SOME PARANORMED SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION OVER N-NORMED SPACES

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Abstract. In this paper we present new classes of sequence spaces using lacunary sequences and a Musielak-Orlicz function over $n$-normed spaces. We examine some topological properties and prove some interesting inclusion relations between them.

1. Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [5] in the mid of 1960’s, while that of $n$-normed spaces one can see in Misiak [14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([6], [7]) and Gunawan and Mashadi [8]. Let $n \in \mathbb{N}$ and $X$ be a linear space over the field $K$, where $K$ is field of real or complex numbers of dimension $d$, where $d \geq n \geq 2$. A real valued function $||\cdot||, \ldots, ||\cdot||$ on $X^n$ satisfying the following four conditions:

1. $||x_1, x_2, \ldots, x_n|| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in $X$;
2. $||x_1, x_2, \ldots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \ldots, x_n|| = |\alpha| \ ||x_1, x_2, \ldots, x_n||$ for any $\alpha \in K$, and
4. $||x + x', x_2, \ldots, x_n|| \leq ||x, x_2, \ldots, x_n|| + ||x', x_2, \ldots, x_n||$

is called an $n$-norm on $X$, and the pair $(X, ||\cdot||, \ldots, ||\cdot||)$ is called an $n$-normed space over the field $K$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the $n$-norm $||x_1, x_2, \ldots, x_n||_E = \text{the volume of the } n\text{-dimensional parallelepiped spanned by the vectors } x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \ldots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Let $(X, ||\cdot||, \ldots, ||\cdot||)$ be an $n$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly

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independent set in $X$. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on $X^{n-1}$ defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \cdots, x_{n-1}, a_i\| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on $X$ with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence $(x_k)$ in a $n$-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$ 

A sequence $(x_k)$ in a $n$-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k \to \infty} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$ 

If every cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the $n$-norm. Any complete $n$-normed space is said to be $n$-Banach space.

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19, Theorem 10.4.2, pp. 183]).

For more details about sequence spaces (see [1], [2], [3], [17], [18]) and references therein.

An Orlicz function $M$ is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let $w$ be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. The space $\ell_M$ is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$ 

It is shown in [10] that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p (p \geq 1)$. The $\Delta_2$-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $M = (M_k)$ of Orlicz function is called a Musielak-Orlicz function (see [13], [16]). A sequence $N = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}, \; k = 1, 2, \ldots$$
is called the complementary function of a Musielak-Orlicz function \( M \). For a given Musielak-Orlicz function \( M \), the Musielak-Orlicz sequence space \( t_M \) and its subspace \( h_M \) are defined as follows

\[
t_M = \left\{ x \in w : I_M(cx) < \infty \text{ for some } c > 0 \right\},
\]

\[
h_M = \left\{ x \in w : I_M(cx) < \infty \text{ for all } c > 0 \right\},
\]

where \( I_M \) is a convex modular defined by

\[
I_M(x) = \sum_{k=1}^{\infty} (M_k)(x_k), x = (x_k) \in t_M.
\]

We consider \( t_M \) equipped with the Luxemburg norm

\[
\|x\| = \inf \left\{ k > 0 : I_M\left(\frac{x}{k}\right) \leq 1 \right\}
\]

or equipped with the Orlicz norm

\[
\|x\|_0 = \inf \left\{ \frac{1}{k} \left(1 + I_M(kx)\right) : k > 0 \right\}.
\]

Let \( \ell_\infty, c \) and \( c_0 \) denotes the sequence spaces of bounded, convergent and null sequences \( x = (x_k) \) respectively. A sequence \( x = (x_k) \in \ell_\infty \) is said to be almost convergent if all Banach limits of \( x = (x_k) \) coincide. In [9], it was shown that

\[
\hat{c} = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{g_r} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s \right\}.
\]

In ([11], [12]) Maddox defined strongly almost convergent sequences. Recall that a sequence \( x = (x_k) \) is strongly almost convergent if there is a number \( L \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.
\]

