Konuralp Journal of Mathematics
Volume 3 No. 1 pp. 12–15 (2015) ©KJM

ON SOME CLASSICAL THEOREMS IN INTUITIONISTIC FUZZY PROJECTIVE PLANE

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Abstract. In this work, we introduce that intuitionistic fuzzy versions of some classical configurations in projective plane are valid in intuitionistic fuzzy projective plane with base Desarguesian or Pappian plane.

1. Introduction

After the introduction of Fuzzy set theory by Zadeh [12] several researches were conducted on generalizations of this theory.

A model of fuzzy projective geometries was introduced by Kuijken, Van Maldeghem and Kerre [10]. This provided a link between the fuzzy versions of classical theories that are very closely related. Also, Kuijken and Van Maldeghem contributed to fuzzy theory by introducing fibered geometries, which is a particular kind of fuzzy geometries [9]. They gave the fibered versions of some classical results in projective planes by using minimum operator. Then the role of the triangular norm in the theory of fibered projective planes and fibered harmonic conjugates and a fibered version of Reidemeister's condition were given in [3] and the fibered version of Menelaus and Ceva's 6-figures was studied in [4]. In these papers, the points and lines of the base geometry mostly have multiple degrees of membership.

Intuitionistic fuzzy set (IFS) was first published by Atanassov [2] and some authors appeared in literature [5], [11]. A model of intuitionistic fuzzy projective geometry and the link between fibered and intuitionistic fuzzy projective geometry were given by Ghassan E. Arif [7].

In the present paper, the intuitionistic fuzzy versions of some classical theorems in projective planes were given.

2. Preliminaries

We first recall the basic notions from the theory of intuitionistic fuzzy geometries and fibered projective geometry. We assume that the reader is familiar with the

2000 *Mathematics Subject Classification.* 20N25.

Key words and phrases. Projective plane, intuitionistic fuzzy projective plane, Desargues theorem, Pappus theorem.

basic notions of fuzzy mathematics, although this is not strictly necessary as the paper is self-contained in this respect.

We denote by $\wedge$ and $\vee$, minimum and maximum operators respectively.

Let $\mathcal{P} = (P, B, \sim)$ be any projective plane with point set P and line set B, i.e., P and B are two disjoint sets endowed with a symmetric relation $\sim$ (called the *incidence relation*) such that the graph $(P \cup B, \sim)$ is a bipartite graph with classes P and B, and such that two distinct points p, q in $\mathcal{P}$ are incident with exactly one line (denoted by $\langle pq \rangle$), every two distinct lines L, M are incident with exactly one point (denoted by $L \cap M$), and every line is incident with at least three points. A set S of *collinear* points is a subset of P each member of which is incident with a common line L. Dually, one defines a set of *concurrent* lines [8].

Definition 2.1. *(see [2])* Let X be a nonempty fixed set. An intuitionistic fuzzy set A on X is an object having the form

$$A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$$

where the function $\lambda : X \to I$ and $\mu : X \to I$ denote the degree of membership (namely, $\lambda(x)$) and the degree of nonmembership (namely, $\mu(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \lambda(x) + \mu(x) \leq 1$ for each $x \in X$. An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ can be written in the $A = \{\langle x, \lambda, \mu\rangle : x \in X\}$, or simply $A = \langle \lambda, \mu \rangle$.

Let $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ and $B = \{\langle x, \delta(x), \gamma(x)\rangle : x \in X\}$ be an intuitionistic fuzzy sets on X . Then,

(a) $\overline{A} = \{\langle x, \mu(x), \lambda(x)\rangle : x \in X\}$ (the complement of A).

(b) $A \cap B = \{\langle x, \lambda(x) \wedge \delta(x),\ \mu(x) \vee \gamma(x)\rangle : x \in X\}$ (the meet of A and B).

(c) $A \cup B = \{\langle x, \lambda(x) \vee \delta(x),\ \mu(x) \wedge \gamma(x)\rangle : x \in X\}$ (the join of A and B).

(d) $A \subseteq B \Leftrightarrow \lambda(x) \leq \delta(x)$ and $\mu(x) \geq \gamma(x)$ for each $x \in X$.

(e) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

(f) $\widetilde{1} = \{\langle x, 1, 0\rangle : x \in X\}$, $\widetilde{0} = \{\langle x, 0, 1\rangle : x \in X\}$.

Definition 2.2. (see [9]) A fibered projective plane FP on the projective plane P consist of a set FP of f- points and a set FB of f-lines , such that every point and line of P is base point and base line of at least one f-point and f-line respectively , (with at least one membership degree different from 1), and such that $\mathcal{FP} = (FP, FB)$ is closed under taking intersections of f-lines and spans of f-points. Finally, a set of f-points are called collinear if each pair of them span the same f-line. Dually, a set of f-lines are called concurrent if each pair of them intersect in the same f-point.

