Konuralp Journal of Mathematics
Volume 3 No. 2 pp. 33–41 (2015) ©KJM

ON A SUBCLASS OF UNIFORMLY QUASI CONVEX FUNCTIONS OF ORDER α

D. VAMSHEE KRISHNA, B. VENKATESWARLU, AND T. RAMREDDY

Abstract. In this paper, we introduce two new classes of analytic functions namely uniformly quasi convex functions of order α and quasi uniformly convex functions of order α denoted by $UQCV(\alpha)$ and $QUCV(\alpha)$ $(0 \leq \alpha < 1)$ respectively and study certain properties of functions belonging to these two classes. Further, we obtain a necessary and sufficient condition for the function $f(z)$ to be in the class $UQCV(\alpha)$. These results are generalized recent results of Rajalakshmi Rajagopal and Selvaraj [7].

1. Introduction and Preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Let S denote the subclass of A which are univalent in U.

Definition 1.1. [3] A function f given in (1.1) is said to be uniformly convex in U, if f is convex and has the property that for every circular arc γ contained in U with centre ξ the arc $f(\gamma)$ is convex. The class of uniformly convex functions is denoted by UCV. The analytical characterization of the function $f \in UCV$ was given by Goodman [3].

Theorem 1.1. [3] *A function f of the form (1.1) is in UCV if and only if* $Re\left\{1 + (z-\xi)\frac{f''(z)}{f'(z)}\right\} > 0, \forall\ (z,\xi) \in U \times U$ *and* $z \neq \xi$.

Theorem 1.2. [8] *A function f of the form (1.1) is in UCV if and only if* $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \forall\ z \in U$.

Date: January 1, 2013 and, in revised form, February 2, 2013.

2010 *Mathematics Subject Classification.* 30C45; 30C50.

Key words and phrases. Analytic function, uniformly quasi convex function, quasi uniformly convex function, positive real function.

Definition 1.2. [5] A function f of the form (1.1) is said to be quasi convex in U if there exists a convex function g in U with $g(0) = 0 = g'(0) - 1$ such that $Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > 0, \forall\ z \in U.$

The class of quasi convex functions is denoted by c^*.

Definition 1.3. [6] A function f of the form (1.1) is said to be close-to- uniformly convex if there exists a uniformly convex function g in U such that $Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0, z\ \in U.$

The class of all close-to-uniformly convex functions is denoted by $CUCV$. The subclasses uniformly quasi convex functions and quasi uniformly convex functions denoted by $UQCV$ and $QUCV$ respectively of S are introduced and studied by Rajalakshmi Rajagopal and Selvaraj [7]. The following Definitions are due to them.

Definition 1.4. A function $f(z)$ in A is said to be uniformly quasi convex in U if there exists a uniformly convex function g in U with $g(0) = 0 = g'(0) - 1$ such that $Re\left\{\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right\} > 0,\ \forall\ z, \xi \in U, z \neq \xi.$

The class of all such functions is denoted by $UQCV$.

Definition 1.5. A function $f(z)$ is A is said to be quasi uniformly convex in U if there exists a uniformly convex function g in U with $g(0) = 0 = g'(0) - 1$ such that $Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > 0,\ \forall\ z \in U.$

The class of all quasi uniformly convex functions is denoted by $QUCV$. From the Definitions 1.4 and 1.5, it is observed that $QUCV \subset UQCV$. Now, we introduce and study certain important properties of the following two classes.

2. Main results:

Definition 2.1. A function $f(z)$ in A is said to be uniformly quasi convex function of order α $(0 \leq \alpha < 1)$ if there exists a uniformly convex function g in U with $g(0) = g'(0) - 1$ such that

$$Re\left\{\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right\} > \alpha,\ \forall\ (z, \xi) \in U \times U \text{ and } z \neq \xi. \tag{2.1}$$

We denote the class of uniformly quasi convex functions of order α by $UQCV(\alpha)$.