By a lacunary sequence \( \theta = (i_r), r = 0, 1, 2, \ldots, \) where \( i_0 = 0, \) we shall mean an increasing sequence of non-negative integers \( g_r = (i_r - i_{r-1}) \to \infty (r \to \infty) \). The intervals determined by \( \theta \) are denoted by \( I_r = (i_r-1,i_r] \) and the ratio \( i_r/i_{r-1} \) will be denoted by \( q_r \). The space of lacunary strongly convergent sequences \( N_\theta \) was defined by Freedman [4] as follows:

\[
N_\theta = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{g_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L \right\}.
\]

Mursaleen and Noman [15] introduced the notion of \( \lambda \)-convergent and \( \lambda \)-bounded sequences as follows:

Let \( \lambda = (\lambda_k)_{k=1}^{\infty} \) be a strictly increasing sequence of positive real numbers tending to infinity i.e.

\[
0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty
\]

and said that a sequence \( x = (x_k) \in w \) is \( \lambda \)-convergent to the number \( L \), called the \( \lambda \)-limit of \( x \) if \( \Lambda_m(x) \to L \) as \( m \to \infty \), where

\[
\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.
\]
The sequence \( x = (x_k) \in w \) is \( \lambda \)-bounded if \( \sup_m |\Lambda_m(x)| < \infty \). It is well known \cite{15} that if \( \lim_m x_m = a \) in the ordinary sense of convergence, then

\[
\lim_m \left( \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.
\]

This implies that

\[
\lim_m |\Lambda_m(x) - a| = \lim_m \left( \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a) \right) = 0
\]

which yields that \( \lim_m \Lambda_m(x) = a \) and hence \( x = (x_k) \in w \) is \( \lambda \)-convergent to \( a \).

Let \( (X, ||\cdot||, \cdots, ||\cdot||) \) be a \( n \)-normed space and \( w(n - X) \) denotes the space of \( X \)-valued sequences. Let \( \mathcal{M} = (\mathcal{M}_k) \) be a Musielak-Orlicz function and \( p = (p_k) \) be a bounded sequence of positive real numbers. Then we define the following sequence spaces in the present paper:

\[
[c, \mathcal{M}, p, \Lambda, ||\cdot||, \cdots, ||]\theta = \left\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ \mathcal{M}_k \left( \frac{||\Lambda_k(x) - L||}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} = 0, \right\}
\]

for some \( \rho > 0 \), \( L \in X \) and for every \( z_1, \cdots, z_{n-1} \in X \},
\]

\[
[c, \mathcal{M}, p, \Lambda, ||\cdot||, \cdots, ||]\theta_0 = \left\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ \mathcal{M}_k \left( \frac{||\Lambda_k(x)||}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} = 0, \right\}
\]

for some \( \rho > 0 \) and for every \( z_1, \cdots, z_{n-1} \in X \}
\]

and

\[
[c, \mathcal{M}, p, \Lambda, ||\cdot||, \cdots, ||]\theta_\infty = \left\{ x = (x_k) \in w(n - X) : \sup_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ \mathcal{M}_k \left( \frac{||\Lambda_k(x)||}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} < \infty, \right\}
\]

for some \( \rho > 0 \) and for every \( z_1, \cdots, z_{n-1} \in X \}
\]

When, \( \mathcal{M}(x) = x \), we get

\[
[c, p, \Lambda, ||\cdot||, \cdots, ||]\theta = \left\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left( \frac{||\Lambda_k(x) - L||}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} = 0, \right\}
\]

for some \( \rho > 0 \), \( L \in X \) and for every \( z_1, \cdots, z_{n-1} \in X \},
\]

\[
[c, p, \Lambda, ||\cdot||, \cdots, ||]\theta_0 = \left\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left( \frac{||\Lambda_k(x)||}{\rho}, z_1, \cdots, z_{n-1} \right) \right]^{p_k} = 0, \right\}
\]

for some \( \rho > 0 \) and for every \( z_1, \cdots, z_{n-1} \in X \}
\]

and
Let \( \{ A_k(x) \} \) be a Musielak-Orlicz function and \( p = (p_k) \) be a bounded sequence of positive real numbers. Then

\[
\{ x = (x_k) \in w(n - X) : \sup_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left( \frac{A_k(x)}{\rho}, z_1, \cdots, z_{n-1} \right)^{p_k} < \infty \}
\]

for some \( \rho > 0 \) and for every \( z_1, \cdots, z_{n-1} \in X \).

