type regularization

$$\begin{cases} u,x_1\in K,(\lambda_n)\subset(0,\infty),(\alpha_n)\subset[0,1],(\varepsilon_n)\subset[0,\infty)\\ \left\langle\overrightarrow{Ty_ny_n}-\frac{1}{\lambda_n}\overrightarrow{y_nx_n},\overrightarrow{y_nz}\right\rangle\geqslant-\varepsilon_n,\quad\forall z\in X,\\ x_{n+1}=\alpha_nu\oplus(1-\alpha_n)y_n,\end{cases} \tag{1.3}$$

with suitable assumptions on the parameter sequences α_n and λ_n, to a fixed point of the nonexpansive mapping, which solves the variational inequality problem (1.1).

In this paper, we build the Mann-type and Ishikawa-type regularization as follow, Mann-type:

$$\begin{cases} x_0\in K,\\ \left\langle\overrightarrow{Ty_ny_n}-\frac{1}{\lambda_n}\overrightarrow{y_nx_n},\overrightarrow{y_nz}\right\rangle\geqslant-\frac{\varepsilon_n}{2},\quad\forall z\in X,\\ x_{n+1}=\alpha_nx_n\oplus(1-\alpha_n)y_n,\end{cases} \tag{1.4}$$

with $(\lambda_n)\subset(0,\infty)$, $(\alpha_n)\subset[0,1]$, $(\varepsilon_n)\subset[0,\infty)$ and Ishikawa-type:

$$\begin{cases} x_0\in K,\\ \left\langle\overrightarrow{Ty_ny_n}-\frac{1}{\lambda_n}\overrightarrow{y_nx_n},\overrightarrow{y_np}\right\rangle\geqslant-\frac{\varepsilon_n}{2},\quad\forall p\in X,\\ u_n=\alpha_nx_n\oplus(1-\alpha_n)y_n,\\ \left\langle\overrightarrow{Tw_nw_n}-\frac{1}{\eta_n}\overrightarrow{w_nu_n},\overrightarrow{w_nq}\right\rangle\geqslant-\frac{\delta_n}{2},\quad\forall q\in X,\\ x_{n+1}=\beta_nx_n\oplus(1-\beta_n)w_n,\end{cases} \tag{1.5}$$

with $(\lambda_n)\subset(0,\infty)$, $(\eta_n)\subset(0,\infty)$, $(\alpha_n)\subset[0,1]$, $(\beta_n)\subset[0,1]$, $(\varepsilon_n)\subset[0,\infty)$ and $(\delta_n)\subset[0,\infty)$. Therefore, we prove the two sequences generated by the algorithm (1.4) and (1.5) are $\Delta-$convergent to a fixed point of the nonexpansive mapping T. Then, we improve and extend their results.

2. Preliminaries

Definition2.1[2] A sequence (x_n) in a complete CAT(0) space (X,d) is said to be $\Delta-$convergent to $x \in X$ if $A((x_{n_k})) = \{x\}$ for every subsequence (x_{n_k}) of (x_n).

Lemma2.2[7] Every bounded sequence in a complete CAT(0) space has a $\Delta-$convergent subsequence.

Lemma2.3[4] Let (X,d) be a CAT(0) space. Then

$$d((1-t)x \oplus ty, z) \leqslant (1-t)d(x,z) + td(y,z)$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Lemma2.4[4] Let (X,d) be a CAT(0) space. Then

$$d^2((1-t)x \oplus ty, z) \leqslant (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y)$$

for all $t \in [0,1]$ and $x, y, z \in X$.

Lemma2.5[5] Let (X,d) be a CAT(0) space and and let (x_n) be a sequence in X. If there exists a nonempty subset F of X verifying the following conditions:

(i) for each $z \in F$, $\lim_n d(x_n, z)$ exists,

(ii) if a subsequence (x_{n_j}) of (x_n) is $\Delta-$convergent to $x \in X$, then $x \in F$, then (x_n) $\Delta-$converges to an element of F.

Lemma2.6[2] Let K be bounded. Then there exists a solution of the variational inequality problem (1.1).