Definition 2.2. A function $f(z)$ in A is said to be uniformly convex of function of order α $(0 \leq \alpha < 1)$ if and only if

$$Re\left\{1 + (z-\xi)\frac{f''(z)}{f'(z)}\right\} > \alpha,\ \forall\ (z, \xi) \in U \times U \text{ and } z \neq \xi. \tag{2.2}$$

The class of such functions is denoted by $UCV(\alpha)$.

Definition 2.3. A function $f(z)$ in A is said to be quasi convex function of order α $(0 \leq \alpha < 1)$ if there exists a convex function g in U with $g(0) = 0 = g'(0) - 1$ such that

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha,\ \forall\ z \in U. \tag{2.3}$$

The class of such functions is denoted by $c^*(\alpha)$.

Definition 2.4. A function $f(z)$ in A is said to be quasi uniformly convex function of order α $(0 \leq \alpha < 1)$ if there exists a uniformly convex function g in U with $g(0) = 0 = g'(0) - 1$ such that

$$Re\left\{\frac{\{z\ f'(z)\ \}'}{g'(z)}\right\} > \alpha,\ \forall\ z \in U. \tag{2.4}$$

The class of all quasi uniformly convex functions is denoted by $QUCV(\alpha)$.

Definition 2.5. A function $f(z)$ in A is said to be close-to-convex function of order α $(0 \leq \alpha < 1)$ if there exists a convex function g in U such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha,\ \forall\ z \in U. \tag{2.5}$$

The class of all close - to- convex functions of order α is denoted by $K(\alpha)$.

Definition 2.6. A function $f(z)$ in A is said to be close-to- uniformly convex function of order α $(0 \leq \alpha < 1)$ if there exists a uniformly convex function g in U such that

$$Re\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha,\ \forall\ z \in U. \tag{2.6}$$

The class of all close - to- uniformly convex functions of order α is denoted by $CUCV(\alpha)$. From the above Definitions, we observe the following conclusions:

1. Choosing $g(z) = f(z)$ in (2.1), where $g(z) \in UCV$, we obtain

$$Re\left\{\frac{\{(z-\xi)f'(z)\}'}{f'(z)}\right\} = Re\left\{1 + (z-\xi)\frac{f''(z)}{f'(z)}\right\} > \alpha,$$
$$\text{for } z \neq \xi \text{ in } |z| < 1 \text{ for } (0 \leq \alpha < 1).$$

   From this result and in view of Definition 2.2, we get

$$UCV(\alpha) \subset UQCV(\alpha) \tag{2.7}$$

2. Taking $\xi = 0$ in (2.1), we obtain

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha,\ \text{for } z \in\ U,\ \text{for } (0 \leq \alpha < 1). \tag{2.8}$$

   From the Definition 2.3, we observe that

$$UQCV(\alpha) \subset c^*(\alpha). \tag{2.9}$$

   From the expressions (2.7) and (2.9), we obtain

$$UCV(\alpha) \subset UQCV(\alpha) \subset c^*(\alpha). \tag{2.10}$$

   Therefore, an immediate consequence of (2.10) is that every uniformly quasi-convex function of order α is univalent.

3. Choosing $t(z) = zf'(z)$ in (2.3), we get

$$Re\left\{\frac{t'(z)}{g'(z)}\right\} > \alpha,\ \text{for } (0 \leq \alpha < 1).$$

(2.11) From the Definition 2.5, we observe that $c^*(\alpha) \subset K(\alpha)$.

4. Taking $\xi = 0$ in (2.1), we obtain

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha, \text{ for } z \in U, \tag{2.12}$$

which implies that $QUCV(\alpha) \subset UQCV(\alpha)$.

