



## CERTAIN NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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**ABSTRACT.** The object of this paper is to obtain certain Hermite-Hadamard type integral inequalities involving general class of fractional integral operators and the fractional integral operators with exponential kernel by using harmonically convex functions.

### 1. INTRODUCTION

Let a real function  $f$  be defined on some nonempty interval  $I$  of a real line  $\mathbb{R}$ . The function  $f$  is said to be convex on  $I$  if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all  $x, y \in I$  and  $t \in [0, 1]$ . We say that  $f$  is concave if  $(-f)$  is convex.

Convexity is an important concept in many branches of mathematics. In particular, many important integral inequalities are based on a convexity assumption of a certain function. For example, the following famous inequality is one of them. Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function defined on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . The following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

It was firstly discovered by Ch. Hermite [2] in 1881 in the journal *Mathesis*. But this inequality was nowhere mentioned in the mathematical literature and was not widely known as Hermite's results. E.F. Beckenbach wrote that this result was proven by J. Hadamard in 1893. In 1974, D.S. Mitrinovic found Hermite's note in *Mathesis*. This inequality known as Hadamard's inequality is now commonly referred as the Hermite-Hadamard inequality. Hermite-Hadamard inequality is

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playing a very important role in all the fields of mathematics. Thus such inequalities were studied extensively by many researchers and a number of the papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications. In recent years, one more dimension has been added to this studies, by introducing various integral inequalities involving fractional integral operators like Riemann-Liouville, Hadamard, Erdelyi-Kober, Katugampola fractional operators and fractional operator with exponential kernel.

A different class of the convexity is introduced by İşcan as the following:

**Definition 1.** [3] Let  $I \subseteq \mathbb{R}/\{0\}$  be a real interval. A function  $f : I \rightarrow \mathbb{R}$  is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(x) + (1-t)f(y) \quad (1.2)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If the inequality in (1.2) is reserved, then  $f$  is said to be harmonically concave.

In [3], İşcan established the following inequalities which is different version of Hermite-Hadamard inequality.

**Theorem 1.** Let  $f : I \subseteq \mathbb{R}/\{0\} \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$  then the following inequalities hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \leq \frac{f(a) + f(b)}{2}. \quad (1.3)$$

We need to recall some definitions and known results.

**Definition 2.** Let  $f \in L_1[a, b]$ . The Riemannn-Liouville fractional integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively.

Here  $\Gamma(t)$  is the Gamma function and its definition is  $\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx$ . It is to be noted that  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ . In the case of  $\alpha = 1$ , the fractional integral reduces to the classical integral.

For more details and properties concerning the fractional integral operators, we refer, for example, to the works [6, 8].

İşcan and Wu [4], recently, using Riemann-Liouville fractional integral, presented Hermite-Hadamard integral inequalities for harmonically convex function as follow:

**Theorem 2.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a function such that  $f \in L[a, b]$  where  $a, b \in I$  with  $a < b$ . If  $f$  is a harmonically convex function on  $[a, b]$ , then the following inequalities for fractional integral hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{a+b}\right)^\alpha \left\{ J_{\frac{1}{a}}^\alpha(fog)\left(\frac{1}{b}\right) + J_{\frac{1}{b}}^\alpha(fog)\left(\frac{1}{a}\right) \right\} \leq \frac{f(a)+f(b)}{2} \tag{1.4}$$

with  $\alpha > 0$  and  $g(x) = 1/x$ .

Lately, Kirane and Torebek [5], have introduced a new class of fractional integrals which are summarized as follows:

**Definition 3.** Let  $f \in L_1[a, b]$ . The fractional integrals  $I_a^\alpha$  and  $I_b^\alpha$  of order  $\alpha \in (0, 1)$  are defined by

$$I_a^\alpha f(x) = \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-s)\right\} f(s) ds, \quad x > a$$

and

$$I_b^\alpha f(x) = \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(s-x)\right\} f(s) ds, \quad x < b$$

respectively.

If  $\alpha = 1$ , then

$$\lim_{\alpha \rightarrow 1} I_a^\alpha f(x) = \int_a^x f(s) ds, \quad \lim_{\alpha \rightarrow 1} I_b^\alpha f(x) = \int_x^b f(s) ds.$$

Therefore the operators  $I_a^\alpha$  and  $I_b^\alpha$  are called a fractional integrals of order  $\alpha$ .