Lemma2.7[2] Let $x \in \mathrm{int}(K)$ (interior of K) be a solution of problem (1.1), then $x \in F(T)$.

Lemma2.8[2] For each $x \in K$ and each $\lambda > 0$, there exists a unique $y \in K$ such that

$$\left\langle \overrightarrow{Tyy} - \frac{1}{\lambda}\overrightarrow{yx}, \overrightarrow{yz} \right\rangle \geqslant 0, \quad \forall z \in X. \tag{2.1}$$

Lemma2.9[6] Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leqslant a_n + b_n$ for all $n \geqslant 1$. If $\sum_n b_n$ converges, then $\lim_n a_n$ exists.

3. Main results

Theorem3.1 Let X be a Hadamard space. $K \subset X$ is convex and closed and $T : K \to X$ is a nonexpansive mapping and $F(T) \neq \emptyset$. The sequence $\{x_n\}$ is generated by (1.4) and let the conditions hold: $\limsup_n \alpha_n < 1$, $\liminf_n \lambda_n > 0$ and $\sum_n \varepsilon_n \lambda_n < \infty$.Then the sequence $\{x_n\}$ is $\Delta-$convergent to a fixed point of T.

Proof. Let $z \in F(T)$, then by (1.4), we have

$$\left\langle \overrightarrow{Ty_n y_n} - \frac{1}{\lambda_n}\overrightarrow{y_n x_n}, \overrightarrow{y_n z} \right\rangle \geqslant -\frac{\varepsilon_n}{2}.$$

Hence, we can obtain

$$\begin{aligned} -\varepsilon_n &\leqslant 2\left\langle \overrightarrow{Ty_n y_n}, \overrightarrow{y_n z} \right\rangle - \frac{2}{\lambda_n}\langle \overrightarrow{y_n x_n}, \overrightarrow{y_n z} \rangle \\ &= d^2(z, Ty_n) - d^2(y_n, Ty_n) - d^2(y_n, z) \\ &\quad - \frac{1}{\lambda_n}(d^2(y_n, z) + d^2(y_n, x_n) - d^2(x_n, z)). \end{aligned}$$

By the nonexpansiveness of T, we get

$$\lambda_n d^2(y_n, Ty_n) + d^2(y_n, x_n) \leqslant d^2(x_n, z) - d^2(y_n, z) + \varepsilon_n \lambda_n, \tag{3.1}$$

which implies

$$d^2(y_n, z) \leqslant d^2(x_n, z) + \varepsilon_n\lambda_n. \tag{3.2}$$

By the lemma 2.4, we have

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\alpha_n x_n \oplus (1-\alpha_n)y_n, z) \\ &\leqslant \alpha_n d^2(x_n, z) + (1-\alpha_n)d^2(y_n, z) - \alpha_n(1-\alpha_n)d^2(x_n, y_n) \\ &\leqslant \alpha_n d^2(x_n, z) + (1-\alpha_n)(d^2(x_n, z) + \varepsilon_n\lambda_n) \\ &= d^2(x_n, z) + (1-\alpha_n)\varepsilon_n\lambda_n \\ &\leqslant \cdots \leqslant d^2(x_1, z) + \sum_{i=1}^{n}(1-\alpha_i)\varepsilon_i\lambda_i. \end{aligned}$$

Therefore, according to the lemma 2.9, we know that $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim_n d(x_n, z)$ exists. Then, let $\lim_n d(x_n, z) = a$. If $a = 0$, it denotes that the sequence $\{x_n\}$ is strongly convergent to a fixed point of T; if $a > 0$, consequently, by (3.2), we have

$$(d(y_n, z) - d(x_n, z))(d(y_n, z) + d(x_n, z)) \leqslant \varepsilon_n\lambda_n,$$

which means

$$d(y_n, z) \leqslant d(x_n, z) + \frac{\varepsilon_n\lambda_n}{d(y_n, z) + d(x_n, z)} \leqslant d(x_n, z) + M\varepsilon_n\lambda_n,$$

where

$$M = \sup_n\{\frac{1}{d(y_n, z) + d(x_n, z)}\}.$$

Since $\lim_n d(x_n, z)$ exists and by the lemma 2.3, we obtain

$$\begin{aligned} 0 &= \lim_n(d(x_{n+1}, z) - d(x_n, z)) \\ &\leqslant \liminf_n(\alpha_n d(x_n, z) + (1-\alpha_n)d(y_n, z) - d(x_n, z)) \\ &= \liminf_n(1-\alpha_n)(d(y_n, z) - d(x_n, z)) \\ &\leqslant \limsup_n(1-\alpha_n)(d(y_n, z) - d(x_n, z)) \\ &\leqslant \limsup_n(1-\alpha_n)M\varepsilon_n\lambda_n = 0. \end{aligned}$$