If we take \( p = (p_k) = 1 \) for all \( k \), then we get

\[
\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \frac{A_k(x)}{\rho}, z_1, \cdots, z_{n-1} \right) = 0
\]

for some \( \rho > 0, L \in X \) and for every \( z_1, \cdots, z_{n-1} \in X \),

\[
\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \frac{A_k(x)}{\rho}, z_1, \cdots, z_{n-1} \right) = 0
\]

for some \( \rho > 0 \) and for every \( z_1, \cdots, z_{n-1} \in X \) and

\[
\{ x = (x_k) \in w(n - X) : \sup_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \frac{A_k(x)}{\rho}, z_1, \cdots, z_{n-1} \right) < \infty
\]

for some \( \rho > 0 \) and for every \( z_1, \cdots, z_{n-1} \in X \).

The following inequality will be used throughout the paper. If \( 0 \leq \inf_k p_k = H_0 \leq p_k \leq \sup_k = H < \infty \), \( K = \max(1, 2^{H-1}) \) and \( H = \sup_k p_k < \infty \), then

\[
|x_k + y_k|^{p_k} \leq K(|x_k|^{p_k} + |y_k|^{p_k}),
\]

for all \( k \in \mathbb{N} \) and \( x, y \in \mathbb{C} \). Also \( |x_k|^{p_k} \leq \max(1, |x_k|^H) \) for all \( x_k \in \mathbb{C} \).

2. Some properties of difference sequence spaces

**Theorem 2.1.** Let \( M = (M_k) \) be a Musielak-Orlicz function and \( p = (p_k) \) be a bounded sequence of positive real numbers. Then \( [c, M, \Lambda, || \cdot ||, \cdots, ||]_0^\infty \), \( [c, M, p, \Lambda, || \cdot ||, \cdots, ||]_0^\infty \), \( [c, M, p, \Lambda, || \cdot ||, \cdots, ||]_0^\infty \) are linear spaces over the field of complex numbers \( C \).

**Proof.** Let \( x = (x_k), y = (y_k) \in [c, M, \Lambda, || \cdot ||, \cdots, ||]_0^\infty \) and \( \alpha, \beta \in \mathbb{C} \). Then there exist positive numbers \( p_1 \) and \( p_2 \) such that

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \frac{|A_k(x)|}{\rho_1}, z_1, \cdots, z_{n-1} \right)^{p_1} = 0,
\]

and

\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M_k \left( \frac{|A_k(x)|}{\rho_2}, z_1, \cdots, z_{n-1} \right)^{p_2} = 0.
\]
Let $\rho_1 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non-decreasing convex function, by using inequality \((1.1)\), we have

$$
\frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(\alpha x + \beta y)}{\rho_3}, z_1, \cdots, z_{n-1}\right) \right]^{p_k}
= \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}\right) \right]^{p_k}
+ \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}\right) \right]^{p_k}

\leq K \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}\right) \right]^{p_k}
+ K \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}\right) \right]^{p_k}

\rightarrow 0 \text{ as } r \rightarrow \infty.
$$

Thus, we have $\alpha x + \beta y \in [c, \mathcal{M}, p, \Lambda, ||\cdot||, \cdots, ||\cdot||]_0$. Hence $[c, \mathcal{M}, p, \Lambda, ||\cdot||, \cdots, ||\cdot||]_0$ is a linear space. Similarly, we can prove that $[c, \mathcal{M}, p, \Lambda, ||\cdot||, \cdots, ||\cdot||]_0$ are linear spaces.

**Theorem 2.2.** For any Musielak-Orlicz function $M = (M_k)$ and a bounded sequence $p = (p_k)$ of positive real numbers, $[c, \mathcal{M}, p, \Lambda, ||\cdot||, \cdots, ||\cdot||]_0$ is a topological linear space paranormed by

$$
g(x) = \inf \left\{ \rho_\pi : \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}\right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1, r \in \mathbb{N} \right\},
$$

where $H = \max(1, \sup_k p_k < \infty)$.

