Hacettepe Journal of Mathematics and Statistics
Volume 47(1) (2018), 203 – 221

Different estimation methods and joint confidence regions for the parameters of a generalized inverted family of distributions

Wenhao Gui*† and Lei Guo ‡

Abstract

In this paper, we deal with the problem of estimating the parameters of a generalized inverted family of distributions. We propose the inverse moment and modified inverse moment estimators of the parameters. The existence and uniqueness of inverse moment and modified inverse moment estimators is derived. Monte Carlo simulations are conducted to compare their performances with maximum-likelihood estimators. Two methods for constructing joint confidence regions for the two parameters are also proposed and their performances are discussed. A numerical example is presented to illustrate the methods.

Keywords: generalized inverted family of distributions, maximum likelihood estimates, existence and uniqueness, inverse moment estimators, joint confidence regions, Monte Carlo simulation.

2000 AMS Classification: AMS 62F99

Received : 17.08.2015 *Accepted* : 02.03.2016 *Doi* : 10.15672/HJMS.201612016315

1. Introduction

In mathematical statistics, a scale family of distributions is a family of univariate distributions $G(\cdot)$ parameterized by a scale parameter λ. It plays an important role in lifetime data analysis. Some representations are Exponential distribution, Half logistic distribution, Rayleigh distribution etc.

By adding a shape parameter, [6] generalized exponential distribution as an alternative to the gamma and Weibull distributions and studied its different properties. The cumulative distribution function is simply the αth power of the standard exponential cumulative distribution. Some references on generalized exponential distribution are [7],

*Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China, Email: whgui@bjtu.edu.cn

†Corresponding Author.

‡Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China, Email: 16121650@bjtu.edu.cn

[14], [8], [4], [11] etc. In a similar manner, [5] and [12] proposed the exponentiated types of distributions such as the exponentiated gamma, exponentiated Fréchet and exponentiated Gumbel distributions.

If Y is a random variable, then $X = 1/Y$ follows an inverted distribution. [1] introduced a generalized version of inverted exponential distribution and used it to model various failure rates, and hence different shapes of aging criteria. They derived statistical and reliability properties of the generalized inverted exponential distribution. Maximum likelihood estimation and least square estimation are used to evaluate the parameters and reliability of the distribution. [10] considered reliability estimation in generalized inverted exponential distribution with progressively type II censored sample.

In this article, we deal with the problem of estimating the parameters of a generalized inverted family of distributions. A random variable X is said to belong to the generalized inverted family of distributions(GIFD) if its cumulative distribution function (cdf) and probability density function (pdf) are respectively given by

$$F(x;\lambda,\alpha) = 1 - \left[G\left(\frac{\lambda}{x}\right)\right]^{\alpha}, \quad x > 0, \tag{1.1}$$

and

$$f(x;\lambda,\alpha) = \frac{\alpha\lambda}{x^2} g\left(\frac{\lambda}{x}\right)\left[G\left(\frac{\lambda}{x}\right)\right]^{\alpha-1}, \quad x > 0, \tag{1.2}$$

where $g(\cdot) = G'(\cdot)$, $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. When $\alpha = 1$, the generalized inverted family of distributions reduces to the inverted distribution.

Some examples of such models are:

(i) Generalized inverted exponential distribution: $GIED(\lambda, \alpha)$ with

$$G(x) = 1 - e^{-x}, \quad x > 0,$$

and the cdf is given by

$$F(x) = 1 - \left[1 - e^{-\frac{\lambda}{x}}\right]^{\alpha}, \quad x > 0.$$

(ii) Generalized inverted Rayleigh distribution: $GIRD(\lambda, \alpha)$ with

$$G(x) = 1 - e^{-x^2}, \quad x > 0,$$

and the cdf is given by

$$F(x) = 1 - \left[1 - e^{-\frac{\lambda}{x^2}}\right]^{\alpha}, \quad x > 0.$$

(iii) Generalized inverted half-logistic distribution: $GIHD(\lambda, \alpha)$ with

$$G(x) = \frac{1 - e^{-x}}{1 + e^{-x}}, \quad x > 0,$$

([3]) and the cdf is given by

$$F(x) = 1 - \left[\frac{1 - e^{-\frac{\lambda}{x}}}{1 + e^{-\frac{\lambda}{x}}}\right]^{\alpha}, \quad x > 0.$$

In this paper, we consider the problem of estimating the two parameters of the generalized inverted family of distributions. The rest of this paper is organized as follows:

In Section 2, we briefly review the maximum-likelihood estimation of the generalized inverted family of distributions. In Section 3, we propose inverse moment and modified inverse moment estimations and study their properties. The conditions for the existence and uniqueness of inverse moment and modified inverse moment estimators of the parameters are established. Joint confidence regions for the two parameters are discussed in

Section 4. Section 5 conducts simulations to compare the estimators and the joint confidence regions for the parameters of the generalized inverted half-logistic distribution which is a member of the family considered. Finally, in Section 6, a numerical example is presented to illustrate the validity of the proposed methods.

2. Maximum likelihood estimation

In this section, we briefly review the classical MLEs of the parameters of GIFD distribution based on a complete sample. Let $X_1, X_2, \cdots, X_n$ be a random sample from $GIFD(\lambda, \alpha)$ with pdf and cdf as (1.2) and (1.1), respectively. The log-likelihood function is given by

$$\ell(\lambda,\alpha) = n\log\alpha + n\log\lambda - 2\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log g\left(\frac{\lambda}{x_i}\right) + (\alpha-1)\sum_{i=1}^{n}\log G\left(\frac{\lambda}{x_i}\right) \tag{2.1}$$

We obtain the score equations as

$$\frac{\partial\ell(\lambda,\alpha)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{g'\left(\frac{\lambda}{x_i}\right)}{x_i g\left(\frac{\lambda}{x_i}\right)} + (\alpha-1)\sum_{i=1}^{n}\frac{g\left(\frac{\lambda}{x_i}\right)}{x_i G\left(\frac{\lambda}{x_i}\right)} = 0, \tag{2.2}$$

$$\frac{\partial\ell(\lambda,\alpha)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log G\left(\frac{\lambda}{x_i}\right) = 0. \tag{2.3}$$

From (2.3) we obtain the MLE of α as a function of λ,

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n}\log G\left(\frac{\lambda}{x_i}\right)}. \tag{2.4}$$

The MLE of λ is the root of the following equation

$$G(\lambda) = \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{g'\left(\frac{\lambda}{x_i}\right)}{x_i g\left(\frac{\lambda}{x_i}\right)} - \left[\frac{n}{\sum_{i=1}^{n}\log G\left(\frac{\lambda}{x_i}\right)} + 1\right]\sum_{i=1}^{n}\frac{g\left(\frac{\lambda}{x_i}\right)}{x_i G\left(\frac{\lambda}{x_i}\right)} = 0. \tag{2.5}$$

Such non-linear equation does not have closed form solution. We can apply numerical method such as Newton-Raphson to compute λ. For more details, see [13].

3. Inverse moment estimation

In general, the moment estimation does not work for estimating the parameters of the GIFD. For example, as for the GIED, the population moments do not exist. In this section, we propose an inverse moment estimation of parameters. Different from the regular method of moments, the idea of the inverse moment estimation (IME) is as follows:

Suppose $X_1, \cdots, X_n$ form a sample from a distribution with unknown parameters, first transform the original sample to a quasi-sample $Y_1, \cdots, Y_n$, where Y_i contains the unknown parameters but its distribution does not depend on the unknown parameters, that is, Y_i is a pivot variable, $i = 1, \cdots, n$. The population moments of the new sample do not dependent on the unknown parameters while the sample moments do. Let the population moments of the quasi-sample equal to the sample moments and solve for the unknown parameters.

Let $X_1, \cdots, X_n$ form a sample from $GIFD(\lambda, \alpha)$ with pdf given in (1.2), it is known that $F(X_i), 1 - F(X_i), i = 1, \cdots, n$ follow uniform distribution $U(0,1)$, and $-\log[1 -$

$F(X_i)], i = 1, \cdots, n$ follow standard exponential distribution $Exp(1)$. By the method of inverse moment estimation, we let

$$\frac{1}{n}\sum_{i=1}^{n}\{-\log[1-F(X_i)]\} = 1, \tag{3.1}$$

that is,

$$-\frac{\alpha}{n}\sum_{i=1}^{n}\log\left[G\left(\frac{\lambda}{x_i}\right)\right] = 1. \tag{3.2}$$

Thus, the IME of α is obtained as a function of λ,

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\log\left[G\left(\frac{\lambda}{x_i}\right)\right]}, \tag{3.3}$$

which is identical to the MLE of α.