Moreover, because

$$\lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \exp\left(-\frac{1-\alpha}{\alpha}(x-s)\right) = \delta(x-s),$$

then

$$\lim_{\alpha \rightarrow 0} I_a^\alpha f(x) = f(x), \quad \lim_{\alpha \rightarrow 0} I_b^\alpha f(x) = f(x).$$

In [7], Raina introduced a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^\sigma(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\dots}(x) = \sum_{k=0}^\infty \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathbf{R}), \tag{1.5}$$

where the coefficients  $\sigma(k)$  ( $k \in \mathbb{N} = \mathbb{N} \cup \{0\}$ ) is a bounded sequence of positive real numbers and  $\mathbf{R}$  is the set of real numbers. With the help of (1.5), Raina [7] and Agarwal *et al.* [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$(\mathcal{J}_{\rho,\lambda,a+;w}^\sigma \varphi)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma[w(x-t)^\rho] \varphi(t) dt \quad (x > a > 0), \tag{1.6}$$

$$(\mathcal{J}_{\rho,\lambda,b-;w}^\sigma \varphi)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma[w(t-x)^\rho] \varphi(t) dt \quad (0 < x < b), \tag{1.7}$$

where  $\lambda, \rho > 0$ ,  $w \in \mathbb{R}$  and  $\varphi(t)$  is such that the integral on the right side exits. In recently some new integral inequalities involving this operator have appeared in the literature (see, e.g., ([1],[9]-[15]).

It is easy to verify that  $\mathcal{J}_{\rho,\lambda,a+;w}^\sigma \varphi(x)$  and  $\mathcal{J}_{\rho,\lambda,b-;w}^\sigma \varphi(x)$  are bounded integral operators on  $L(a, b)$ , if

$$\mathfrak{M} := \mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] < \infty. \quad (1.8)$$

In fact, for  $\varphi \in L(a, b)$ , we have

$$\|\mathcal{J}_{\rho,\lambda,a+;w}^\sigma \varphi(x)\|_1 \leq \mathfrak{M}(b-a)^\lambda \|\varphi\|_1 \quad (1.9)$$

and

$$\|\mathcal{J}_{\rho,\lambda,b-;w}^\sigma \varphi(x)\|_1 \leq \mathfrak{M}(b-a)^\lambda \|\varphi\|_1, \quad (1.10)$$

where

$$\|\varphi\|_p := \left( \int_a^b |\varphi(t)|^p dt \right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient  $\sigma(k)$ . For instance the classical Riemann-Liouville fractional integrals  $J_{a+}^\alpha$  and  $J_{b-}^\alpha$  of order  $\alpha$  follow easily by setting  $\lambda = \alpha$ ,  $\sigma(0) = 1$  and  $w = 0$  in (1.6) and (1.7).

Here, motivated by the works in ([4],[5],[7]), we aim at establishing certain new Hermite-Hadamard type inequalities associated with the fractional integral operators with exponential kernel and a general class of fractional integral operators by using harmonically convex functions. Relevant connections of the results presented here are also pointed out.

## 2. MEAN RESULTS

Firstly, we will present Hermite-Hadamard inequalities for harmonically convex function via fractional integral operators with exponential kernel. We henceforth in denote,  $\mathcal{A} = \frac{1-\alpha}{\alpha} \frac{b-a}{ab}$  for  $\alpha \in (0, 1)$ .

**Theorem 3.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function with  $0 \leq a < b$  and  $f \in L_1[a, b]$ . If  $f$  is a harmonically convex function on  $[a, b]$ , then the following inequalities for fractional integral operators with exponential kernel hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1-\alpha}{2[1-\exp(-\mathcal{A})]} \left[ I_{\frac{1}{a}}^\alpha (f \circ g) \left(\frac{1}{b}\right) + I_{\frac{1}{b}}^\alpha (f \circ g) \left(\frac{1}{a}\right) \right] \leq \frac{f(a) + f(b)}{2}.$$

*Proof.* Since  $f$  is a harmonically convex function on  $[a, b]$ , we have for all  $x, y \in [a, b]$