**Proof.** Clearly $g(x) \geq 0$ for $x = (x_k) \in [c, \mathcal{M}, p, \Lambda, ||\cdot||, \cdots, ||\cdot||]_0$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$
\inf \left\{ \rho_\pi : \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}\right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1, r \in \mathbb{N} \right\} = 0.
$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon(0 < \rho_\epsilon < \epsilon)$ such that

$$
\left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}\right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1.
$$

Thus

$$
\left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(x)}{\epsilon}, z_1, \cdots, z_{n-1}\right) \right]^{p_k} \right)^{\frac{1}{p}} \leq \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k\left( \frac{\Lambda_k(x)}{\rho_\epsilon}, z_1, \cdots, z_{n-1}\right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1,
$$
for each \( r \). Suppose that \( x \neq 0 \) for each \( k \in N \). This implies that \( \Lambda_k(x) \neq 0 \), for each \( k \in N \). Let \( \epsilon \to 0 \), then \( \|\frac{\Lambda_k(x)}{\epsilon}, z_1, \cdots, z_{n-1}\| \to \infty \). It follows that

\[
\left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(x)}{\epsilon}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} \to \infty,
\]

which is a contradiction. Therefore, \( \Lambda_k(x) = 0 \) for each \( k \) and thus \( x = 0 \) for each \( k \in N \). Let \( \rho_1 > 0 \) and \( \rho_2 > 0 \) be such that

\[
\left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1
\]

and

\[
\left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1
\]

for each \( r \). Let \( \rho = \rho_1 + \rho_2 \). Then, by Minkowski’s inequality, we have

\[
\left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(x+y)}{\rho}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} + \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq \leq 1
\]

Since \( \rho \)'s are non-negative, so we have

\[
g(x+y) = \inf \left\{ \rho^{\frac{1}{p_k}} : \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(x+y)}{\rho}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1, r \in N \right\},
\]

\[
\leq \inf \left\{ \rho_1^{\frac{1}{p_k}} : \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1, r \in N \right\}
\]

\[
+ \inf \left\{ \rho_2^{\frac{1}{p_k}} : \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \|\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_k} \right)^{\frac{1}{p}} \leq 1, r \in N \right\}
\]

Therefore,

\[
g(x+y) \leq g(x) + g(y).
\]

Finally, we prove that the scalar multiplication is continuous. Let \( \mu \) be any complex number. By definition,
Then for all fixed $x > 0$, we have

$$g(\mu x) = \inf \left\{ \rho^{pr} : \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left| \frac{\Lambda_k(x)}{\rho}, z_1, \ldots, z_{n-1} \right| \right) \right] \right)^{pr} \leq 1, r \in \mathbb{N} \right\}.$$ 

Hence

$$g(\mu x) = \inf \left\{ (|\mu|t)^{pr} : \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left| \frac{\Lambda_k(x)}{t}, z_1, \ldots, z_{n-1} \right| \right) \right] \right)^{pr} \leq 1, r \in \mathbb{N} \right\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{pr} \leq \max(1, |\mu|^{sup_{pr}})$, we have

$$g(\mu x) \leq \max(1, |\mu|^{sup_{pr}}) \inf \left\{ t^{pr} : \left( \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left| \frac{\Lambda_k(x)}{t}, z_1, \ldots, z_{n-1} \right| \right) \right] \right)^{pr} \leq 1, r \in \mathbb{N} \right\}.$$ 

So, the fact that scalar multiplication is continuous follows from the above inequality.

This completes the proof of the theorem. □

**Theorem 2.3.** Let $M = (M_k)$ be a Musielak-Orlicz function. If $\sup_{k} [M_k(x)]^{pr} < \infty$ for all fixed $x > 0$, then $\left[ c, M, p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\sup}^{\rho} \subset \left[ c, M, p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\infty}^{\rho}$.