3.1. Lemma. Let $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ be the order statistics from the standard exponential distribution. Then, the random variables $W_1, W_2, \cdots, W_n$, where

$$W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)}), \quad i = 1, 2, \cdots, n \tag{3.4}$$

with $Z_{(0)} \equiv 0$, are independent and follow standard exponential distributions.

Proof. The proof can be found in [2]. □

3.2. Lemma. Let $W_1, W_2, \cdots, W_n$ be i.i.d. standard exponential variables, $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i, i = 1, 2, \cdots, n-1$, $U_n = W_1 + \cdots + W_n$, then
(1) $U_1, U_2, \cdots, U_n$ are independent;
(2) $U_1, U_2, \cdots, U_{n-1}$ follow the uniform distribution $U(0, 1)$;
(3) $2U_n$ follows $\chi^2(2n)$.

Proof. The proof can be found in [17]. □

Now we determine the IME of λ. For the sample $X_1, \cdots, X_n$ from $GIFD(\lambda, \alpha)$, consider the order statistics $X_{(1)} \leq \cdots \leq X_{(n)}$, we have

$$-\log\left[1-F(X_{(1)})\right] \leq \cdots \leq -\log\left[1-F(X_{(n)})\right], \tag{3.5}$$

are n order statistics from standard exponential distribution.

Let $Z_{(i)} = -\alpha\log G\left(\frac{\lambda}{x_{(i)}}\right), i = 1, \cdots, n$. Thus, $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ are the first n order statistics from the standard exponential distribution. By Lemma 3.1, $W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)}), i = 1, 2, \cdots, n$ form a sample from standard exponential distribution.

Let $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i, i = 1, 2, \cdots, n-1$, $U_n = W_1 + \cdots + W_n$, by Lemma 3.2, we have

$$-2\sum_{i=1}^{n-1}\log U_i = -2\sum_{i=1}^{n-1} i\log(S_i/S_{i+1}) = 2\sum_{i=1}^{n-1}\log(S_n/S_i) \sim \chi^2(2n-2), \tag{3.6}$$

where

$$\begin{aligned}
\frac{S_n}{S_i} &= \frac{W_1 + \cdots + W_n}{W_1 + \cdots + W_i} \\
&= \frac{Z_{(1)} + Z_{(2)} + \cdots + Z_{(n)}}{Z_{(1)} + Z_{(2)} + \cdots + Z_{(i-1)} + (n-i+1)Z_{(i)}} \\
&= \frac{\log G\left(\frac{\lambda}{x_{(1)}}\right) + \log G\left(\frac{\lambda}{x_{(2)}}\right) + \cdots + \log G\left(\frac{\lambda}{x_{(n)}}\right)}{\log G\left(\frac{\lambda}{x_{(1)}}\right) + \cdots + \log G\left(\frac{\lambda}{x_{(i-1)}}\right) + (n-i+1)\log G\left(\frac{\lambda}{x_{(i)}}\right)}.
\end{aligned}$$

Noting that the mean of $\chi^2(2n-2)$ is $2n-2$. Thus, we obtain an inverse moment equation for λ as follows:

$$\sum_{i=1}^{n-1}\log\left[\frac{\log G\left(\frac{\lambda}{x_{(1)}}\right)+\log G\left(\frac{\lambda}{x_{(2)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(n)}}\right)}{\log G\left(\frac{\lambda}{x_{(1)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(i-1)}}\right)+(n-i+1)\log G\left(\frac{\lambda}{x_{(i)}}\right)}\right] = n-1. \tag{3.7}$$

Solve the equation and we obtain the inverse estimate $\hat{\lambda}_{IME}$ of λ. Plugging $\hat{\lambda}_{IME}$ into (3.3), we obtain the inverse estimate $\hat{\alpha}_{IME}$. In addition, considering that the mode of $\chi^2(2n-2)$ is $2n-4$, we can obtain a modified equation for λ:

$$\sum_{i=1}^{n-1}\log\left[\frac{\log G\left(\frac{\lambda}{x_{(1)}}\right)+\log G\left(\frac{\lambda}{x_{(2)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(n)}}\right)}{\log G\left(\frac{\lambda}{x_{(1)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(i-1)}}\right)+(n-i+1)\log G\left(\frac{\lambda}{x_{(i)}}\right)}\right] = n-2. \tag{3.8}$$

Solve the equation and we obtain the modified inverse estimate $\hat{\lambda}_{MIME}$ of λ. Plugging $\hat{\lambda}_{MIME}$ into (3.3), we obtain the modified inverse estimate $\hat{\alpha}_{MIME}$.

In the following, we prove the existence and uniqueness of the root in the equation (3.7) and (3.8).

3.3. Theorem. Let $W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)}), i = 1, 2, \cdots, n$ form a sample from standard exponential distribution, $S_i = W_1 + \cdots + W_i$, then for $t > 0$, equation $\sum_{i=1}^{n-1}\log(S_n/S_i) = t$ has a unique positive solution if the following conditions are satisfied:

$$\begin{cases} (1)\lim_{\lambda\to 0^+}\frac{\log G\left(\frac{\lambda}{a}\right)}{\log G\left(\frac{\lambda}{b}\right)} = 1, \quad \text{for} \quad a>0, b>0. \\ (2)\lim_{\lambda\to\infty}\frac{\log G\left(\frac{\lambda}{a}\right)}{\log G\left(\frac{\lambda}{b}\right)} = +\infty, \quad \text{for} \quad a>b>0. \\ (3)\lim_{\lambda\to\infty}\frac{\log G\left(\frac{\lambda}{a}\right)}{\log G\left(\frac{\lambda}{b}\right)} = 0, \quad \text{for} \quad b>a>0. \\ (4)\text{For} \quad t>0, f(t) = \frac{t[g'(t)G(t)-g^2(t)]}{g(t)G(t)} \quad \text{is a decreasing function of} \quad t. \end{cases} \tag{3.9}$$

Proof.

$$\begin{aligned} & \lim_{\lambda\to 0}\frac{S_n}{S_i} \\ = & \lim_{\lambda\to 0}\frac{\log G\left(\frac{\lambda}{x_{(1)}}\right)+\log G\left(\frac{\lambda}{x_{(2)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(n)}}\right)}{\log G\left(\frac{\lambda}{x_{(1)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(i-1)}}\right)+(n-i+1)\log G\left(\frac{\lambda}{x_{(i)}}\right)} \\ = & \lim_{\lambda\to 0}\frac{[\log G\left(\frac{\lambda}{x_{(1)}}\right)+\log G\left(\frac{\lambda}{x_{(2)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(n)}}\right)]/\log G\left(\frac{\lambda}{x_{(n)}}\right)}{[\log G\left(\frac{\lambda}{x_{(1)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(i-1)}}\right)+(n-i+1)\log G\left(\frac{\lambda}{x_{(i)}}\right)]/\log G\left(\frac{\lambda}{x_{(n)}}\right)} \\ = & 1. \end{aligned}$$

Thus, $\lim_{\lambda\to 0}\sum_{i=1}^{n-1}\log(S_n/S_i) = 0$. On the other hand,

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{S_n}{S_i}\\
=\quad & 1+\lim_{\lambda\to\infty}\frac{W_{i+1}+\cdots+W_n}{W_1+W_2+\cdots+W_i}\\
=\quad & 1+\lim_{\lambda\to\infty}\frac{\log G\left(\frac{\lambda}{x_{(i+1)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(n)}}\right)-(n-i)\log G\left(\frac{\lambda}{x_{(i)}}\right)}{\log G\left(\frac{\lambda}{x_{(1)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(i)}}\right)+(n-i)\log G\left(\frac{\lambda}{x_{(i)}}\right)}\\
=\quad & 1+\lim_{\lambda\to\infty}\frac{[\log G\left(\frac{\lambda}{x_{(i+1)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(n)}}\right)-(n-i)\log G\left(\frac{\lambda}{x_{(i)}}\right)]/\log G\left(\frac{\lambda}{x_{(i)}}\right)}{[\log G\left(\frac{\lambda}{x_{(1)}}\right)+\cdots+\log G\left(\frac{\lambda}{x_{(i)}}\right)+(n-i)\log G\left(\frac{\lambda}{x_{(i)}}\right)]/\log G\left(\frac{\lambda}{x_{(i)}}\right)}\\
=\quad & +\infty.
\end{aligned}$$

Thus, $\lim_{\lambda\to\infty}\sum_{i=1}^{n-1}\log(S_n/S_i)=\infty$. Therefore, for $t>0$, equation $\sum_{i=1}^{n-1}\log(S_n/S_i)=t$ has one positive solution. For the uniqueness of the solution, we consider the derivative of S_n/S_i with respect to λ.