By the assumptions $\limsup_n \alpha_n < 1$, we get

$$\lim_n(d(y_n, z) - d(x_n, z)) = 0.$$

Also, by the assumptions and (3.1), we have

$$\lim_n d(y_n, Ty_n) = 0, \quad \lim_n d(y_n, x_n) = 0.$$

Therefore, by the nonexpansiveness of T, we may obtain

$$\begin{aligned} \lim_n d(x_n, Tx_n) &\leqslant \lim_n(d(x_n, y_n) + d(y_n, Ty_n) + d(Ty_n, Tx_n)) \\ &\leqslant \lim_n(d(x_n, y_n) + d(y_n, Ty_n) + d(y_n, x_n)) = 0, \end{aligned}$$

which implies

$$\lim_n d(x_n, Tx_n) = 0.$$

Thus, if a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ is $\Delta-$convergent to $q \in X$, then

$$d(x_{n_j}, Tx_{n_j}) \to 0.$$

Hence, by $\Delta-$demiclosedness of nonexpansive mappings, we have $q \in F(T)$. Finally, by the lemma 2.5, the sequence $\{x_n\}$ is $\Delta-$convergent to $z \in F(T)$. This completes the proof. □

Theorem3.2 Let X be a Hadamard space. $K \subset X$ is convex and closed and $T : K \to X$ is a nonexpansive mapping and $F(T) \neq \emptyset$. The sequence $\{x_n\}$ is generated by (1.5) and let the conditions hold: $\limsup_n \beta_n < 1$, $\liminf_n \lambda_n > 0$, $\liminf_n \eta_n > 0$, $\sum_n \varepsilon_n \lambda_n < \infty$ and $\sum_n \delta_n \eta_n < \infty$.Then the sequence $\{x_n\}$ is $\Delta-$convergent to a fixed point of T.

Proof. Let $z \in F(T)$, then by (1.5), we have

$$\left\langle \overrightarrow{Ty_n y_n} - \frac{1}{\lambda_n} \overrightarrow{y_n x_n}, \overrightarrow{y_n z} \right\rangle \geqslant -\frac{\varepsilon_n}{2}.$$

Hence, we can obtain

$$\begin{aligned} -\varepsilon_n &\leqslant 2 \left\langle \overrightarrow{Ty_n y_n}, \overrightarrow{y_n z} \right\rangle - \frac{2}{\lambda_n} \langle \overrightarrow{y_n x_n}, \overrightarrow{y_n z} \rangle \\ &= d^2(z, Ty_n) - d^2(y_n, Ty_n) - d^2(y_n, z) \\ &\quad - \frac{1}{\lambda_n}(d^2(y_n, z) + d^2(y_n, x_n) - d^2(x_n, z)). \end{aligned}$$

By the nonexpansiveness of T, we get

$$\lambda_n d^2(y_n, Ty_n) + d^2(y_n, x_n) \leqslant d^2(x_n, z) - d^2(y_n, z) + \varepsilon_n \lambda_n, \tag{3.3}$$

which implies

$$d^2(y_n, z) \leqslant d^2(x_n, z) + \varepsilon_n \lambda_n. \tag{3.4}$$

Similarly, we obtain

$$\eta_n d^2(w_n, Tw_n) + d^2(u_n, w_n) \leqslant d^2(u_n, z) - d^2(w_n, z) + \delta_n \eta_n, \tag{3.5}$$