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x) + f(y)}{2}.$$

For  $x = \frac{ab}{tb+(1-t)a}$ ,  $y = \frac{ab}{ta+(1-t)b}$ , we obtain

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)}{2}. \tag{2.1}$$

Multiplying both sides of (2.1) by  $\exp(-\mathcal{A}t)$ , then integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we get

$$\begin{aligned} & \int_0^1 f\left(\frac{2ab}{a+b}\right) \exp(-\mathcal{A}t) dt \\ & \leq \frac{1}{2} \left\{ \int_0^1 \exp(-\mathcal{A}t) f\left(\frac{ab}{tb+(1-t)a}\right) dt + \int_0^1 \exp(-\mathcal{A}t) f\left(\frac{ab}{ta+(1-t)b}\right) dt \right\}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \frac{2(1 - \exp(-\mathcal{A}))}{\mathcal{A}} f\left(\frac{2ab}{a+b}\right) \\ & \leq \frac{ab}{b-a} \left[ \int_{\frac{1}{b}}^{\frac{1}{a}} \exp\left\{-\frac{1-\alpha}{\alpha} \left(\frac{b-a}{ab}\right) \left(s - \frac{1}{b}\right) \left(\frac{ab}{b-a}\right)\right\} f\left(\frac{1}{s}\right) ds \right. \\ & \quad \left. + \int_{\frac{1}{b}}^{\frac{1}{a}} \exp\left\{-\frac{1-\alpha}{\alpha} \left(\frac{b-a}{ab}\right) \left(\frac{1}{a} - s\right) \left(\frac{ab}{b-a}\right)\right\} f\left(\frac{1}{s}\right) ds \right] \\ & = \frac{ab\alpha}{b-a} \left[ \frac{1}{\alpha} \int_{\frac{1}{b}}^{\frac{1}{a}} \exp\left\{-\frac{1-\alpha}{\alpha} \left(s - \frac{1}{b}\right)\right\} f\left(\frac{1}{s}\right) ds \right. \\ & \quad \left. + \frac{1}{\alpha} \int_{\frac{1}{b}}^{\frac{1}{a}} \exp\left\{-\frac{1-\alpha}{\alpha} \left(\frac{1}{a} - s\right)\right\} f\left(\frac{1}{s}\right) ds \right] \\ & = \frac{ab\alpha}{b-a} \left[ I_{\frac{1}{a}}^{\alpha}(f \circ g)\left(\frac{1}{b}\right) + I_{\frac{1}{b}}^{\alpha}(f \circ g)\left(\frac{1}{a}\right) \right] \end{aligned}$$

where  $g(x) = \frac{1}{x}$  and the first inequality is proved.

For the proof of the second inequality in (1.2), we first note that if  $f$  is a harmonically convex function then, for  $t \in [0, 1]$ , it yields

$$f\left(\frac{ab}{tb+(1-t)a}\right) \leq tf(b) + (1-t)f(a)$$

and

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(a) + (1-t)f(b).$$

By adding these inequalities we have

$$f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right) \leq f(a) + f(b). \tag{2.2}$$

Then multiplying both sides of (2.2) by  $\exp(-At)$ , and integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\begin{aligned} & \int_0^1 \exp(-At) f\left(\frac{ab}{tb+(1-t)a}\right) dt + \int_0^1 \exp(-At) f\left(\frac{ab}{ta+(1-t)b}\right) dt \\ & \leq [f(a) + f(b)] \int_0^1 \exp(-At) dt. \end{aligned}$$

Using the similar arguments as above we can show that

$$\frac{1-\alpha}{(1-\exp(-\mathcal{A}))} \left[ I_{\frac{1}{a}}^\alpha(fog)\left(\frac{1}{b}\right) + I_{\frac{1}{b}}^\alpha(fog)\left(\frac{1}{a}\right) \right] \leq [f(a) + f(b)].$$

So, the proof is completed.  $\square$

Now, using a general fractional integral operators introduced by Raina [7] and Agarwal *et al.* [1], we will prove Hermite-Hadamard inequalities for harmonically convex functions.