Noting that, for $i=1,\cdots,n$,

$$\begin{aligned}
W_i &= (n-i+1)\alpha\left\{\log G\left(\frac{\lambda}{x_{(i-1)}}\right)-\log G\left(\frac{\lambda}{x_{(i)}}\right)\right\},\\
\frac{dW_i}{d\lambda} &= (n-i+1)\alpha\left[\frac{g\left(\frac{\lambda}{x_{(i-1)}}\right)}{x_{(i-1)}G\left(\frac{\lambda}{x_{(i-1)}}\right)}-\frac{g\left(\frac{\lambda}{x_{(i)}}\right)}{x_{(i)}G\left(\frac{\lambda}{x_{(i)}}\right)}\right]\\
&= W_i\frac{\frac{g\left(\frac{\lambda}{x_{(i-1)}}\right)}{x_{(i-1)}G\left(\frac{\lambda}{x_{(i-1)}}\right)}-\frac{g\left(\frac{\lambda}{x_{(i)}}\right)}{x_{(i)}G\left(\frac{\lambda}{x_{(i)}}\right)}}{\log G\left(\frac{\lambda}{x_{(i-1)}}\right)-\log G\left(\frac{\lambda}{x_{(i)}}\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\frac{S_n}{S_i}\right)' &= \left(1+\frac{W_{i+1}+\cdots+W_n}{W_1+\cdots+W_i}\right)'\\
&= \frac{1}{(\sum_{k=1}^{i}W_k)^2}\sum_{j=i+1}^{n}\sum_{k=1}^{i}[W_j'W_k-W_jW_k']\\
&= \frac{1}{\lambda(\sum_{k=1}^{i}W_k)^2}\sum_{j=i+1}^{n}\sum_{k=1}^{i}W_jW_k[A(\lambda)-B(\lambda)],
\end{aligned}$$

where

$$A(\lambda)=\frac{\frac{\lambda g\left(\frac{\lambda}{x_{(j-1)}}\right)}{x_{(j-1)}G\left(\frac{\lambda}{x_{(j-1)}}\right)}-\frac{\lambda g\left(\frac{\lambda}{x_{(j)}}\right)}{x_{(j)}G\left(\frac{\lambda}{x_{(j)}}\right)}}{\log G\left(\frac{\lambda}{x_{(j-1)}}\right)-\log G\left(\frac{\lambda}{x_{(j)}}\right)},$$

and

$$B(\lambda)=\frac{\frac{\lambda g\left(\frac{\lambda}{x_{(k-1)}}\right)}{x_{(k-1)}G\left(\frac{\lambda}{x_{(k-1)}}\right)}-\frac{\lambda g\left(\frac{\lambda}{x_{(k)}}\right)}{x_{(k)}G\left(\frac{\lambda}{x_{(k)}}\right)}}{\log G\left(\frac{\lambda}{x_{(k-1)}}\right)-\log G\left(\frac{\lambda}{x_{(k)}}\right)}.$$

By Cauchy's mean-value theorem, for $j = i+1, \cdots, n$, $k = 1, \cdots, i$, there exist $\xi_1 \in (\frac{\lambda}{X_{(j)}}, \frac{\lambda}{X_{(j-1)}})$ and $\xi_2 \in (\frac{\lambda}{X_{(k)}}, \frac{\lambda}{X_{(k-1)}})$ such that

$$\begin{aligned} A(\lambda) &= 1 + \frac{\xi_1[g'(\xi_1)G(\xi_1) - g^2(\xi_1)]}{g(\xi_1)G(\xi_1)}, \\ B(\lambda) &= 1 + \frac{\xi_2[g'(\xi_2)G(\xi_2) - g^2(\xi_2)]}{g(\xi_2)G(\xi_2)}. \end{aligned}$$

Note that $\xi_1 < \xi_2$, therefore, $A(\lambda) - B(\lambda) > 0$, $\left(\frac{S_n}{S_i}\right)' > 0$, thus $\sum_{i=1}^{n-1} \log(S_n/S_i)$ is a strictly increasing function of λ, $\sum_{i=1}^{n-1} \log(S_n/S_i) = t$ has a unique positive solution. □

3.1. Remark. For general scale family of distributions $G(\cdot)$, the conditions (3.9) in Theorem 3.3 are satisfied. For example, as for the Generalized inverted exponential distribution: $GIED(\lambda, \alpha)$ with $G(x) = 1 - e^{-x}$, $\quad x > 0$. It is easy to verify the following conditions:

(1) $\lim_{\lambda \to 0^+} \frac{\log[1-e^{-\frac{\lambda}{a}}]}{\log[1-e^{-\frac{\lambda}{b}}]} = 1$, for $a > 0, b > 0$.

(2) $\lim_{\lambda \to \infty} \frac{\log[1-e^{-\frac{\lambda}{a}}]}{\log[1-e^{-\frac{\lambda}{b}}]} = +\infty$, for $a > b > 0$.

(3) $\lim_{\lambda \to \infty} \frac{\log[1-e^{-\frac{\lambda}{a}}]}{\log[1-e^{-\frac{\lambda}{b}}]} = 0$, for $b > a > 0$.

(4) For $t > 0$, $f(t) = \frac{t[g'(t)G(t) - g^2(t)]}{g(t)G(t)} = \frac{e^t t}{1-e^t}$ is a decreasing function of t.

4. Joint confidence regions for λ and α

Let $X_1, X_2, \cdots, X_n$ form a sample from the GIFD distribution $GIFD(\lambda, \alpha)$, and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics from this sample. Let $Z_{(i)} = -\alpha \log G\left(\frac{\lambda}{x_{(i)}}\right), i = 1, \cdots, n$. Thus, $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ are the first n order statistics from the standard exponential distribution. By Lemma 3.1, $W_i = (n-i+1)(Z_{(i)} - Z_{(i-1)}), i = 1, 2, \cdots, n$ form a sample from standard exponential distribution. Let $S_i = W_1 + \cdots + W_i$, $U_i = (S_i/S_{i+1})^i, i = 1, 2, \cdots, n-1$, $U_n = W_1 + \cdots + W_n$. Hence,

$$V = 2S_1 = 2W_1 = 2nZ_{(1)} = -2n\alpha \log\left(\frac{\lambda}{x_{(1)}}\right) \sim \chi^2(2), \tag{4.1}$$

and

$$U = 2(S_n - S_1) = 2\sum_{i=2}^{n} W_i = 2[Z_{(1)} + \cdots + Z_{(n)} - nZ_{(1)}] \sim \chi^2(2n-2). \tag{4.2}$$

We can find that U and V are independent. Define

$$T_1 = \frac{U/(2n-2)}{V/2} = \frac{S_n - S_1}{(n-1)S_1} \sim F(2n-2, 2), \tag{4.3}$$

and

$$T_2 = U + V = 2S_n \sim \chi^2(2n). \tag{4.4}$$

We obtain that T_1 and T_2 are independent using the known bank-post office story ([15]) in statistics.

Let $F_\gamma(v_1, v_2)$ denote the percentile of F distribution with left-tail probability γ and v_1 and v_2 degrees of freedom. Let $\chi^2_\gamma(v)$ denote the percentile of χ^2 distribution with left-tail probability γ and v degrees of freedom.

By using the pivotal variables T_1 and T_2, a joint confidence region for the two parameters λ and α can be constructed as follows.

4.1. Theorem. (Method 1) Let $X_1, X_2, \cdots, X_n$ form a sample from the GIFD distribution $GIFD(\lambda, \alpha)$, then, based on the pivotal variables T_1 and T_2, a $100(1-\gamma)\%$ joint confidence region for the two parameters λ and α is determined by the following inequalities:

$$\left\{ \begin{array}{c} \lambda_L \leq \lambda \leq \lambda_U \\ \dfrac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log G\left(\frac{\lambda}{x_{(i)}}\right)} \leq \alpha \leq \dfrac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log G\left(\frac{\lambda}{x_{(i)}}\right)}, \end{array} \right. \tag{4.5}$$

where λ_L is the root of λ for the equation $T_1 = F_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2, 2)$ and λ_U is the root of λ for the equation $T_1 = F_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2, 2)$.