**Proof.** Let $x = (x_k) \in \left[ c, M, p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\infty}^{\rho}$. There exists some positive $\rho_1$ such that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left| \frac{\Lambda_k(x)}{\rho_1}, z_1, \ldots, z_{n-1} \right| \right) \right]^{pk} = 0.$$ 

Define $\rho = 2\rho_1$. Since $M = (M_k)$ is non-decreasing and convex, by using inequality (1.1), we have

$$\sup_{r} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left| \frac{\Lambda_k(x)}{\rho}, z_1, \ldots, z_{n-1} \right| \right) \right]^{pk}$$

$$= \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left| \frac{\Lambda_k(x)}{\rho}, z_1, \ldots, z_{n-1} \right| \right) \right]^{pk}$$

$$\leq K \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} \left[ \frac{1}{2pk} M_k \left( \left| \frac{\Lambda_k(x)}{\rho_1}, z_1, \ldots, z_{n-1} \right| \right) \right]^{pk}$$

$$+ K \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} \left[ \frac{1}{2pk} M_k \left( \left| \frac{L}{\rho_1}, z_1, \ldots, z_{n-1} \right| \right) \right]^{pk}$$

$$\leq K \sup_{r} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left| \frac{L}{\rho_1}, z_1, \ldots, z_{n-1} \right| \right) \right]^{pk}$$

$$< \infty.$$ 

Hence $x = (x_k) \in \left[ c, M, p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\infty}^{\rho}$. □

**Theorem 2.4.** Let $0 < \inf p_k = g \leq p_k \leq \sup p_k = H < \infty$ and $M = (M_k)$, $M' = (M'_k)$ are Musielak-Orlicz functions satisfying $\Delta_2$-condition, then we have

(i) $\left[ c, M', p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\sup}^{\rho} \subset \left[ c, M \circ M', p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\sup}^{\rho}$,

(ii) $\left[ c, M', p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\infty}^{\rho} \subset \left[ c, M \circ M', p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\infty}^{\rho}$,

(iii) $\left[ c, M', p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\infty}^{\rho} \subset \left[ c, M \circ M', p, \Lambda, ||\cdot, \cdot, \cdot\| \right]_{\infty}^{\rho}$.
Proof. Let \( x = (x_k) \in [c, \mathcal{M}', p, \Lambda, || \cdot ||_0]^\theta \). Then we have
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} [M_k^r\left(|| \frac{\lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} ||\right)]_{pk} = 0, \text{ for some } L.
\]
Let \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( M_k(t) < \epsilon \) for \( 0 \leq t \leq \delta \). Let
\[
y_k = M_k^r\left(|| \frac{\lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} ||\right) \text{ for all } k \in \mathbb{N}.
\]
We can write
\[
\frac{1}{h_r} \sum_{k \in I_r} [M_k(y_k)]_{pk} = \frac{1}{h_r} \sum_{k \in I_r, y_k \leq \delta} [M_k(y_k)]_{pk} + \frac{1}{h_r} \sum_{k \in I_r, y_k > \delta} [M_k(y_k)]_{pk}.
\]
Since \( \mathcal{M} = (M_k) \) satisfies \( \Delta_2 \)-condition, we have
\[
\frac{1}{h_r} \sum_{k \in I_r, y_k \leq \delta} [M_k(y_k)]_{pk} \leq [M_k(1)]^H \frac{1}{h_r} \sum_{k \in I_r, y_k \leq \delta} [M_k(y_k)]_{pk} \leq [M_k(2)]^H \frac{1}{h_r} \sum_{k \in I_r, y_k \leq \delta} [M_k(y_k)]_{pk}
\]
(2.1)
For \( y_k > \delta \)
\[
y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}.
\]
Since \( \mathcal{M} = (M_k) \) is non-decreasing and convex, it follows that
\[
M_k(y_k) < M_k\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2} M_k(2) + \frac{1}{2} M_k\left(\frac{2y_k}{\delta}\right).
\]
Since \( (M_k) \) satisfies \( \Delta_2 \)-condition, we can write
\[
M_k(y_k) < \frac{1}{2} T \frac{y_k}{\delta} M_k(2) + \frac{1}{2} T \frac{y_k}{\delta} M_k(2) = T \frac{y_k}{\delta} M_k(2).
\]
Hence,
\[
\frac{1}{h_r} \sum_{k \in I_r, y_k > \delta} [M_k(y_k)]_{pk} \leq \max\left(1, \left(\frac{T M_k(2)}{\delta}\right)^H\right) \frac{1}{h_r} \sum_{k \in I_r, y_k > \delta} [(y_k)]_{pk}^p
\]
from equations (2.1) and (2.2), we have
\[
x = (x_k) \in [c, \mathcal{M} \circ \mathcal{M}', p, \Lambda, || \cdot ||_0]^\theta.
\]
This completes the proof of (i). Similarly, we can prove that
\[
[c, \mathcal{M}'_0^\theta] \subset [c, \mathcal{M} \circ \mathcal{M}'_0^\theta]
\]
and
\[
[c, \mathcal{M}_\infty^\theta] \subset [c, \mathcal{M} \circ \mathcal{M}_\infty^\theta]
\]
\[
\square
\]
Corollary 2.1. Let \( 0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty \) and \( M = (M_k) \) be a Musielak-Orlicz function satisfying \( \Delta_2 \)-condition, then we have
\[
[c, \mathcal{M}, p, \Lambda, || \cdot ||_0]^\theta \subset [c, \mathcal{M}, p, \Lambda, || \cdot ||_0^\theta]
\]
and
\[
[c, \mathcal{M}, p, \Lambda, || \cdot ||_\infty^\theta] \subset [c, \mathcal{M}, p, \Lambda, || \cdot ||_\infty^\theta].
\]
Proof. Taking $M'(x) = x$ in Theorem 2.4, we get the required result. \qed