Lemma 2.1. *If $g(z) \in UCV$, then*

$$|g'(z)| \leq \frac{1}{1-r}, \; \textit{for } |z| = r < 1, \; z \in U.$$

Proof. Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \Leftrightarrow g'(z) = 1 + \sum_{n=2}^{\infty} nb_n z^{n-1}$. Taking modulus on both sides of $g'(z)$, using the facts $|a+b| \leq |a| + |b|$ and $|ab| = |a||b|$, we get

$$|g'(z)| = \left|1 + \sum_{n=2}^{\infty} nb_n z^{n-1}\right| \Leftrightarrow |g'(z)| \leq \left[1 + \sum_{n=2}^{\infty} n|b_n|r^{n-1}\right]. \tag{2.13}$$

For the function $g(z) \in UCV$ (according to Goodman [2]), we have

$$|b_n| \leq \frac{1}{n}, \; \forall n \geq 2. \tag{2.14}$$

Simplifying the expressions (2.13) and (2.14), we obtain

$$|g'(z)| \leq \left[1 + \sum_{n=2}^{\infty} r^{n-1}\right] = \frac{1}{1-r}.$$

Hence the Lemma. □

Theorem 2.1. *Let $f(z)$ be in A. Then f is uniformly quasi-convex function of order α $(0 \leq \alpha < 1)$ if and only if* $Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha + \left|\frac{zf''(z)}{g'(z)}\right|$.

Proof. Let $f(z) \in UQCV(\alpha)$ $(0 \leq \alpha < 1)$. By virtue of Definition 2.1, there exists uniformly convex function $g(z) \in U$ such that

$$Re\left\{\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right\} > \alpha, \; \forall \, (z,\xi) \in U \times U \text{ and } z \neq \xi. \tag{2.15}$$

$$\Leftrightarrow Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} > \alpha + Re\left\{\frac{\xi f''(z)}{g'(z)}\right\}. \tag{2.16}$$

If we choose $\xi = ze^{i\beta}$ in a suitable way, for some real β, we get

$$Re\left\{\frac{\xi f''(z)}{g'(z)}\right\} = \left|\frac{ze^{i\beta} f''(z)}{g'(z)}\right| = \left|\frac{zf''(z)}{g'(z)}\right|. \tag{2.17}$$

From the expressions (2.16) and (2.17), we obtain

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} \geq \alpha + \left|\frac{zf''(z)}{g'(z)}\right|. \tag{2.18}$$

Hence, the condition is necessary.

Conversely, suppose the condition given by (2.18) is true.
Let ξ be an arbitrary but fixed point in the unit disc U. Since the quotient of two analytic functions, whose real part is harmonic and hence the function $Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\}$ becomes harmonic, provided $g(z) \in UCV$.

Therefore, by the minimum principle it is enough to show that the result is true for $|z| = \rho > |\xi|,\ \rho < 1$. From (2.18), for $|\xi| < |z| = \rho < 1$ and using the fact $Re(z) \le |z|$, we get

$$Re\left\{\frac{\{zf'(z)\}'}{g'(z)}\right\} \ge \left[\alpha + \left|\frac{zf''(z)}{g'(z)}\right|\right] > \left[\alpha + \left|\frac{\xi f''(z)}{g'(z)}\right|\right] \ge \left[\alpha + Re\left\{\frac{\xi f''(z)}{g'(z)}\right\}\right].$$

$$\Leftrightarrow\ Re\Big\{\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\Big\}\ \ge\ \alpha,$$

which shows that $f(z) \in UQCV(\alpha)$. Hence the condition is sufficient. □

Remark 2.1. Since $\frac{\{zf'(z)\}'}{g'(z)}$ is analytic in $|z| < 1$ and maps 0 to 1, the open mapping theorem implies that equality in (2.18) is not possible.

