ON SOME ČEBYŠEV TYPE INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE \((h_1, h_2)\)-CONVEX ON THE CO-ORDINATES

B. MEFTAH AND K. BOUKERRIOU\(^*\)

Abstract. The aim of this paper is to establish some new Čebyšev type inequalities involving functions whose mixed partial derivatives are \((h_1, h_2)\)-convex on the co-ordinates.

1. Introduction

In 1882, Čebyšev [4] gave the following inequality:

\[
|T(f, g)| \leq \frac{1}{12} (b - a)^2 \|f'\|_\infty \|g'\|_\infty
\]

where \(f, g : [a, b] \to \mathbb{R}\) are absolutely continuous functions, whose first derivatives \(f'\) and \(g'\) are bounded,

\[
T(f, g) = \frac{1}{b - a} \int_a^b f(x) g(x) \, dx - \left( \frac{1}{b - a} \int_a^b f(x) \, dx \right) \left( \frac{1}{b - a} \int_a^b g(x) \, dx \right),
\]

and \(\|\cdot\|_\infty\) denotes the norm in \(L_\infty[a, b]\) defined as \(\|f\|_\infty = \text{ess sup}_{t \in [a, b]} |f(t)|\).

During the past few years many researchers have given considerable attention to the inequality (1.1), various generalizations, extensions and variants of this inequality have appeared in the literature, see [1, 3, 6, 8, 9, 10]. Recently, Guezane-Lakoud and Aissaoui [6] established new Čebyšev type inequalities similar to (1.1) for functions \(f, g\) defined on bidimensional intervals \(\Delta = [a, b] \times [c, d] \subset [0, \infty)^2\) whose mixed partial derivatives \(f_{st}\) and \(g_{st}\) are integrable and bounded. The authors of the paper [12] further extend these results in special cases when the mixed partial derivatives belong to certain classes of functions that generalize convex function on the co-ordinates.

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The main purpose of this work is to obtain new Čebyšev type inequalities involving functions whose mixed partial derivatives are \((h_1, h_2)\)-convex on the co-ordinates.

2. Preliminaries

Throughout this paper we denote by \(\Delta\) the bidimensional interval in \([0, \infty)^2\), \(\Delta := [a, b] \times [c, d]\) with \(a < b\) and \(c < d\), \(k = (b - a)(d - c)\) and \(f_{\lambda\alpha}\) for \(\frac{\partial^2 f}{\partial x \partial y}\).

**Definition 2.1** ([5]). A function \(f : \Delta \to \mathbb{R}\) is said to be convex on the co-ordinates on \(\Delta\), if the following inequality
\[
f (\lambda x + (1 - \lambda) t, \alpha y + (1 - \alpha) v) \leq \lambda f(x, y) + \lambda (1 - \alpha) f(x, v) + (1 - \lambda) \alpha f(t, y) + (1 - \lambda) (1 - \alpha) f(t, v)
\]
holds for all \(\lambda, \alpha \in [0, 1]\) and \((x, y), (x, v), (t, y), (t, v) \in \Delta\).

Clearly, every convex mapping \(f : \Delta \to \mathbb{R}\) is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex.

**Definition 2.2** ([2]). A function \(f : \Delta \to \mathbb{R}\) is said to be \(s\)-convex in the second sense on the co-ordinates on \(\Delta\), if the following inequality
\[
f (\lambda x + (1 - \lambda) t, \alpha y + (1 - \alpha) v) \leq \lambda^s \alpha^s f(x, y) + \lambda^s (1 - \alpha)^s f(x, v) + (1 - \lambda)^s \alpha^s f(t, y) + (1 - \lambda)^s (1 - \alpha)^s f(t, v)
\]
holds for all \(\lambda, \alpha \in [0, 1]\) and \((x, y), (x, v), (t, y), (t, v) \in \Delta\), for some fixed \(s \in (0, 1]\).

\(s\)-convexity on the co-ordinates does not imply the \(s\)-convexity, that is there exist functions which are \(s\)-convex on the co-ordinates but are not \(s\)-convex.