**Theorem 2.5.** Let $M = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:

(i) $[c, p, \Lambda, ||\cdot, \cdot, \cdot||_0^{\theta}] \subset [c, M, p, \Lambda, ||\cdot, \cdot, \cdot||]_0^{\theta}$

(ii) $[c, p, \Lambda, ||\cdot, \cdot, \cdot||_0^{\theta}] \subset [c, M, p, \Lambda, ||\cdot, \cdot, \cdot||]_0^{\theta}$

(iii) $\sup \frac{1}{r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{pk} < \infty \ (t, \rho > 0)$.

Proof. (i) $\Rightarrow$ (ii) The proof is obvious in view of the fact that $[c, p, \Lambda, ||\cdot, \cdot, \cdot||_0^{\theta}] \subset [c, M, p, \Lambda, ||\cdot, \cdot, \cdot||]_0^{\theta}$.

(ii) $\Rightarrow$ (iii) Let $[c, p, \Lambda, ||\cdot, \cdot, \cdot||_0^{\theta}] \subset [c, M, p, \Lambda, ||\cdot, \cdot, \cdot||]_0^{\theta}$. Suppose that (iii) does not hold. Then for some $t, \rho > 0$

$$\sup \frac{1}{r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{pk} = \infty$$

and therefore we can find a subinterval $I_{r(j)}$ of the set of interval $I_r$ such that

$$(2.3) \quad \frac{1}{h_{r(j)}} \sum_{k \in I_{r(j)}} [M_k(\frac{j^{-1}}{\rho})]^{pk} > j, j = 1, 2,$$

Define the sequence $x = (x_k)$ by

$$\Lambda_k(x) = \begin{cases} j^{-1}, & k \in I_{r(j)} \\ 0, & k \not\in I_{r(j)} \end{cases} \text{ for all } s \in \mathbb{N}.$$

Then $x = (x_k) \in [c, p, \Lambda, ||\cdot, \cdot, \cdot||_0^{\theta}]$ but by equation (2.3), $x \not\in [c, M, p, \Lambda, ||\cdot, \cdot, \cdot||]_0^{\theta}$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold and $x = (x_k) \in [c, p, \Lambda, ||\cdot, \cdot, \cdot||_0^{\theta}]$. Suppose that $x \not\in [c, M, p, \Lambda, ||\cdot, \cdot, \cdot||]_0^{\theta}$.