Theorem 2.2. *If $f(z) \in UQCV(\alpha)$ $(0 \le \alpha < 1)$ then*

$$|a_n| \le \frac{1}{n}\left[(1-\alpha) + \frac{\alpha}{(2n-1)}\right],\ n \ge 2.$$

Proof. Let $f(z) \in UQCV(\alpha)$, from the Definition 2.1, there exists uniformly convex function g given in Lemma 2.1, such that

$$Re\left[\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right] > \alpha,\ \forall\ (z,\xi) \in U \times U \text{ and } z \ne \xi$$

$$\Leftrightarrow Re\left[\frac{(z-\xi)f''(z) + f'(z)}{g'(z)}\right] > \alpha. \tag{2.19}$$

Choosing $\xi = -z$ in (2.19), it takes the form

$$Re\left[\frac{2zf''(z)}{g'(z)} + \frac{f'(z)}{g'(z)}\right] > \alpha. \tag{2.20}$$

Let $p(z) = \frac{2zf''(z)}{g'(z)} + \frac{f'(z)}{g'(z)}$, which is incompatible with $p(z) = \frac{1+(1-2\alpha)w(z)}{1-w(z)}$, where $w(z)$ is schwarz's function in the unit disc U and $p(z) = \sum_{n=0}^{\infty} p_n z^n$ with $p_0 = 1$, then we have

$$2zf''(z) + f'(z) = p(z)g'(z). \tag{2.21}$$

Replacing $f'(z), f''(z), g'(z)$ and $p(z)$ by their equivalent expressions in series in (2.21), after simplifying, we get

$$1 + \sum_{n=2}^{\infty} \{2n(n-1) + n\}\, a_n z^{n-1} = \big\{1 + p_1 z + p_2 z^2 + \cdots + p_{n-1} z^{n-1} \tag{2.22}$$
$$+ p_n z^n + \cdots\big\} \times \big\{1 + 2b_2 z + 3b_3 z^2 + \cdots + (n-1)b_{n-1} z^{n-2} + nb_n z^{n-1} + \cdots\big\}.$$

Equating the coefficient of z^{n-1} on both sides of (2.22), we have

$$\big[2n(n-1) + n\big]a_n = [nb_n + p_1(n-1)b_{n-1} + p_2(n-2)b_{n-2} + \tag{2.23}$$
$$\cdots + p_{n-2}2b_2 + p_{n-1}].$$

Taking the modulus on both sides of (2.23) and using the facts, for the functions with positive real part, $|p_0| = 1,\ |p_n| \le 2(1-\alpha),\ \forall\ n \ge 1$ with $0 \le \alpha < 1$ and the result from (2.14), which simplifies to

$$\begin{aligned} n(2n-1)|a_n| &\leq [(1-\alpha)(2n-1)+\alpha] \\ \Leftrightarrow |a_n| &\leq \frac{1}{n}\left[(1-\alpha)+\frac{\alpha}{2n-1}\right],\ \forall\ n\geq 2. \end{aligned}$$

Hence the Theorem. □

Theorem 2.3. *If* $f \in UQCV(\alpha)$ $(0 \leq \alpha < 1)$ *then*

$$|2f'(z)-f(z)| \leq \left[\frac{2(1-\alpha)r}{1-r}+(1-2\alpha)\ \log(1-r)\right],\ \textit{for}\ |z|=r<1.$$

Proof. Let $f \in UQCV(\alpha)$, from the Definition 2.1, there exists a uniformly convex function g such that

$$Re\left[\frac{\{(z-\xi)f'(z)\}'}{g'(z)}\right] > \alpha,\ z,\ \xi \in U,\ \text{where}\ z \neq \xi$$

$$\Rightarrow Re\left[\frac{(z-\xi)f''(z)+f'(z)}{g'(z)}\right] > \alpha \tag{2.24}$$

Choosing $\xi = -z$ in (2.24), we get

$$Re\left[\frac{2zf''(z)}{g'(z)}+\frac{f'(z)}{g'(z)}\right] > \alpha. \tag{2.25}$$