**Theorem 4.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function such that  $f \in L[a, b]$ , where  $a, b \in \mathbb{R}$  with  $a < b$ . If  $f$  is a harmonically convex function on  $[a, b]$ , then the following inequalities for fractional integral operators holds:*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right) \\ & \leq \left(\frac{ab}{b-a}\right)^\lambda \frac{1}{2\mathcal{F}_{\rho,\lambda+1}^\sigma[w\left(\frac{b-a}{ab}\right)^\rho]} \left[ J_{\rho,\lambda,\frac{1}{a}^-;w}^\alpha(fog)\left(\frac{1}{b}\right) + J_{\rho,\lambda,\frac{1}{b}^+;w}^\alpha(fog)\left(\frac{1}{a}\right) \right] \\ & \leq \left(\frac{f(a)+f(b)}{2}\right) \end{aligned} \quad (2.3)$$

where  $\lambda > 0$ ,  $g(x) = \frac{1}{x}$ .

*Proof.* For  $t \in [0, 1]$ , let  $x = \frac{ab}{tb+(1-t)a}$ ,  $y = \frac{ab}{ta+(1-t)b}$ . The harmonically convexity of  $f$  yields

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)}{2}. \quad (2.4)$$

Multiplying both sides of (2.4) by  $t^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma[w\left(\frac{b-a}{ab}\right)^\rho t^\rho]$ , then integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\begin{aligned} & 2f\left(\frac{2ab}{a+b}\right) \int_0^1 t^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[w\left(\frac{b-a}{ab}\right)^\rho t^\rho\right] dt \\ & \leq \int_0^1 t^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[w\left(\frac{b-a}{ab}\right)^\rho t^\rho\right] f\left(\frac{ab}{tb+(1-t)a}\right) dt \\ & \quad + \int_0^1 t^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[w\left(\frac{b-a}{ab}\right)^\rho t^\rho\right] f\left(\frac{ab}{ta+(1-t)b}\right) dt. \end{aligned}$$

The following integrals calculated by using (1.5), we have

$$\begin{aligned}
\int_0^1 t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho t^\rho \right] dt &= \int_0^1 t^{\lambda-1} \left[ \sum_{k=0}^{\infty} \frac{\sigma(k) w^k \left( \frac{b-a}{ab} \right)^{\rho k}}{\Gamma(\rho k + \lambda)} t^{\rho k} \right] dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k) w^k \left( \frac{b-a}{ab} \right)^{\rho k}}{\Gamma(\rho k + \lambda)} \int_0^1 t^{\lambda+\rho k-1} dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k) w^k \left( \frac{b-a}{ab} \right)^{\rho k}}{\Gamma(\rho k + \lambda + 1)} \\
&= \mathcal{F}_{\rho,\lambda+1}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho \right],
\end{aligned}$$