Proof. $T_1 = \frac{1}{n-1} \frac{\log G\left(\frac{\lambda}{x_{(1)}}\right) + \cdots \log G\left(\frac{\lambda}{x_{(n)}}\right) - n \log G\left(\frac{\lambda}{x_{(1)}}\right)}{n \log G\left(\frac{\lambda}{x_{(1)}}\right)}$ is a function of λ and does not depend on α. From Theorem 3.3, we have $\lim_{\lambda \to 0} T_1 = \frac{1}{n-1} \lim_{\lambda \to 0}(\frac{S_n}{S_1} - 1) = 0$, $\lim_{\lambda \to \infty} T_1 = \frac{1}{n-1} \lim_{\lambda \to \infty}(\frac{S_n}{S_1} - 1) = \infty$, $T_1' = \frac{1}{n-1}(\frac{S_n}{S_1})' > 0$. Therefore, for any $t > 0$, equation $T_1 = t$ has a unique positive root of λ.

$$\begin{aligned} 1-\gamma &= P(F_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2,2) \leq T_1 \leq F_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2,2)) \\ &\quad \times P(\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n) \leq T_2 \leq \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)) \\ &= P\Big(F_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2,2) \leq T_1 \leq F_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2,2)), \\ &\quad \chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n) \leq T_2 \leq \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)\Big) \\ &= P\left(\lambda_L \leq \lambda \leq \lambda_U, \frac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log G\left(\frac{\lambda}{x_{(i)}}\right)} \leq \alpha \leq \frac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log G\left(\frac{\lambda}{x_{(i)}}\right)}\right). \end{aligned}$$

□

On the other hand, by Lemma 3.2, we have

$$T_3 = -2\sum_{i=1}^{n-1} \log U_i = -2\sum_{i=1}^{n-1} i \log(S_i/S_{i+1}) = 2\sum_{i=1}^{n-1} \log(S_n/S_i) \sim \chi^2(2n-2). \tag{4.6}$$

T_2 and T_3 are also independent. By using the pivotal variables T_2 and T_3, a joint confidence region for the two parameters λ and α can be constructed as follows.

4.2. Theorem. (Method 2) Let $X_1, X_2, \cdots, X_n$ form a sample from the GIFD distribution $GIFD(\lambda, \alpha)$, then, based on the pivotal variables T_2 and T_3, a $100(1-\gamma)\%$ joint confidence region for the two parameters λ and α is determined by

$$\left\{ \begin{array}{c} \lambda_L^* \leq \lambda \leq \lambda_U^* \\ \dfrac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log G\left(\frac{\lambda}{x_{(i)}}\right)} \leq \alpha \leq \dfrac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log G\left(\frac{\lambda}{x_{(i)}}\right)}, \end{array} \right. \tag{4.7}$$

where λ_L^* is the root of λ for the equation $T_3 = \chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2)$ and λ_U^* is the root of λ for the equation $T_3 = \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2)$.

Proof. $T_3 = 2\sum_{i=1}^{n-1}\log(S_n/S_i)$ is a function of λ and does not depend on α. From Theorem 3.3, for any $s > 0$, equation $T_3 = s$ has a unique positive root of λ.

$$\begin{aligned}
1-\gamma &= \sqrt{1-\gamma}\sqrt{1-\gamma} \\
&= P(\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2) \leq T_3 \leq \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2)) \\
&\quad \times P(\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n) \leq T_2 \leq \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)) \\
&= P\Bigg(\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2) \leq T_3 \leq \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2), \\
&\quad \chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n) \leq T_2 \leq \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)\Bigg) \\
&= P\left(\lambda_L^* \leq \lambda \leq \lambda_U^*, \frac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log G\left(\frac{\lambda}{x_{(i)}}\right)} \leq \alpha \leq \frac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log G\left(\frac{\lambda}{x_{(i)}}\right)}\right).
\end{aligned}$$

□

5. Application to generalized inverted half-logistic distribution and simulation study

In this section, we consider a member of the generalized inverted family distributions, namely generalized inverted half-logistic distribution $GIHD(\lambda, \alpha)$. Its cdf and pdf are respectively given by

$$F(x) = 1 - \left[\frac{1-e^{-\frac{\lambda}{x}}}{1+e^{-\frac{\lambda}{x}}}\right]^{\alpha}, \quad x > 0, \tag{5.1}$$

and

$$f(x) = \frac{2\alpha\lambda e^{\lambda/x}}{x^2\left(e^{\frac{2\lambda}{x}}-1\right)}\left(\frac{e^{\lambda/x}-1}{e^{\lambda/x}+1}\right)^{\alpha}, \quad x > 0. \tag{5.2}$$

The log-likelihood function is given by

$$\begin{aligned}
L(\lambda,\alpha) &= \alpha\sum_{i=1}^n \log\left(e^{\frac{\lambda}{x_i}}-1\right) - \alpha\sum_{i=1}^n \log\left(e^{\frac{\lambda}{x_i}}+1\right) + \lambda\sum_{i=1}^n \frac{1}{x_i} - \sum_{i=1}^n \log\left(e^{\frac{2\lambda}{x_i}}-1\right) \\
&\quad -2\sum_{i=1}^n \log x_i + n\log\alpha + n\log\lambda + n\log 2
\end{aligned} \tag{5.3}$$

The score equations are as follows:

$$\frac{\partial L(\lambda,\alpha)}{\partial\lambda} = \alpha\sum_{i=1}^n \frac{2e^{\frac{\lambda}{x_i}}}{x_i\left(e^{\frac{2\lambda}{x_i}}-1\right)} - \sum_{i=1}^n \frac{2e^{\frac{2\lambda}{x_i}}}{x_i\left(e^{\frac{2\lambda}{x_i}}-1\right)} + \sum_{i=1}^n \frac{1}{x_i} + \frac{n}{\lambda} = 0, \tag{5.4}$$

$$\frac{\partial L(\lambda,\alpha)}{\partial\alpha} = \sum_{i=1}^n \log\left(e^{\frac{\lambda}{x_i}}-1\right) - \sum_{i=1}^n \log\left(e^{\frac{\lambda}{x_i}}+1\right) + \frac{n}{\alpha} = 0. \tag{5.5}$$

From (5.5) we obtain the MLE of α as a function of λ,

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^n \log\left(\frac{e^{\frac{\lambda}{x_i}}+1}{e^{\frac{\lambda}{x_i}}-1}\right)}. \tag{5.6}$$

The MLE of λ is the root of the following equation

$$\frac{n}{\sum_{i=1}^{n}\log\left(\frac{e^{\frac{\lambda}{x_i}}+1}{e^{\frac{\lambda}{x_i}}-1}\right)}\sum_{i=1}^{n}\frac{2e^{\frac{\lambda}{x_i}}}{x_i\left(e^{\frac{2\lambda}{x_i}}-1\right)}-\sum_{i=1}^{n}\frac{2e^{\frac{2\lambda}{x_i}}}{x_i\left(e^{\frac{2\lambda}{x_i}}-1\right)}+\sum_{i=1}^{n}\frac{1}{x_i}+\frac{n}{\lambda}=0.$$

(5.7)

The IME of α is obtained as a function of λ,

$$\hat{\alpha}=\frac{n}{\sum_{i=1}^{n}\log\left(\frac{e^{\frac{\lambda}{x_i}}+1}{e^{\frac{\lambda}{x_i}}-1}\right)}. \tag{5.8}$$

which is identical to the MLE of α. The inverse estimate $\hat{\lambda}_{IME}$ of λ is the root of the following equation:

$$\sum_{i=1}^{n-1}\log[\frac{\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(2)}}}}{1+e^{-\frac{\lambda}{x_{(2)}}}}\right)+\cdots+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(n)}}}}{1+e^{-\frac{\lambda}{x_{(n)}}}}\right)}{\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)+\cdots+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(i-1)}}}}{1+e^{-\frac{\lambda}{x_{(i-1)}}}}\right)+(n-i+1)\log\left(\frac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)}]$$
$$=n-1.$$

(5.9)

The modified inverse estimate $\hat{\lambda}_{MIME}$ of λ is the root of the following equation:

$$\sum_{i=1}^{n-1}\log[\frac{\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(2)}}}}{1+e^{-\frac{\lambda}{x_{(2)}}}}\right)+\cdots+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(n)}}}}{1+e^{-\frac{\lambda}{x_{(n)}}}}\right)}{\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)+\cdots+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(i-1)}}}}{1+e^{-\frac{\lambda}{x_{(i-1)}}}}\right)+(n-i+1)\log\left(\frac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)}]$$
$$=n-2.$$

(5.10)

For $G(x)=\frac{1-e^{-x}}{1+e^{-x}}, \quad x>0$, it is easy to verify that the conditions (3.9) are satisfied. The proposed estimators exist and are unique.