Then

$$(2.4) \quad \sup \frac{1}{r} \sum_{k \in I_r} [M_k(\frac{||\Lambda_k(x)||_0, z_1, \cdot, z_{n-1}||}{\rho})]^{pk} = \infty$$

Let $t = ||\Lambda_k(x), z_1, \cdot, z_{n-1}||$ for each $k$, then by equations (2.4)

$$\sup \frac{1}{r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})] = \infty,$$

which contradicts (iii). Hence (i) must hold. \qed

**Theorem 2.6.** Let $1 \leq p_k \leq \sup p_k < \infty$ and $M = (M_k)$ be a Musielak Orlicz function. Then the following statements are equivalent:

(i) $[c, M, p, \Lambda, ||\cdot, \cdot, \cdot||_0^{\theta}] \subset [c, M, p, \Lambda, ||\cdot, \cdot, \cdot||]_0^{\theta}$

(ii) $[c, M, p, \Lambda, ||\cdot, \cdot, \cdot||_0^{\theta}] \subset [c, M, p, \Lambda, ||\cdot, \cdot, \cdot||]_0^{\theta}$

(iii) $\inf \frac{1}{r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{pk} > 0 \ (t, \rho > 0)$.

Proof. (i) $\Rightarrow$ (ii) It is trivial.

(ii) $\Rightarrow$ (iii) Let (ii) hold. Suppose that (iii) does not hold. Then

$$\inf \frac{1}{r} \sum_{k \in I_r} [M_k(\frac{t}{\rho})]^{pk} = 0 \ (t, \rho > 0),$$
so we can find a subinterval $I_{r(j)}$ of the set of interval $I_r$ such that

**Formula (2.5)**

$$\frac{1}{h_{r(j)}} \sum_{k \in I_{r(j)}} \left[ M_k \left( \frac{j}{p} \right) \right]^{p_k} < j^{-1}, \quad j = 1, 2, \ldots$$

Define the sequence $x = (x_k)$ by

$$\Lambda_k(x) = \begin{cases} j, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases} \text{ for all } s \in \mathbb{N}.$$

Thus by equation (2.5), $x = (x_k) \in [c, \mathcal{M}, p, \Lambda, ||\cdot||, \ldots, ||\cdot||^\rho]$, but by equation (2.3), $x = (x_k) \notin [c, p, \Lambda, ||\cdot||, \ldots, ||\cdot||^\rho]$, which contradicts (ii). Hence (iii) must hold.

(iii) ⇒ (i) Let (iii) hold and suppose that $x = (x_k) \in [c, \mathcal{M}, p, \Lambda, ||\cdot||, \ldots, ||\cdot||^\rho]$, i.e.,

**Formula (2.6)**

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{\Lambda_k(x)}{\rho}, z_1, \ldots, z_{n-1} || \right) \right]^{p_k} = 0, \text{ for some } \rho > 0.$$

Again, suppose that $x = (x_k) \notin [c, p, \Lambda, ||\cdot||, \ldots, ||\cdot||^\rho]$. Then, for some number $\epsilon > 0$ and a subinterval $I_{r(j)}$ of the set of interval $I_r$, we have $||\Lambda_k(x), z_1, \ldots, z_{n-1} || \geq \epsilon$ for all $k \in \mathbb{N}$ and some $s \geq s_0$. Then, from the properties of the Orlicz function, we can write

$$M_k \left( \frac{\Lambda_k(x)}{\rho}, z_1, \ldots, z_{n-1} \right) \geq M_k \left( \frac{\epsilon}{\rho} \right)^{p_k}$$

and consequently by (2.6)

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \frac{\epsilon}{\rho} \right) \right]^{p_k} = 0,$$

which contradicts (iii). Hence (i) must hold. \qed

**Theorem 2.7.** Let $0 < p_k \leq q_k$ for all $k \in \mathbb{N}$ and $\left( \frac{q_k}{p_k} \right)$ be bounded. Then, $[c, \mathcal{M}, q, \Lambda, ||\cdot||, \ldots, ||\cdot||^\rho] \subset [c, \mathcal{M}, p, \Lambda, ||\cdot||, \ldots, ||\cdot||^\rho]$.