Definition 2.3. (see [9]) Let $\mathcal{P}$ be a projective plane, $a \in P$ and $\alpha \in]0, 1]$. Then an f-point (a, α) is the following fuzzy set on the point set P of $\mathcal{P}$:

$$(a, \alpha) : P \to [0, 1] : \begin{cases} a \to \alpha & \\ x \to 0 & \text{if } x \in P \backslash \{a\}. \end{cases}$$

Dually, one defines in the same way the f-line (L, β) for $L \in B$ and $\beta \in]0, 1]$.

The real number α above is called the membership degree of the f-point (a, α), while the point a is called the base point of it. Similarly for f-lines.

Two f-lines (L, α) and (M, β), with $\alpha \wedge \beta > 0$, intersect in the unique f-point $(L \cap M, \alpha \wedge \beta)$. Dually, the f-points (a, λ) and (b, μ), with $\lambda \wedge \mu > 0$, span the unique f-line $(\langle a, b \rangle, \lambda \wedge \mu)$.

In the above definitions, the $\wedge$ was originally meant to be the minimum operator, but in [3] was considered any triangular norm.

Definition 2.4. (see [7]) An intuitionistic fuzzy set $A = \{\langle x, \lambda(x), \mu(x)\rangle : x \in X\}$ on n -dimensional projective space S is an intuitionistic fuzzy n - dimensional projective space on S if $\lambda(p) \geq \lambda(q) \wedge \lambda(r)$ and $\mu(p) \leq \mu(q) \vee \mu(r)$, for any three collinear points p, q, r of A we denoted $[A, S]$.

The projective space S is called the base projective space of $[A, S]$ if $[A, S]$ is an intuitionistic fuzzy point , line , plane , ... ,we use base point , base line , base plane,... , respectively.

Definition 2.5. (see [7]) Consider the projective plane $\mathcal{P} = (P, B, I)$. Suppose $a \in P$ and $\alpha, \beta \in [0, 1]$. The IF-point (a, α, β) is the following intuitionistic fuzzy set on the point set P of $\mathcal{P}$:

$$(a, \alpha, \beta) : P \to [0, 1] : \begin{cases} a \to \alpha, \ a \to \beta & \\ x \to 0 & \text{if } x \in P \backslash \{a\}. \end{cases}$$

The point a is called the base point of the IF-point (a, α, β). An IF-line (L, α, β) with base line L is defined in a similar way .

The IF-lines (L, α, β) and (M, σ, ω) intersect in the unique IF- point $(L \cap M, \alpha \wedge \sigma, \beta \vee \omega)$. The IF-points (a, α, β) and (b, σ, ω) span the unique IF-line $(\langle a, b\rangle, \alpha \wedge \sigma, \beta \vee \omega)$.

Definition 2.6. (see [7]) Suppose $\mathcal{P}$ is a projective plane $\mathcal{P} = (P, B, I)$. The intuitionistic fuzzy set $Z = \langle \lambda, \mu \rangle$ on $P \cup B$ is an intuitionistic fuzzy projective plane on $\mathcal{P}$ if :

1) $\lambda(L) \geq \lambda(p) \wedge \lambda(q)$ and $\mu(L) \leq \mu(p) \vee \mu(q)$, $\forall p, q$: $\langle p, q\rangle = L$
2) $\lambda(p) \geq \lambda(L) \wedge \lambda(M)$ and $\mu(p) \leq \mu(L) \vee \mu(M)$, $\forall L, M$: $L \cap M = p$.

The intuitionistic fuzzy projective plane can be considered as an ordinary projective plane, where to every point (and only to points) one (and only one) degrees of membership and nonmembership are assigned.

3. Some Properties of the $\mathcal{IFP}$ with base plane $\mathcal{P}$

We now consider some classical configurations in $\mathcal{P}$ and extend them to intuitionistic fuzzy projective planes. Firstly, we look at the Desargues configuration in an intuitionistic projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$ that is Desarguesian.

Theorem 3.1. *Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with base plane $\mathcal{P}$ that is Desarguesian. Choose three IF-points $(a_i, \alpha_i, \alpha_i')$, $i \in \{1, 2, 3\}$with noncollinear base points, and three other f- points (b_i, β_i, β_i'), $i \in \{1, 2, 3\}$with noncollinear base points, such that the f-lines $(\langle a_i, b_i\rangle, \alpha_i \wedge \beta_i, \alpha_i' \vee \beta_i')$, for $i \in \{1, 2, 3\}$, meet in an IF-point (p, γ, η) of $\mathcal{IFP}$, with $a_i \neq b_i \neq p \neq a_i$. Then the three IF-points $(c_{\{i,j\}}, \gamma_{\{i,j\}}, \gamma'_{\{i,j\}})$ obtained by intersecting $(\langle a_i, a_j\rangle, \alpha_i \wedge \alpha_j, \alpha_i' \vee \alpha_j')$ and $(\langle b_i, b_j\rangle, \beta_i \wedge \beta_j, \beta_i' \vee \beta_j')$, for $i \neq j$ and $i, j \in \{1, 2, 3\}$), are collinear.*