Based on method 1, the $100(1-\gamma)\%$ joint confidence region for the parameters (λ,α) is determined by the following inequalities:

$$\begin{cases} \lambda_L\le\lambda\le\lambda_U \\ \dfrac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^{n}\log\left(\frac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)}\le\alpha\le\dfrac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^{n}\log\left(\frac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)}, \end{cases} \tag{5.11}$$

where λ_L is the root of λ for the equation $T_1=F_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2,2)$ and λ_U is the root of λ for the equation $T_1=F_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2,2)$. Here

$$T_1=\frac{1}{n-1}\frac{\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)+\cdots\log\left(\frac{1-e^{-\frac{\lambda}{x_{(n)}}}}{1+e^{-\frac{\lambda}{x_{(n)}}}}\right)-n\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)}{n\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)}.$$

Based on method 2, the $100(1-\gamma)\%$ joint confidence region for the parameters (λ, α) is determined by the following inequalities:

$$(5.12)\qquad \begin{cases} \lambda_L^* \le \lambda \le \lambda_U^* \\ \dfrac{\chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left(\frac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)} \le \alpha \le \dfrac{\chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n)}{-2\sum_{i=1}^n \log\left(\frac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)}. \end{cases}$$

where λ_L^* is the root of λ for the equation $T_3 = \chi^2_{\frac{1-\sqrt{1-\gamma}}{2}}(2n-2)$ and λ_U^* is the root of λ for the equation $T_3 = \chi^2_{\frac{1+\sqrt{1-\gamma}}{2}}(2n-2)$. Here

$$T_3 = 2\sum_{i=1}^{n-1} \log\left[\frac{\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(2)}}}}{1+e^{-\frac{\lambda}{x_{(2)}}}}\right)+\cdots+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(n)}}}}{1+e^{-\frac{\lambda}{x_{(n)}}}}\right)}{\log\left(\frac{1-e^{-\frac{\lambda}{x_{(1)}}}}{1+e^{-\frac{\lambda}{x_{(1)}}}}\right)+\cdots+\log\left(\frac{1-e^{-\frac{\lambda}{x_{(i-1)}}}}{1+e^{-\frac{\lambda}{x_{(i-1)}}}}\right)+(n-i+1)\log\left(\frac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)}\right].$$

5.1. Comparison of the three estimation methods. In this section, we conduct simulations to compare the performances of the MLEs, IMEs and MIMEs mainly with respect to their biases and mean squared errors (MSE's), for various sample sizes and true parametric values. R source code for the simulations is available upon request.

The random data X from the $GIHD(\lambda, \alpha)$ distribution can be generated as follows: $X = -\lambda/\log\left(\frac{1-U^{1/\alpha}}{1+U^{1/\alpha}}\right)$, where U follows uniform distribution over $[0,1]$. We obtain $\hat{\lambda}_{MLE}$ by solving equation (5.7) and $\hat{\alpha}_{MLE}$ by (5.6). The $\hat{\lambda}_{IME}$ and $\hat{\lambda}_{MIME}$ can be obtained by solving (5.9) and (5.10) respectively. The $\hat{\alpha}_{IME}$ and $\hat{\alpha}_{MIME}$ can be obtained from (5.8).

We consider sample sizes $n = 30, 40, 50, 60, 80, 100$ and $\alpha = 2.0, 2.5, 3.0, 3.5, 4.0$. We take the scale parameter $\lambda = 1$ in all our computations without loss of generality. For each combination of sample size n and parameter α, we generate a sample of size n from $GIHD(\lambda = 1, \alpha)$, and estimate the parameters λ and α by the MLE, IME, MIME methods. The average values of $\hat{\alpha}/\alpha$ and $\hat{\lambda}/1 = \hat{\lambda}$ as well as the corresponding MSEs over 1000 replications are computed and reported.

For different cases, Table 1 reports the average values of $\hat{\alpha}/\alpha$ and the corresponding MSE is reported within parenthesis. Figure 1a, 1b 1c and 1d show the relative biases and the MSEs of the three estimators of α for sample sizes $n = 40$ and $n = 80$. Figure 1e and 1f show the relative biases and the MSEs of the three estimators of α for $\alpha = 3.0$. The other cases are similar.

For different cases, Table 2 reports the average values of $\hat{\lambda}/\lambda = \hat{\lambda}$ and the corresponding MSE is reported within parenthesis. Figure 2a, 2b 2c and 2d show the relative biases and the MSEs of the three estimators of λ for sample sizes $n = 40$ and $n = 80$. Figure 2e and 2f show the relative biases and the MSEs of the three estimators of λ for $\alpha = 3.0$. The other cases are similar.

From Table 1 and 2, we find that

- The average biases and relative MSEs of $\hat{\alpha}/\alpha$ increase as α goes up. The average biases and relative MSEs of $\hat{\lambda}$ decrease as α goes up.
- Considering only MSE's, the estimation of α's are more accurate for smaller values.
- The average relative biases and MSEs for the three methods decrease as sample size n increases as expected. The asymptotic unbiasedness of all the estimators are verified.

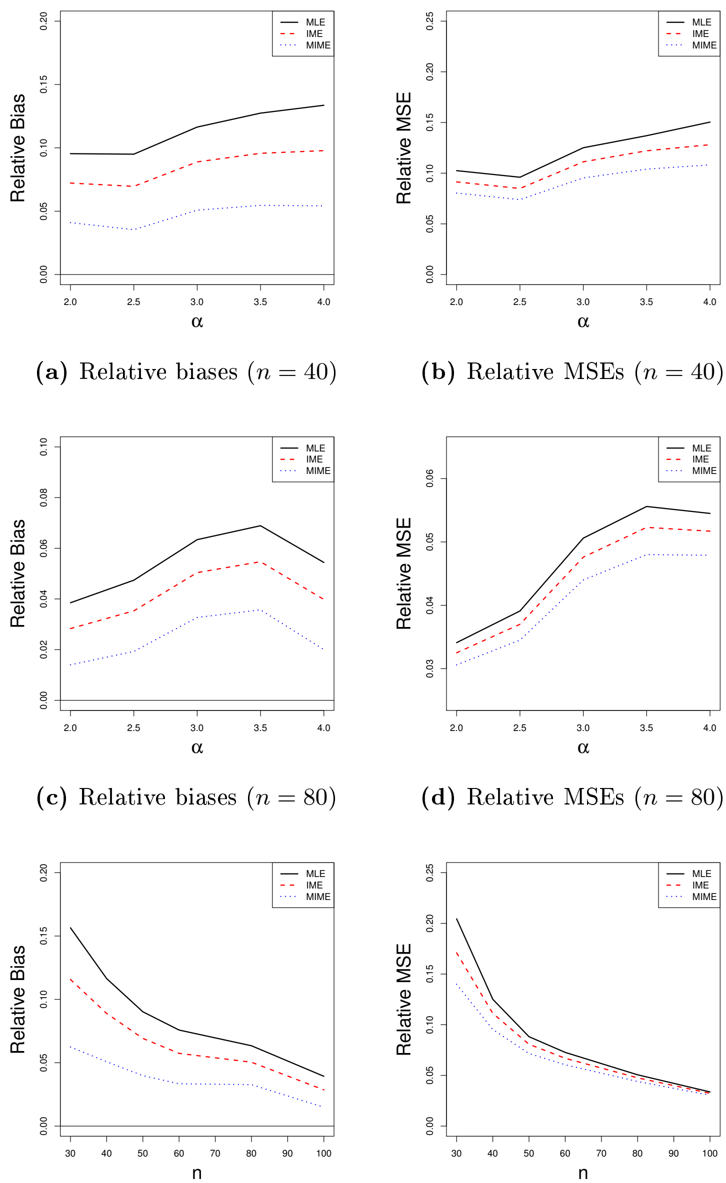
Table 1. Average relative estimates and MSEs of α