which also implies

$$d^2(w_n, z) \leqslant d^2(u_n, z) + \delta_n \eta_n. \tag{3.6}$$

By the lemma 2.4, we get

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\beta_n x_n + (1-\beta_n) w_n, z) \\ &\leqslant \beta_n d^2(x_n, z) + (1-\beta_n) d^2(w_n, z) - \beta_n(1-\beta_n) d^2(x_n, w_n) \\ &\leqslant \beta_n d^2(x_n, z) + (1-\beta_n) d^2(u_n, z) + (1-\beta_n)\delta_n \eta_n, \end{aligned}$$

and

$$\begin{aligned} d^2(u_n, z) &= d^2(\alpha_n x_n + (1-\alpha_n) y_n, z) \\ &\leqslant \alpha_n d^2(x_n, z) + (1-\alpha_n) d^2(y_n, z) - \alpha_n(1-\alpha_n) d^2(x_n, y_n) \\ &\leqslant \alpha_n d^2(x_n, z) + (1-\alpha_n)(d^2(x_n, z) + \varepsilon_n \lambda_n) \\ &= d^2(x_n, z) + (1-\alpha_n)\varepsilon_n \lambda_n, \end{aligned}$$

then, we obtain

$$\begin{aligned} d^2(x_{n+1}, z) &\leqslant \beta_n d^2(x_n, z) + (1-\beta_n)(d^2(x_n, z) + (1-\alpha_n)\varepsilon_n \lambda_n) + (1-\beta_n)\delta_n \eta_n \\ &= d^2(x_n, z) + (1-\beta_n)(1-\alpha_n)\varepsilon_n \lambda_n + (1-\beta_n)\delta_n \eta_n \\ &\leqslant \cdots \leqslant d^2(x_1, z) + \sum\nolimits_{i=1}^{n} (1-\beta_i)(1-\alpha_i)\varepsilon_i \lambda_i + \sum\nolimits_{i=1}^{n} (1-\beta_i)\delta_i \eta_i. \end{aligned}$$

Therefore, according to the lemma 2.9, we know that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{w_n\}$ are bounded and $\lim_n d(x_n, z)$ exists. Then, let $\lim_n d(x_n, z) = a$. If $a = 0$, it denotes that the sequence $\{x_n\}$ is strongly convergent to a fixed point of T; if $a > 0$, consequently, by (3.4) and (3.6), we have

$$(d(y_n, z) - d(x_n, z))(d(y_n, z) + d(x_n, z)) \leqslant \varepsilon_n \lambda_n$$

and

$$(d(w_n, z) - d(u_n, z))(d(w_n, z) + d(u_n, z)) \leqslant \delta_n \eta_n,$$

which mean

$$d(y_n, z) \leqslant d(x_n, z) + \frac{\varepsilon_n \lambda_n}{d(y_n, z) + d(x_n, z)} \leqslant d(x_n, z) + M_1 \varepsilon_n \lambda_n$$

and

$$d(w_n, z) \leqslant d(u_n, z) + \frac{\delta_n \eta_n}{d(w_n, z) + d(u_n, z)} \leqslant d(u_n, z) + M_2 \delta_n \eta_n,$$

where

$$M_1 = \sup_n \{\frac{1}{d(y_n, z) + d(x_n, z)}\}, M_2 = \sup_n \{\frac{1}{d(w_n, z) + d(u_n, z)}\}.$$

In the following, we consider three possible cases for the sequence $\{\alpha_n\}$.

(1)$\limsup_n \alpha_n < 1$.

Since $\lim_n d(x_n, z)$ exists and by the lemma 2.3, we obtain

$$\begin{aligned}
0 &= \lim_n (d(x_{n+1}, z) - d(x_n, z)) \\
&\leqslant \liminf_n (\beta_n d(x_n, z) + (1-\beta_n) d(w_n, z) - d(x_n, z)) \\
&= \liminf_n (1-\beta_n)(d(w_n, z) - d(x_n, z)) \\
&\leqslant \liminf_n (1-\beta_n)(d(u_n, z) + M_2 \delta_n \eta_n - d(x_n, z)) \\
&\leqslant \liminf_n (1-\beta_n)(\alpha_n d(x_n, z) + (1-\alpha_n) d(y_n, z) + M_2 \delta_n \eta_n - d(x_n, z)) \\
&= \liminf_n (1-\beta_n)(1-\alpha_n)(d(y_n, z) - d(x_n, z) + \frac{M_2 \delta_n \eta_n}{1-\alpha_n}) \\
&\leqslant \limsup_n (1-\beta_n)(1-\alpha_n)(d(y_n, z) - d(x_n, z) + \frac{M_2 \delta_n \eta_n}{1-\alpha_n}) \\
&\leqslant \limsup_n (1-\beta_n)(1-\alpha_n)(M_1 \varepsilon_n \lambda_n + \frac{M_2 \delta_n \eta_n}{1-\alpha_n}) = 0.
\end{aligned}$$