$$\begin{aligned}
\int_0^1 t^{\lambda-1} \mathcal{F}_{\rho,\lambda+1}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho t^\rho \right] f \left( \frac{ab}{tb + (1-t)a} \right) dt &= \int_{\frac{1}{b}}^{\frac{1}{a}} \left( s - \frac{1}{b} \right)^{\lambda-1} \left( \frac{ab}{b-a} \right)^{\lambda-1} \\
&\times \left[ \sum_{k=0}^{\infty} \frac{\sigma(k) w^k \left( \frac{b-a}{ab} \right)^{\rho k}}{\Gamma(\rho k + \lambda)} \left( s - \frac{1}{b} \right)^{\rho k} \left( \frac{ab}{b-a} \right)^{\rho k} \right] \left( \frac{1}{s} \right) \left( \frac{ab}{b-a} \right) ds \\
&= \left( \frac{ab}{b-a} \right)^\lambda \int_{\frac{1}{b}}^{\frac{1}{a}} \left( s - \frac{1}{b} \right)^{\lambda-1} \times \left[ \sum_{k=0}^{\infty} \frac{\sigma(k) w^k \left( \frac{b-a}{ab} \right)^{\rho k}}{\Gamma(\rho k + \lambda)} \left( s - \frac{1}{b} \right)^{\rho k} \right] \left( \frac{1}{s} \right) ds \\
&= \left( \frac{ab}{b-a} \right)^\lambda \int_{\frac{1}{b}}^{\frac{1}{a}} \left( s - \frac{1}{b} \right)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma \left[ w \left( s - \frac{1}{b} \right)^\rho \right] \left( \frac{1}{s} \right) ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 t^{\lambda-1} \mathcal{F}_{\rho,\lambda+1}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho t^\rho \right] f \left( \frac{ab}{ta + (1-t)b} \right) dt &= \int_{\frac{1}{b}}^{\frac{1}{a}} \left( \frac{1}{a} - s \right)^{\lambda-1} \left( \frac{ab}{b-a} \right)^{\lambda-1} \\
&\times \left[ \sum_{k=0}^{\infty} \frac{\sigma(k) w^k \left( \frac{b-a}{ab} \right)^{\rho k}}{\Gamma(\rho k + \lambda)} \left( \frac{1}{a} - s \right)^{\rho k} \left( \frac{ab}{b-a} \right)^{\rho k} \right] \left( \frac{1}{s} \right) \left( \frac{ab}{b-a} \right) ds \\
&= \left( \frac{ab}{b-a} \right)^\lambda \int_{\frac{1}{b}}^{\frac{1}{a}} \left( \frac{1}{a} - s \right)^{\lambda-1} \\
&\times \left[ \sum_{k=0}^{\infty} \frac{\sigma(k) w^k \left( \frac{b-a}{ab} \right)^{\rho k}}{\Gamma(\rho k + \lambda)} \left( \frac{1}{a} - s \right)^{\rho k} \right] \left( \frac{1}{s} \right) ds \\
&= \left( \frac{ab}{b-a} \right)^\lambda \int_{\frac{1}{b}}^{\frac{1}{a}} \left( \frac{1}{a} - s \right)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma \left[ w \left( \frac{1}{a} - s \right)^\rho \right] \left( \frac{1}{s} \right) ds.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} & 2\mathcal{F}_{\rho,\lambda+1}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho \right] f \left( \frac{2ab}{a+b} \right) \\ & \leq \left( \frac{ab}{a+b} \right)^\lambda \left[ J_{\rho,\lambda,\frac{1}{a}^-;w}^\alpha (f \circ g) \left( \frac{1}{b} \right) + J_{\rho,\lambda,\frac{1}{b}^+;w}^\alpha (f \circ g) \left( \frac{1}{a} \right) \right] \end{aligned}$$

where  $g(x) = \frac{1}{x}$  and the first inequality is proved. For the proof of the second inequality in (2.3), we first note that if  $f$  is a harmonically convex function, then for  $t \in [a, b]$ , we have

$$f \left( \frac{ab}{tb + (1-t)a} \right) + f \left( \frac{ab}{ta + (1-t)b} \right) \leq f(a) + f(b). \quad (2.5)$$

Then multiplying both sides of (2.5) by  $t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [w \left( \frac{b-a}{ab} \right)^\rho t^\rho]$  and integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\begin{aligned} & \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho t^\rho \right] f \left( \frac{ab}{tb + (1-t)a} \right) dt \\ & + \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho t^\rho \right] f \left( \frac{ab}{ta + (1-t)b} \right) dt \\ & \leq [f(a) + f(b)] \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho t^\rho \right] dt. \end{aligned}$$

Using the similar arguments as above we can show that

$$\begin{aligned} & \left( \frac{ab}{a+b} \right)^\lambda \left[ J_{\rho,\lambda,\frac{1}{a}^-;w}^\alpha (f \circ g) \left( \frac{1}{b} \right) + J_{\rho,\lambda,\frac{1}{b}^+;w}^\alpha (f \circ g) \left( \frac{1}{a} \right) \right] \\ & \leq \mathcal{F}_{\rho,\lambda}^\sigma \left[ w \left( \frac{b-a}{ab} \right)^\rho t^\rho \right] [f(a) + f(b)]. \end{aligned}$$

So, the proof is completed.  $\square$

**Remark 1.** If in Theorem 4, we get  $\lambda = \alpha$ ,  $\sigma(0) = 1$ ,  $w = 0$ , then the inequalities (2.3) become the inequalities (1.4).

**Remark 2.** If in Theorem 4, we get  $\lambda = 1$ ,  $\sigma(0) = 1$ ,  $w = 0$ , then the inequalities (2.3) become the inequalities (1.3).

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