n	Methods	$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
	MLE	1.1388 (0.1374)	1.1386 (0.1750)	1.1565 (0.2045)	1.1689 (0.2303)	1.1897 (0.2769)
30	IME	1.1066 (0.1182)	1.1027 (0.1506)	1.1157 (0.1710)	1.1236 (0.1930)	1.1405 (0.2288)
	MIME	1.0626 (0.0985)	1.0542 (0.1250)	1.0624 (0.1399)	1.0661 (0.1562)	1.0783 (0.1830)
	MLE	1.0954 (0.1025)	1.0950 (0.0960)	1.1164 (0.1251)	1.1274 (0.1370)	1.1336 (0.1504)
40	IME	1.0722 (0.0914)	1.0696 (0.0849)	1.0889 (0.1112)	1.0957 (0.1221)	1.0978 (0.1281)
	MIME	1.0410 (0.0804)	1.0354 (0.0739)	1.0508 (0.0953)	1.0546 (0.1040)	1.0542 (0.1082)
	MLE	1.0640 (0.0606)	1.0899 (0.0933)	1.0902 (0.0882)	1.0955 (0.1050)	1.1065 (0.1295)
50	IME	1.0466 (0.0555)	1.0696 (0.0847)	1.0692 (0.0807)	1.0708 (0.0955)	1.0798 (0.1173)
	MIME	1.0228 (0.0503)	1.0423 (0.0756)	1.0398 (0.0716)	1.0392 (0.0844)	1.0459 (0.1029)
	MLE	1.0661 (0.0520)	1.0728 (0.0599)	1.0758 (0.0727)	1.0897 (0.0871)	1.0624 (0.0672)
60	IME	1.0516 (0.0480)	1.0553 (0.0549)	1.0574 (0.0669)	1.0674 (0.0783)	1.0427 (0.0623)
	MIME	1.0317 (0.0438)	1.0332 (0.0498)	1.0334 (0.0605)	1.0413 (0.0703)	1.0161 (0.0564)
	MLE	1.0385 (0.0341)	1.0474 (0.0391)	1.0634 (0.0506)	1.0689 (0.0556)	1.0544 (0.0545)
80	IME	1.0283 (0.0325)	1.0353 (0.0370)	1.0504 (0.0476)	1.0547 (0.0523)	1.0398 (0.0517)
	MIME	1.0140 (0.0306)	1.0193 (0.0345)	1.0327 (0.0440)	1.0357 (0.0480)	1.0200 (0.0479)
	MLE	1.0349 (0.0287)	1.035 (0.0298)	1.0393 (0.0337)	1.0445 (0.0376)	1.0457 (0.0421)
100	IME	1.0268 (0.0273)	1.0262 (0.0287)	1.0286 (0.0323)	1.0335 (0.0360)	1.0333 (0.0400)
	MIME	1.0155 (0.0260)	1.0136 (0.0273)	1.0149 (0.0305)	1.0187 (0.0338)	1.0177 (0.0376)

Table 2. Average relative estimates and MSEs of λ

n	Methods	$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
	MLE	1.0529 (0.0362)	1.0394 (0.0299)	1.0463 (0.0331)	1.0490 (0.0300)	1.0484 (0.0317)
30	IME	1.0303 (0.0330)	1.0170 (0.0273)	1.0234 (0.0299)	1.0261 (0.0269)	1.0271 (0.0293)
	MIME	0.9981 (0.0305)	0.9861 (0.0259)	0.9929 (0.0280)	0.9959 (0.0249)	0.9973 (0.0272)
	MLE	1.0403 (0.0224)	1.0367 (0.0235)	1.0391 (0.0232)	1.0378 (0.0223)	1.0234 (0.0205)
40	IME	1.0237 (0.0205)	1.0205 (0.0221)	1.0226 (0.0218)	1.0208 (0.0203)	1.0075 (0.0195)
	MIME	0.9999 (0.0192)	0.9975 (0.0209)	1.0000 (0.0205)	0.9985 (0.0192)	0.9857 (0.0189)
	MLE	1.0291 (0.0210)	1.0362 (0.0185)	1.0305 (0.0180)	1.0307 (0.0165)	1.0293 (0.0184)
50	IME	1.0158 (0.0198)	1.0232 (0.0174)	1.0170 (0.0171)	1.0180 (0.0156)	1.0156 (0.0172)
	MIME	0.9969 (0.0190)	1.0048 (0.0164)	0.9991 (0.0164)	1.0004 (0.0149)	0.9981 (0.0165)
	MLE	1.0255 (0.0162)	1.0326 (0.0167)	1.0227 (0.0131)	1.0211 (0.0133)	1.0268 (0.0137)
60	IME	1.0150 (0.0156)	1.0220 (0.0158)	1.0119 (0.0126)	1.0096 (0.0127)	1.0156 (0.0131)
	MIME	0.9994 (0.0150)	1.0068 (0.0150)	0.9971 (0.0121)	0.9950 (0.0123)	1.0012 (0.0125)
	MLE	1.0152 (0.0122)	1.0122 (0.0105)	1.0192 (0.0103)	1.0179 (0.0102)	1.0110 (0.0089)
80	IME	1.0067 (0.0118)	1.0034 (0.0100)	1.0109 (0.0099)	1.0091 (0.0098)	1.0034 (0.0087)
	MIME	0.9951 (0.0116)	0.9922 (0.0099)	0.9999 (0.0096)	0.9982 (0.0095)	0.9928 (0.0086)
	MLE	1.0150 (0.0092)	1.0128 (0.0083)	1.0126 (0.0070)	1.0190 (0.008)	1.0102 (0.0075)
100	IME	1.0087 (0.0090)	1.0064 (0.0081)	1.0062 (0.0068)	1.0127 (0.0078)	1.0034 (0.0074)
	MIME	0.9995 (0.0087)	0.9975 (0.0079)	0.9974 (0.0067)	1.0040 (0.0075)	0.9948 (0.0073)

- MLE and IME overestimate both of the two parameters α and λ. MIME overestimates only α.



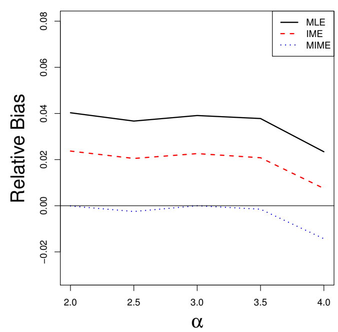
(a) Relative biases ($n = 40$) **(b)** Relative MSEs ($n = 40$)

(c) Relative biases ($n = 80$) **(d)** Relative MSEs ($n = 80$)

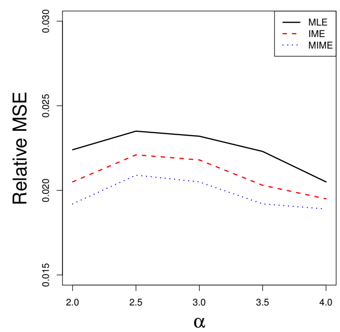
(e) Relative biases ($\alpha = 3.0$) **(f)** Relative MSEs ($\alpha = 3.0$)

Figure 1. Average relative biases and MSEs of α.

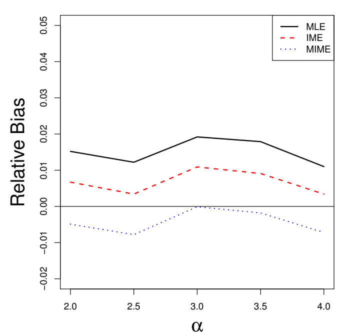
As far as the biases and MSEs are concerned, MIME works the best in all the cases considered for estimating the two parameters. Its performance is followed by IME and



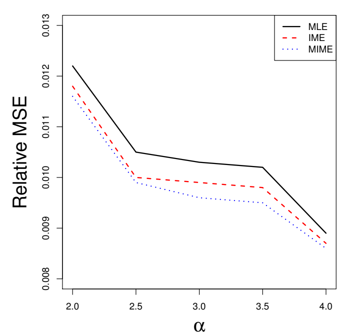
(a) Relative biases ($n = 40$)



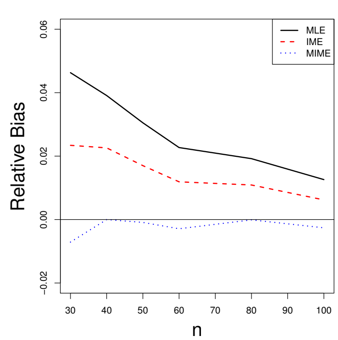
(b) Relative MSEs ($n = 40$)



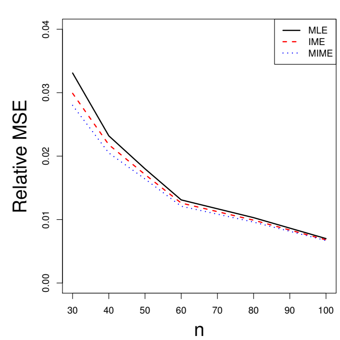
(c) Relative biases ($n = 80$)



(d) Relative MSEs ($n = 80$)



(e) Relative biases ($\alpha = 3.0$)



(f) Relative MSEs ($\alpha = 3.0$)

Figure 2. Average relative biases and MSEs of λ.