**Definition 2.3** ([7]). Let \(h : J \subseteq \mathbb{R} \to \mathbb{R}\) be a positive function. A mapping \(f : \Delta \to \mathbb{R}\) is said to be \(h\)-convex on \(\Delta\), if the following inequality
\[
f (\alpha x + (1 - \alpha) t, \alpha y + (1 - \alpha) v) \leq h(\alpha) f(x, y) + h(1 - \alpha) f(t, v)
\]
holds, for all \((x, y), (t, v) \in \Delta\) and \(\alpha \in (0, 1)\).

**Definition 2.4** ([7]). A function \(f : \Delta \to \mathbb{R}\) is said to be \((h_1, h_2)\)-convex on the coordinates on \(\Delta\), if the following inequality
\[
f (\lambda x + (1 - \lambda) t, \alpha y + (1 - \alpha) v) \leq h_1(\lambda) h_2(\alpha) f(x, y) + h_1(\lambda) h_2(1 - \alpha) f(x, v) + h_1(1 - \lambda) h_2(\alpha) f(t, y) + h_1(1 - \lambda) h_2(1 - \alpha) f(t, v)
\]
holds for all \(\lambda, \alpha \in [0, 1]\) and \((x, y), (x, v), (t, y), (t, v) \in \Delta\).

\(h\)-convexity on the co-ordinates does not imply the \(h\)-convexity, that is there exist functions which are \(h\)-convex on the co-ordinates but are not \(h\)-convex.

**Lemma 2.1** (Lemma 1. [11]). Let \(f : \Delta \to \mathbb{R}\) be a partial differentiable mapping on \(\Delta\) in \(\mathbb{R}^2\). If \(f_{\lambda\alpha} \in L_1(\Delta)\), then for any
Let theorem 3.1.

\[ f(x, y) = \frac{1}{b-a} \int_a^b f(t, y) dt + \frac{1}{d-c} \int_c^d f(x, v) dv - \frac{1}{k} \int_a^b \int_c^d f(t, v) dv dt \]

\[ + \frac{1}{k} \int_a^b \int_c^d (x-t)(y-v) \]


(2.5)

\[ \times \left( \int_0^1 \int_0^1 f_{\lambda} (\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v) d\alpha d\lambda \right) dv dt \]

3. Main result

**Theorem 3.1.** Let \( h_i : J_i \subseteq \mathbb{R} \to \mathbb{R} \) be positive functions, for \( i = 1, 2 \). \( f, g : \Delta \to \mathbb{R} \) be partially differentiable functions, such that their second derivatives \( f_{\lambda} \) and \( g_{\lambda} \) are integrable on \( \Delta \). If \( |f_{\lambda}| \) and \( |g_{\lambda}| \) are \((h_1, h_2)\)-convex on the co-ordinates, then we have

\[ |T(f, g)| \leq \frac{49}{3600} k^2 \left( \int_0^1 h_1(\lambda) d\lambda \right)^2 \left( \int_0^1 h_2(\alpha) d\alpha \right)^2 MN \]

where

\[ T(f, g) = \frac{1}{k} \int_a^b \int_c^d f(x, y) g(x, y) dy dx - \frac{(b-a)}{k^2} \int_a^b \int_c^d g(x, y) \left( \int_f (t, y) dt \right) dy dx \]

\[ - \frac{(b-a)}{k^2} \int_a^b \int_c^d g(x, y) \left( \int_f (x, v) dv \right) dy dx \]

\[ + \frac{1}{k^2} \left( \int_a^b \int_c^d f(x, y) dy dx \right) \left( \int_a^b \int_c^d g(t, v) dv dt \right) \]

\[ M = \text{ess sup}_{x, t \in [a, b], y, v \in [c, d]} \{ |f_{\lambda} (x, y)| + |f_{\lambda} (x, v)| + |f_{\lambda} (t, y)| + |f_{\lambda} (t, v)| \}, \]

\[ N = \text{ess sup}_{x, t \in [a, b], y, v \in [c, d]} \{ |g_{\lambda} (x, y)| + |g_{\lambda} (x, v)| + |g_{\lambda} (t, y)| + |g_{\lambda} (t, v)| \} \]

and \( k = (b-a)(d-c) \).