Put $p(z) = \frac{2zf''(z)}{g'(z)}+\frac{f'(z)}{g'(z)}$ in (2.25), which takes the form $\text{Re}(p(z)) > \alpha$ $(0 \leq \alpha < 1)$ so that we can have

$$[2zf''(z)+f'(z)] = p(z)g'(z) \Leftrightarrow [2zf'(z)-f(z)]' = p(z)g'(z).$$

Taking modulus on both sides, we get

$$\big|[2zf'(z)-f(z)]'\big| = \big|p(z)\big|\big|g'(z)\big|.$$

Using the known result for $|p(z)|$ (according to Goodman [2]) and Lemma 2.1, resolving into partial fractions on the right hand side, we obtain

$$\left|[2zf'(z)-f(z)]'\right| \leq \left[\frac{1+(1-2\alpha)r}{(1-r)^2}\right] = \left[\frac{2\alpha-1}{1-r}+\frac{2(1-\alpha)}{(1-r)^2}\right],\ \text{for}\ |z|=r<1. \tag{2.26}$$

On integrating along a line segment from 0 to $|z| = r$ in (2.26) and using the fact $|f(z)| \leq \int_0^z |f'(z)||dz|$, which simplifies to give

$$[2zf'(z)-f(z)] \leq \left[\frac{2(1-\alpha)r}{1-r}+(1-2\alpha)\log(1-r)\right],\quad (0 \leq \alpha < 1).$$

Hence the Theorem. □

Theorem 2.4. $f(z) \in QUCV(\alpha) \Leftrightarrow zf' \in CUCV(\alpha)$ $(0 \leq \alpha < 1)$.

Proof. Let $f(z) \in QUCV(\alpha)$, from the Definition 2.4 , we have

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \alpha. \tag{2.27}$$

Choosing $zf'(z) = F(z)$ in (2.27), we get

$$Re\left[\frac{F'(z)}{g'(z)}\right] >\ \alpha,\quad \text{for}\ |z|<1.$$

From the Definition 2.6, we conclude that $F = zf' \in CUCV(\alpha)$.

Conversely, let $F = zf' \in CUCV(\alpha$, from the Definition 2.6, we have

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \alpha, \; |z| < 1.$$

In view of Definition 2.4, we conclude that $f(z) \in QUCV(\alpha)$.
Hence the Theorem. □

Theorem 2.5. *If* $f \in QUCV(\alpha)$ *then* $f \in CUCV(\alpha)$.

Proof. Let $f \in QUCV(\alpha)$, then by a result obtained by Libera [4], we have

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \alpha \Leftrightarrow \left[\frac{zf'(z)}{g(z)}\right] > \alpha, \; z \in U, \tag{2.28}$$

where $g \in UCV$, which is also in S_p, denotes the class of parabolic star like functions introduced by Ronning [9]. Geometrically S_p is the class of functions f given (1.1), for which $\frac{zf'(z)}{f(z)}$ takes its value in the interior of the parabola in the right half plane symmetric about the real axis with vertex at $\left(\frac{1}{2}, 0\right)$.

$$\text{Put } h(z) = \int_0^z \frac{g(t)}{t} dt \Leftrightarrow h'(z) = \frac{g(z)}{z} \Leftrightarrow g(z) = zh'(z) \in S_p. \tag{2.29}$$

By the relation between UCV and S_p given in terms of the Alexander type Theorem [1] by Ronning [8], we have

$$zh'(z) \in S_p \Leftrightarrow h(z) \in UCV.$$

Simplifying the relations (2.28) and (2.29), we obtain

$$Re\left[\frac{f'(z)}{h'(z)}\right] > \alpha, \; z \in U \text{ for } (0 \leq \alpha < 1).$$

Since $h(z) \in UCV$, from the Definition 2.7, we conclude that $f(z) \in CUCV(\alpha)$.
Hence the Theorem. □

Remark 2.2. From the Theorems 2.4 and 2.5, we conclude that
if $f(z) \in QUCV(\alpha)$ then both $f(z)$ and $zf'(z)$ belongs to $CUCV(\alpha)$.