MLE, especially for small sample sizes. The three methods are close for larger sample sizes. Considering all the points, MIME is recommended for estimating both the parameters of the $GIHD(\lambda, \alpha)$ distribution.

Table 3. Average relative estimates and MSEs of α and λ when $\alpha = 2.0$

n	Methods	$\lambda = 1.0$	$\lambda = 1.2$	$\lambda = 1.5$	$\lambda = 1.8$	$\lambda = 2.0$
				Average relative estimates and MSEs of α		
	MLE	1.0664 (0.0541)	1.0644 (0.0486)	1.0597 (0.0482)	1.055 (0.0519)	1.0639 (0.0508)
60	IME	1.0526 (0.0505)	1.0503 (0.0452)	1.0454 (0.0446)	1.041 (0.0487)	1.0495 (0.0473)
	MIME	1.0327 (0.0462)	1.0305 (0.0412)	1.0257 (0.0409)	1.0214 (0.0449)	1.0296 (0.0433)
	MLE	1.0511 (0.0384)	1.0446 (0.0366)	1.0437 (0.0311)	1.042 (0.0362)	1.0431 (0.0351)
80	IME	1.0404 (0.0363)	1.0344 (0.0347)	1.0333 (0.0295)	1.0316 (0.0344)	1.0326 (0.0335)
	MIME	1.0258 (0.0339)	1.0200 (0.0325)	1.0189 (0.0276)	1.0172 (0.0324)	1.0181 (0.0314)
	MLE	1.0379 (0.0280)	1.0355 (0.0281)	1.0251 (0.0256)	1.0373 (0.0279)	1.0247 (0.0263)
100	IME	1.0296 (0.0267)	1.0270 (0.0267)	1.0175 (0.0247)	1.0292 (0.0268)	1.0165 (0.0251)
	MIME	1.0182 (0.0253)	1.0156 (0.0253)	1.0062 (0.0237)	1.0178 (0.0255)	1.0053 (0.0241)
				Average relative estimates and MSEs of λ		
	MLE	1.0197 (0.0125)	1.029 (0.0137)	1.0203 (0.0132)	1.0178 (0.0122)	1.019 (0.0131)
60	IME	1.0106 (0.012)	1.0201 (0.0131)	1.0109 (0.0127)	1.0089 (0.0119)	1.0100 (0.0125)
	MIME	0.9973 (0.0117)	1.0067 (0.0125)	0.9975 (0.0124)	0.9956 (0.0116)	0.9966 (0.0122)
	MLE	1.0202 (0.0114)	1.0272 (0.0129)	1.0198 (0.0125)	1.0142 (0.0117)	1.0175 (0.0101)
80	IME	1.0119 (0.0109)	1.0191 (0.0123)	1.0118 (0.0121)	1.0063 (0.0114)	1.0095 (0.0098)
	MIME	1.0003 (0.0106)	1.0074 (0.0117)	1.0001 (0.0118)	0.9947 (0.0112)	0.9978 (0.0095)
	MLE	1.0154 (0.0087)	1.0182 (0.0088)	1.011 (0.0086)	1.018 (0.0083)	1.0079 (0.0079)
100	IME	1.0089 (0.0085)	1.0115 (0.0084)	1.0046 (0.0084)	1.0115 (0.008)	1.0015 (0.0078)
	MIME	0.9996 (0.0083)	1.0022 (0.0082)	0.9953 (0.0083)	1.0022 (0.0078)	0.9923 (0.0078)

In addition, we fix the shape parameter $\alpha = 2.0$. We consider sample sizes $n = 60, 80, 100$ and $\lambda = 1.0, 1.2, 1.5, 1.8, 2.0$. For each combination of sample size n and parameter λ, we generate a sample of size n from $GIHD(\lambda, \alpha = 2.0)$, and estimate the parameters λ and α by the MLE, IME, MIME methods. The average values of $\hat{\alpha}/2.0$ and $\hat{\lambda}/\lambda = \hat{\lambda}$ as well as the corresponding MSEs over 1000 replications are computed and reported.

For different cases, Table 3 reports the average values and the corresponding MSE is reported within parenthesis. We find that

- The average biases and relative MSEs of $\hat{\alpha}/\alpha$ and $\hat{\lambda}$ remain unchanged as λ goes up.
- The average relative biases and MSEs for the three methods decrease as sample size n increases as expected. The asymptotic unbiasedness of all the estimators are verified.
- MLE and IME overestimate both of the two parameters α and λ. MIME overestimates only α.

5.2. Comparison of the two joint confidence regions. In this section, we conduct simulations to compare the two methods to construct the confidence joint regions of the two parameters λ and α.

First, we assess the precisions of the two methods of interval estimators for the parameter λ. We take sample sizes $n = 30, 40, 50, 60, 80, 100$ and $\alpha = 2.0, 2.5, 3.0, 3.5, 4.0$. We take $\lambda = 1$ in all our computations. For each combination of sample size n and parameter α, we generate a sample of size n from $GIHD(\lambda = 1, \alpha)$, and estimate the parameters λ by the two proposed methods (5.11) and (5.12).

Table 4. Results of the methods for constructing intervals for λ with confidence level 0.95

n	Methods		$\alpha=2.0$	$\alpha=2.5$	$\alpha=3.0$	$\alpha=3.5$	$\alpha=4.0$
30	(1)	Mean width	1.244	1.2272	1.2189	1.2051	1.1919
		Coverage rate	0.963	0.955	0.961	0.954	0.96
	(2)	Mean width	0.6692	0.6511	0.6377	0.6295	0.6242
		Coverage rate	0.966	0.956	0.95	0.941	0.954
40	(1)	Mean width	1.154	1.1494	1.1482	1.119	1.1268
		Coverage rate	0.949	0.96	0.941	0.949	0.954
	(2)	Mean width	0.5782	0.5616	0.5478	0.5397	0.5394
		Coverage rate	0.951	0.949	0.958	0.948	0.948
50	(1)	Mean width	1.1029	1.1074	1.0676	1.0795	1.0682
		Coverage rate	0.947	0.94	0.958	0.942	0.946
	(2)	Mean width	0.517	0.5024	0.4897	0.4847	0.4739
		Coverage rate	0.95	0.949	0.952	0.946	0.947
60	(1)	Mean width	1.0462	1.0417	1.0325	1.0329	1.032
		Coverage rate	0.951	0.954	0.954	0.947	0.955
	(2)	Mean width	0.4678	0.4545	0.444	0.4399	0.4336
		Coverage rate	0.951	0.956	0.946	0.944	0.952
80	(1)	Mean width	0.9919	0.9759	0.9663	0.9692	0.9582
		Coverage rate	0.943	0.947	0.965	0.96	0.954
	(2)	Mean width	0.4047	0.3901	0.3828	0.3774	0.3733
		Coverage rate	0.948	0.944	0.959	0.948	0.952
100	(1)	Mean width	0.954	0.9399	0.9381	0.9305	0.929
		Coverage rate	0.95	0.951	0.943	0.943	0.948
	(2)	Mean width	0.3603	0.3492	0.3427	0.336	0.3338
		Coverage rate	0.958	0.956	0.941	0.939	0.946

The mean widths as well as the coverage rates over 1000 replications are computed and reported. Here the coverage rate is defined as the rate of the confidence intervals that contain the true value $\lambda = 1$ among these 1,000 confidence intervals. The results are reported in Table 4.

It is observed that:

- The mean widths of the intervals decrease as sample sizes n increase as expected.
- The mean widths of the intervals decrease as the parameter α increases.
- The coverage rates of the two methods are close to the nominal level 0.95.