**Proof.** Let $x \in [c, \mathcal{M}, q, \Lambda, ||\cdot||, \ldots, ||\cdot||^\rho]$. Write

$$t_k = \left[ M_k \left( \frac{||\Lambda_k(x) - L, z_1, \ldots, z_{n-1} ||}{\rho} \right) \right]^{q_k}$$

and $\mu_k = \frac{p_k}{q_k}$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \leq 1$ for $k \in \mathbb{N}$. Take $0 < \mu < \mu_k$ for $k \in \mathbb{N}$. Define the sequences $(u_k)$ and $(v_k)$ as follows: For $t_k \geq 1$, let $u_k = t_k$ and $v_k = 0$ and for $t_k < 1$, let $u_k = 0$ and $v_k = t_k$. Then clearly for all $k \in \mathbb{N}$, we have

$$t_k = u_k + v_k, \quad \mu_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}$$

Now it follows that $u_k^{\mu_k} \leq u_k \leq t_k$ and $v_k^{\mu_k} \leq v_k^{\mu_k}$. Therefore,

$$\frac{1}{h_r} \sum_{k \in I_r} \mu_k^{\mu_k} \leq \frac{1}{h_r} \sum_{k \in I_r} (u_k^{\mu_k} + v_k^{\mu_k}) \leq \frac{1}{h_r} \sum_{k \in I_r} t_k + \frac{1}{h_r} \sum_{k \in I_r} v_k^{\mu_k}.$$
Now for each \( k \),
\[
\frac{1}{h_r} \sum_{k \in I_r} t_k^\mu = \frac{1}{h_r} \left( \sum_{k \in I_r} \left( \frac{1}{h_r} v_k \right)^\mu \left( \frac{1}{h_r} \right)^{1-\mu} \right) \\
\leq \left( \sum_{k \in I_r} \left[ \left( \frac{1}{h_r} v_k \right)^\mu \right] \right)^{\frac{1}{\mu}} \left( \sum_{k \in I_r} \left[ \left( \frac{1}{h_r} \right)^{1-\mu} \right]^{\frac{1}{1-\mu}} \right)^{1-\mu} \\
= \left( \frac{1}{h_r} \sum_{k \in I_r} v_k \right)^\mu
\]
and so
\[
\frac{1}{h_r} \sum_{k \in I_r} t_k^\mu \leq \frac{1}{h_r} \sum_{k \in I_r} t_k + \left( \frac{1}{h_r} \sum_{k \in I_r} v_k \right)^\mu.
\]
Hence \( x \in [ c,\mathcal{M},p,\Lambda,||,\cdots,||]^\theta \).

\[ \square \]

**Theorem 2.8.** (a) If \( 0 < \inf p_k \leq p_k \leq 1 \) for all \( k \in N \), then
\[ [ c,\mathcal{M},p,\Lambda,||,\cdots,||]^\theta \subset [ c,\mathcal{M},\Lambda,||,\cdots,||]^\theta. \]

(b) If \( 1 \leq p_k \leq \sup p_k < \infty \) for all \( k \in N \). Then
\[ [ c,\mathcal{M},\Lambda,||,\cdots,||]^\theta \subset [ c,\mathcal{M},p,\Lambda,||,\cdots,||]^\theta. \]

**Proof.** (a) Let \( x \in [ c,\mathcal{M},p,\Lambda,||,\cdots,||]^\theta \), then
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} = 0.
\]
Since \( 0 < \inf p_k \leq p_k \leq 1 \). This implies that
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right] \\
\leq \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k},
\]
therefore,
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right] = 0.
\]
This shows that \( x \in [ c,\mathcal{M},p,\Lambda,||,\cdots,||]^\theta \). Therefore,
\[ [ c,\mathcal{M},p,\Lambda,||,\cdots,||]^\theta \subset [ c,\mathcal{M},\Lambda,||,\cdots,||]^\theta. \]
This completes the proof.

(b) Let \( p_k \geq 1 \) for each \( k \) and \( \sup p_k < \infty \). Let \( x \in [ c,p,\Lambda,||,\cdots,||]^\theta \). Then for each \( \epsilon > 0 \) there exists a positive integer \( N \) such that
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} = 0 < 1.
\]
Since \( 1 \leq p_k \leq \sup p_k < \infty \), we have...
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} \\
\leq \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[ M_k \left( \left\| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right] \\
= 0 \\
< 1.
\]
Therefore \( x \in [c, M, p, \Lambda, ||\cdot||, \cdots, ||\cdot||]^{\theta} \).

References


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