By the condition $\limsup_n \alpha_n < 1$ and assumption $\limsup_n \beta_n < 1$, we get

$$\lim_n (d(y_n, z) - d(x_n, z)) = 0.$$

Also, by the assumptions and (3.3), we have

$$\lim_n d(y_n, Ty_n) = 0, \quad \lim_n d(y_n, x_n) = 0.$$

Therefore, by the nonexpansiveness of T, we may obtain

$$\begin{aligned}
\lim_n d(x_n, Tx_n) &\leqslant \lim_n (d(x_n, y_n) + d(y_n, Ty_n) + d(Ty_n, Tx_n)) \\
&\leqslant \lim_n (d(x_n, y_n) + d(y_n, Ty_n) + d(y_n, x_n)) = 0,
\end{aligned}$$

which implies

$$\lim_n d(x_n, Tx_n) = 0.$$

Thus, if a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ is $\Delta-$convergent to $q \in X$, then

$$d(x_{n_j}, Tx_{n_j}) \to 0.$$

Hence, by $\Delta-$demiclosedness of nonexpansive mappings, we have $q \in F(T)$. Finally, by the lemma 2.5, the sequence $\{x_n\}$ is $\Delta-$convergent to $z \in F(T)$.

(2)$\limsup_n \alpha_n = 1$.

Hence, there exists a $\{\alpha_{n_j}\}$ of $\{\alpha_n\}$ satisfied $\limsup_j \alpha_{n_j} < 1$. Similar to (1), there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_j (d(y_{n_j}, z) - d(x_{n_j}, z)) = 0.$$

Also, by the assumptions and (3.3), we still have

$$\lim_j d(y_{n_j}, Ty_{n_j}) = 0, \lim_j d(y_{n_j}, x_{n_j}) = 0.$$

Therefore, we also obtain

$$\lim_j d(x_{n_j}, Tx_{n_j}) = 0.$$

Thus, if the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ is $\Delta-$convergent to $q \in X$, then

$$d(x_{n_j}, Tx_{n_j}) \to 0.$$

Hence, by $\Delta-$demiclosedness of nonexpansive mappings, we have $q \in F(T)$. Finally, by the lemma 2.5, the sequence $\{x_n\}$ is $\Delta-$convergent to $z \in F(T)$.

(3)$\lim_n \alpha_n = 1$.

Since $\lim_n d(x_n, z)$ exists and by the lemma 2.3, we obtain

$$\begin{aligned}
0 &= \lim_n (d(x_{n+1}, z) - d(x_n, z)) \\
&\leqslant \liminf_n (\beta_n d(x_n, z) + (1-\beta_n) d(w_n, z) - d(x_n, z)) \\
&= \liminf_n (1-\beta_n)(d(w_n, z) - d(x_n, z)) \\
&\leqslant \liminf_n (1-\beta_n)(d(u_n, z) + M_2\delta_n\eta_n - d(x_n, z)) \\
&\leqslant \limsup_n (1-\beta_n)(d(u_n, z) + M_2\delta_n\eta_n - d(x_n, z)) \\
&\leqslant \limsup_n (1-\beta_n)(\alpha_n d(x_n, z) + (1-\alpha_n) d(y_n, z) + M_2\delta_n\eta_n - d(x_n, z)) \\
&\leqslant \limsup_n (1-\beta_n)(1-\alpha_n)(d(y_n, z) - d(x_n, z)) + \limsup_n (1-\beta_n) M_2\delta_n\eta_n \\
&= \limsup_n (1-\beta_n) M_2\delta_n\eta_n = 0.
\end{aligned}$$