Theorem 2.6. *If* $f \in QUCV(\alpha)$ $(0 \leq \alpha < 1)$ *then*

$$|a_n| \leq \frac{1}{n^2}\big[2n(1-\alpha) + (2\alpha - 1)\big], \; \forall \; n \geq 2.$$

Proof. Let $f \in QUCV(\alpha)$, from the Definition 2.4, there exists uniformly convex function g in U such that

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \alpha, \quad z \in U. \tag{2.30}$$

Choosing $p(z) = \dfrac{\{zf'(z)\}'}{g'(z)}$ in (2.30), we can have

$$Re(p(z)) > \alpha, \text{ so that } \{zf'(z)\}' = p(z)g'(z).$$

Applying the same procedure described in Theorem 2.2, we obtain

$$|a_n| \leq \frac{1}{n^2}\big[(1-\alpha)(2n-1)+\alpha\big],\ \forall\ n \geq 2. \tag{2.31}$$

Hence the Theorem. □

Theorem 2.7. *If $f \in QUCV(\alpha)$ $(0 \leq \alpha < 1)$, then*

$$|zf'(z)| \leq \left[\frac{2(1-\alpha)r}{1-r} + (1-2\alpha)\log(1-r)\right], for\ |z| \leq r < 1.$$

Proof. Let $f \in QUCV(\alpha)$, from the Definition 2.1, we have

$$Re\left[\frac{\{zf'(z)\}'}{g'(z)}\right] > \alpha \tag{2.32}$$

$$\text{Put} p(z) = \frac{\{zf'(z)\}'}{g'(z)} \text{in (2.32), we get } Re\{p(z)\} > \alpha,$$

so that, we can have

$$\{zf'(z)\}' = p(z)g'(z). \tag{2.33}$$

Taking modulus on both sides of (2.33), which takes the form

$$|zf'(z))'| = |p(z)||g'(z)|.$$

Applying the same procedure described in Theorem 2.3, we obtain

$$|zf'(z)| \leq \left[\frac{2(1-\alpha)r}{(1-r)} + (1-2\alpha)\log(1-r)\right].$$

Hence the Theorem. □

Acknowledgement: The authors would like to express sincere thanks to the esteemed Referee(s) for their careful readings, valuable suggestions and comments, which helped them to improve the presentation of the paper.

References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Annals. Math.*, 17(1915), 12 -22.
[2] A. W. Goodman, On uniformly convex functions, *Ann. Polon. Math.*, 56(1991), 87 - 92.
[3] A. W. Goodman, *Univalent functions vol. I and vol. II*, Mariner Publishing Comp. Inc., Tampa, Florida, 1983.
[4] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.,* 16(1965), 755 -758.
[5] K. I. Noor, On quasi convex functions and related topics, *Int. J. Math. Sci.,* 10(2)(1987), 241 - 258.
[6] K. S. Padmanabhan, On certain subclasses of Bazilevic functions, *Ind. J. Maths.*, 39(3)(1997).
[7] Rajalakshmi Rajagopal and C. Selvaraj, On a class of uniformly quasi-convex functions, *Bull. Calcutta Math. Soc.,* 95(2003), 199 -206.
[8] F. Ronning, A survey on uniformly convex and uniformly starlike functions, *Ann. Univ. Mariae Curie - Sklodowska Sect. A,* 47(1993), 123 - 134.
[9] F. Ronning, Uniformly convex functions and corresponding class of star like functions, *Proc. Amer. Math. Soc.,* 118(1) (1993), 189 - 196.

Department of Mathematics, GIT, GITAM University, Visakhapatnam- 530 045, A.P., India.
E-mail address: vamsheekrishna1972@gmail.com

Department of Mathematics, GIT, GITAM University, Visakhapatnam- 530 045, A.P., India.
E-mail address: bvlmaths@gmail.com

Department of Mathematics, Kakatiya University, Warangal- 506 009, T. S., India.
E-mail address: reddytr2@gmail.com