Proof. One calculates that $\gamma = \alpha_i \wedge \alpha_j \wedge \beta_i \wedge \beta_j$ and $\gamma' = \alpha_i' \vee \alpha_j' \vee \beta_i' \vee \beta_j'$ for $\{i, j\} \subseteq \{1, 2, 3\}$, with $i \neq j$. Now, the membership degree of the line spanned by $(c_{\{i,j\}}, \gamma_{\{i,j\}}, \gamma'_{\{i,j\}})$ and $(c_{\{i,k\}}, \gamma_{\{i,k\}}, \gamma'_{\{i,k\}})$, with $\{i, j, k\} = \{1, 2, 3\}$, is equal to

$$\alpha_i \wedge \alpha_i \wedge \alpha_j \wedge \alpha_k \wedge \beta_i \wedge \beta_i \wedge \beta_j \wedge \beta_k = \gamma \wedge \gamma \text{ and } \alpha_i' \vee \alpha_i' \vee \alpha_j' \vee \alpha_k' \vee \beta_i' \vee \beta_i' \vee \beta_j' \vee \beta_k' = \eta \vee \eta$$

which is independent of i. $\square$

The Pappus' theorem was fuzzified using minimum operator in [9]. Now, we give intuitionistic fuzzy version of Pappus theorem as the following:

Theorem 3.2. *Suppose we have an intuitionistic fuzzy projective plane $\mathcal{IFP}$ with Pappian base plane $\mathcal{P}$. Choose two different lines L_1 and L_2 in $\mathcal{P}$. Choose two triples of $IF-$points $(a_i, \alpha_i, \alpha_i')$ and (b_i, β_i, β_i') with a_i on L_1 and b_i on L_2, $i = 1, 2, 3$ and such that no three of the base points a_1, a_2, b_1, b_2 are collinear. Then the three intersection $IF-$points $(c_1, \gamma_1, \gamma_1') = (a_2b_3 \cap a_3b_2, \alpha_2 \wedge \alpha_3 \wedge \beta_2 \wedge \beta_3, \alpha_2' \vee \alpha_3' \vee \beta_2' \vee \beta_3')$, $(c_2, \gamma_2, \gamma_2') = (a_1b_3 \cap a_3b_1, \alpha_1 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_3, \alpha_1' \vee \alpha_3' \vee \beta_1' \vee \beta_3')$ and $(c_3, \gamma_3, \gamma_3') = (a_1b_2 \cap a_2b_1,\ \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2,\ \alpha_1' \vee \alpha_2' \vee \beta_1' \vee \beta_2')$ are collinear.*

Proof. Since $IF-$points $(a_i, \alpha_i, \alpha_i')$ and (b_i, β_i, β_i') are $IF-$collinear, $\alpha_1 \wedge \alpha_2 = \alpha_1 \wedge \alpha_3 = \alpha_2 \wedge \alpha_3, \alpha_1' \vee \alpha_2' = \alpha_1' \vee \alpha_3' = \alpha_2' \vee \alpha_3'$, and $\beta_1 \wedge \beta_2 = \beta_1 \wedge \beta_3 = \beta_2 \wedge \beta_3$, $\beta_1' \vee \beta_2' = \beta_1' \vee \beta_3' = \beta_2' \vee \beta_3'$, $i = 1, 2, 3$. $\gamma_1 \wedge \gamma_2 = \alpha_2 \wedge \alpha_3 \wedge \beta_2 \wedge \beta_3 \wedge \alpha_1 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_3$, $\gamma_1 \wedge \gamma_3 = \alpha_2 \wedge \alpha_3 \wedge \beta_2 \wedge \beta_3 \wedge \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2$ and $\gamma_2 \wedge \gamma_3 = \alpha_1 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_3 \wedge \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2$. Also, $\gamma_1' \vee \gamma_2' = \alpha_2' \vee \alpha_3' \vee \beta_2' \vee \beta_3' \vee \alpha_1' \vee \alpha_3' \vee \beta_1' \vee \beta_3'$, $\gamma_1' \vee \gamma_3' = \alpha_2' \vee \alpha_3' \vee \beta_2' \vee \beta_3' \vee \alpha_1' \vee \alpha_2' \vee \beta_1' \vee \beta_2'$ and $\gamma_2' \vee \gamma_3' = \alpha_1' \vee \alpha_3' \vee \beta_1' \vee \beta_3' \vee \alpha_1' \vee \alpha_2' \vee \beta_1' \vee \beta_2'$. So, it is clear that $\gamma_1 \wedge \gamma_2 = \gamma_1 \wedge \gamma_3 = \gamma_2 \wedge \gamma_3$, $\gamma_1' \vee \gamma_2' = \gamma_1' \vee \gamma_3' = \gamma_2' \vee \gamma_3'$. □

Conclusion: In the present paper, we have considered Desargues and Pappus configurations in projective plane $\mathcal{P}$. We have seen that intuitionistic fuzzy versions of them automatically holds. In further investigation, when using other triangular norms, it will be given contribution to the intuitionistic fuzzy projective geometry and other classical theorems of projective geometry will be extended to the intuitionistic fuzzy projective geometry.

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