By the assumption $\limsup_n \beta_n < 1$, we get

$$\lim_n (d(u_n, z) - d(x_n, z)) = 0.$$

In fact, it is easy to prove

$$\lim_n (d(w_n, z) - d(x_n, z)) = 0,$$

hence, we may obtain

$$\lim_n (d(w_n, z) - d(u_n, z)) = 0.$$

Then, by the assumptions and (3.5), we have

$$\lim_n d(w_n, Tw_n) = 0, \lim_n d(u_n, w_n) = 0.$$

Therefore, by the nonexpansiveness of T, we may obtain

$$\begin{aligned}
\lim_n d(x_n, Tx_n) &\leqslant \lim_n (d(x_n, u_n) + d(u_n, w_n) + d(w_n, Tw_n) + d(Tw_n, Tx_n)) \\
&\leqslant \lim_n ((1-\alpha_n)d(x_n, y_n) + d(u_n, w_n) + d(w_n, Tw_n) + d(w_n, x_n)) \\
&\leqslant \lim_n (1-\alpha_n)d(x_n, y_n) + \lim_n d(u_n, w_n) + \lim_n d(w_n, Tw_n) \\
&\quad + \limsup_n d(w_n, x_n) \\
&= \limsup_n d(w_n, x_n).
\end{aligned}$$

By the lemma 2.4 again, we have

$$\begin{aligned}
d^2(x_{n+1}, z) &= d^2(\beta_n x_n + (1-\beta_n)w_n, z) \\
&\leqslant \beta_n d^2(x_n, z) + (1-\beta_n)d^2(w_n, z) - \beta_n(1-\beta_n)d^2(x_n, w_n),
\end{aligned}$$

which implies

$$\begin{aligned}
\beta_n(1-\beta_n)d^2(x_n, w_n) &\leqslant \beta_n d^2(x_n, z) + (1-\beta_n)d^2(w_n, z) - d^2(x_{n+1}, z) \\
&= d^2(x_n, z) - d^2(x_{n+1}, z) \\
&\quad + (1-\beta_n)(d^2(w_n, z) - d^2(x_n, z)).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\lim_n \beta_n(1-\beta_n)d^2(x_n, w_n) &\leqslant \lim_n (d^2(x_n, z) - d^2(x_{n+1}, z)) \\
&\quad + \lim_n (1-\beta_n)(d^2(w_n, z) - d^2(x_n, z)),
\end{aligned}$$

which implies

$$0 \leqslant \lim_n \beta_n(1-\beta_n)d^2(x_n, w_n) \leqslant 0,$$

therefore, we obtain

$$\lim_n \beta_n(1-\beta_n)d^2(x_n, w_n) = 0.$$

By the assumption, we know $\limsup_n \beta_n < 1$. Hence, as long as $\limsup_n \beta_n \neq 0$, then there exists a subsequence $\{\beta_{n_j}\}$ of $\{\beta_n\}$ such that $\liminf_j \beta_{n_j} > 0$, we have

$$\lim_j \beta_{n_j}(1-\beta_{n_j})d^2(x_{n_j}, w_{n_j}) = 0,$$

which implies

$$\lim_j d(x_{n_j}, w_{n_j}) = 0.$$

Thus, there also exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfied

$$\begin{aligned}
\lim_j d(x_{n_j}, Tx_{n_j}) &\leqslant limsup_j\, d(w_{n_j}, x_{n_j}) \\
&= lim_j\, d(w_{n_j}, x_{n_j}) = 0,
\end{aligned}$$

which implies

$$\lim_{j} d(x_{n_j}, Tx_{n_j}) = 0.$$

Thus, if the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ is $\Delta-$convergent to $q \in X$, then

$$d(x_{n_j}, Tx_{n_j}) \to 0.$$

Hence, by $\Delta-$demiclosedness of nonexpansive mappings, we have $q \in F(T)$. Finally, by the lemma 2.5, the sequence $\{x_n\}$ is $\Delta-$convergent to $z \in F(T)$. This completes the proof. □

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