Considering the mean widths, the interval estimate of λ obtained in method 2 performs better than that obtained in method 1. Method 2 for constructing the interval estimate of λ is recommended.

Second, we consider the two joint confidence regions and the empirical coverage rates and expected areas. The results of the methods for constructing joint confidence regions for (λ, α) with confidence level $\gamma = 0.95$ are reported in Table 5. It shows that

Table 5. Results of the methods for constructing joint confidence regions for (λ, α) with confidence level $\gamma = 0.95$.

n	Methods		$\alpha = 2.0$	$\alpha = 2.5$	$\alpha = 3.0$	$\alpha = 3.5$	$\alpha = 4.0$
30	(1)	Mean area	4.4386	6.1655	8.1152	10.4965	13.6872
		Coverage rate	0.94	0.947	0.953	0.961	0.948
	(2)	Mean area	1.5284	1.8815	2.2449	2.6492	3.1533
		Coverage rate	0.948	0.932	0.947	0.943	0.95
40	(1)	Mean area	3.2269	4.4034	5.6075	7.1126	8.7009
		Coverage rate	0.942	0.948	0.945	0.956	0.95
	(2)	Mean area	1.0639	1.3254	1.5224	1.8016	2.0885
		Coverage rate	0.945	0.945	0.945	0.949	0.95
50	(1)	Mean area	2.5839	3.4272	4.482	5.4883	6.8003
		Coverage rate	0.957	0.954	0.954	0.954	0.962
	(2)	Mean area	0.8351	0.995	1.2057	1.3713	1.6237
		Coverage rate	0.944	0.961	0.948	0.95	0.958
60	(1)	Mean area	2.1611	2.8147	3.6691	4.6292	5.7979
		Coverage rate	0.948	0.955	0.937	0.95	0.952
	(2)	Mean area	0.6661	0.7938	0.9727	1.1323	1.3009
		Coverage rate	0.95	0.949	0.927	0.96	0.958
80	(1)	Mean area	1.6647	2.2265	2.9026	3.5014	4.3953
		Coverage rate	0.952	0.956	0.948	0.954	0.946
	(2)	Mean area	0.487	0.5913	0.7123	0.8195	0.9451
		Coverage rate	0.956	0.948	0.944	0.951	0.953
100	(1)	Mean area	1.3895	1.8321	2.3926	2.9028	3.5519
		Coverage rate	0.955	0.947	0.949	0.951	0.938
	(2)	Mean area	0.3867	0.4662	0.5571	0.6382	0.7336
		Coverage rate	0.947	0.939	0.954	0.956	0.941

- The mean areas of the joint regions decrease as sample sizes n increase as expected.
- The mean areas of the joint regions increase as the parameter α increases.
- The coverage rates of the two methods are close to the nominal level 0.95.

Considering the mean areas, the joint region of (λ, α) obtained in method 2 performs better than that obtained in method 1. Method 2 is recommended.

6. Real illustrative example

In this section, We consider a real dataset. This dataset from [9] contains 30 successive values for precipitation (in inches) in March for the Minneapolis/St. Paul area over a 30-year period. The observed values are as follows:

0.77 , 1.74 , 0.81 , 1.20 , 1.95 , 1.20 , 0.47 , 1.43 , 3.37 , 2.20 , 3.00 , 3.09 , 1.51 , 2.10 , 0.52 , 1.62 , 1.31 , 0.32 , 0.59 , 0.81 , 2.81 , 1.87 , 1.18 , 1.35 , 4.75 , 2.48 , 0.96 , 1.89 , 0.90 , 2.05.

This dataset has been previously analyzed by [16] etc. Here we fit the dataset with generalized inverted half-logistic distribution. The MLEs of the parameters are $\hat{\lambda}_{MLE} = 2.4410$ and $\hat{\alpha}_{MLE} = 2.2225$ with log-likelihood value -40.5046. The Kolmogorov-Smirnov distance and its corresponding $p-$value are $D = 0.1333$ and $p = 0.9525$, respectively. GIHD fits the data well.

The inverse moment and modified inverse moment estimates are given as follows:

$$\hat{\lambda}_{IME} = 2.3344, \hat{\alpha}_{IME} = 2.0983, \hat{\lambda}_{MIME} = 2.2577, \hat{\alpha}_{MIME} = 2.0120.$$

Based on method 2, the 95% joint confidence region for the parameters (λ, α) is determined by the following inequalities:

$$\begin{cases} 1.4791 \leq \lambda \leq 3.3142 \\ \dfrac{-19.1026}{\sum_{i=1}^{30} \log\left(\dfrac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)} \leq \alpha \leq \dfrac{-43.5574}{\sum_{i=1}^{30} \log\left(\dfrac{1-e^{-\frac{\lambda}{x_{(i)}}}}{1+e^{-\frac{\lambda}{x_{(i)}}}}\right)} \end{cases}.$$

Figure 3 show the 95% joint confidence regions of (λ, α) based on method 2.

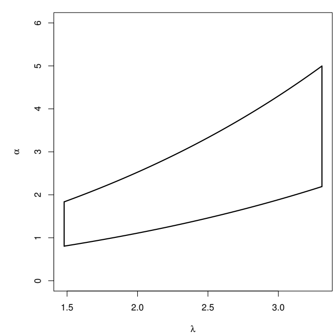


Figure 3. The 95% joint confidence region of (λ, α).

Acknowledgements The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper. Wenhao Gui's work was partially supported by the program for the Fundamental Research Funds for the Central Universities (NO: 2014RC042; 2015JBM109).

References

[1] Abouammoh, A. and Alshingiti, A. M. *Reliability estimation of generalized inverted exponential distribution*, Journal of Statistical Computation and Simulation **79**(11), 1301–1315, 2009.

[2] Arnold, B., Balakrishnan, N. and Nagaraja, H. *A First Course in Order Statistics*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), 1992.

[3] Balakrishnan, N. *Order statistics from the half logistic distribution*, Journal of Statistical Computation & Simulation **20**(4), 287-309, 1985.

[4] Chen, D. G. and Lio, Y. L. *Parameter estimations for generalized exponential distribution under progressive type I interval censoring*, Computational Statistics & Data Analysis **54**(6), 1581-1591, 2010.

[5] Gupta, R. C., Gupta, P. L. and Gupta, R. D. *Modeling failure time data by Lehmann alternatives*, Communication in Statistics-Theory and Methods **27**(4), 887-904, 1998.

[6] Gupta, R. D. and Kundu, D. *Generalized exponential distributions*, Australian & New Zealand Journal of Statistics **41**(2), 173-188, 1999.

[7] Gupta, R. D. and Kundu, D. *Generalized exponential distribution: different method of estimations*, Journal of Statistical Computation & Simulation**69**(4), 315-337, 2001.

[8] Gupta, R. D. and Kundu, D. *Generalized exponential distribution: existing results and some recent developments*, Journal of Statistical Planning & Inference **137**(11), 3537-3547, 2007.

[9] Hinkley, D. *On quick choice of power transformation*', Journal of the Royal Statistical Society. Series C **26** (1), 67-69, 1977.

[10] Krishna, H. and Kumar, K. *Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample*, Journal of Statistical Computation and Simulation **83**(6), 1007-1019, 2013.

[11] Kundu, D. and Pradhan, B. *Estimating the parameters of the generalized exponential distribution in presence of hybrid censoring*, Communications in Statistics: Theory and Methods **38**(12), 2030-2041, 2009.

[12] Nadarajah, S. and Kotz, S. *The exponentiated type distributions*, Acta Applicandae Mathematicae **92**(2), 97-111, 2006.

[13] Potdar, K. G. and Shirke, D. T. *Inference for the parameters of generalized inverted family of distributions*, Probstat Forum **6**, 18-28, 2013.

[14] Raqab, M. M. and Ahsanullah, M. *Estimation of the location and scale parameters of generalized exponential distribution based on order statistics*, Journal of Statistical Computation & Simulation **69**(2), 109-123, 2001.

[15] Ross, S. M. *Introduction to probability models(eleventh edition)*, Academic press, 2014.

[16] Seo, J.I. and Kang, S.B. *Notes on the exponentiated half logistic distribution*, Applied Mathematical Modelling **39**(21), 6491-6500, 2015.

[17] Wang, B. X. *Statistical inference for Weibull distribution*, Chinese Journal of Applied Probability & Statistics **8**(4), 357-364, 1992.