**Proof.** Let \( F, G, \tilde{F} \) and \( \tilde{G} \) be defined as follows

\[ F = f(x, y) - \frac{1}{b-a} \int_a^b f(t, y) dt - \frac{1}{d-c} \int_c^d f(x, v) dv + \frac{1}{k} \int_a^b \int_c^d f(t, v) dv dt \]

\[ G = g(x, y) - \frac{1}{b-a} \int_a^b g(t, y) dt - \frac{1}{d-c} \int_c^d g(x, v) dv + \frac{1}{k} \int_a^b \int_c^d g(t, v) dv dt \]

\[ \tilde{F} = \frac{1}{k} \int_a^b \int_c^d (x-t)(y-v) \times \left( \int_0^1 \int_0^1 f_{\lambda} (\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v) d\alpha d\lambda \right) dv dt \]

ON SOME ČEBYŠEV TYPE INEQUALITIES FOR FUNCTIONS WHOSE ...
\[
\tilde{G} = \frac{1}{k} \int_{a}^{b} \int_{c}^{d} (x-t)(y-v) \times \left( \int_{0}^{1} \int_{0}^{1} g_{\lambda \alpha} (\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v) \, d\alpha \, d\lambda \right) \, dv \, dt.
\]

By Lemma 2.1, we have \( F = \tilde{F} \) and \( G = \tilde{G} \), then

\[ FG = \tilde{F} \tilde{G}. \]

Integrating (3.3) over \( \Delta \), with respect to \( x, y \), multiplying the resultant equality by \( \frac{1}{k} \), using Fubini’s Theorem and modulus, we get

\[
|T(f,g)| = \frac{1}{k^3} \left| \int_{a}^{b} \int_{c}^{d} \int_{a}^{b} \int_{c}^{d} (x-t)(y-v)
\times \left( \int_{0}^{1} \int_{0}^{1} f_{\lambda \alpha} (\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v) \, d\alpha \, d\lambda \right) \, dv \, dt \right|
\times \left( \int_{0}^{1} \int_{0}^{1} g_{\lambda \alpha} (\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v) \, d\alpha \, d\lambda \right) \, dv \, dt \right| \, dy \, dx
\leq \frac{1}{k^3} \left| \int_{a}^{b} \int_{c}^{d} \int_{a}^{b} \int_{c}^{d} |x-t||y-v|
\times \left( \int_{0}^{1} \int_{0}^{1} |f_{\lambda \alpha} (\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v)| \, d\alpha \, d\lambda \right) \, dv \, dt \right|
\times \left( \int_{0}^{1} \int_{0}^{1} |g_{\lambda \alpha} (\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v)| \, d\alpha \, d\lambda \right) \, dv \, dt \right| \, dy \, dx.
\]

Using the \((h_1,h_2)\)-convexity and taking into account that

\[
\int_{a}^{b} \left( \int_{a}^{b} |x-t| \, dt \right)^2 \, dx = \frac{7}{60} (b-a)^5,
\]
\[
\int_{c}^{d} \left( \int_{c}^{d} |y-v| \, dv \right)^2 \, dy = \frac{7}{60} (d-c)^5,
\]
\[
\begin{align*}
\int_0^1 h_1(1 - \lambda)d\lambda &= \int_0^1 h_1(\lambda)d\lambda \quad \text{and} \quad \int_0^1 h_2(1 - \alpha)d\alpha = \int_0^1 h_2(\alpha)d\alpha,
\end{align*}
\]
we obtain

\[
|T(f,g)| \leq \frac{1}{k^3} \left( \int_0^1 h_1(\lambda)d\lambda \right)^2 \left( \int_0^1 h_2(\alpha)d\alpha \right)^2 \times \int_a^b \int_c^d \left( \int_a^b \int_c^d |x - t||y - v| \right.
\]
\[
\left. \times |f_{\lambda\alpha}(x,y)| + |f_{\lambda\alpha}(x,v)| + |g_{\lambda\alpha}(t,y)| + |g_{\lambda\alpha}(t,v)| \right) dvdt
\]
\[
\times \left[ \int_a^b \left( \int_a^b |x - t||y - v| \right) dvdt \right] dydx
\]
\[
\leq \frac{MN}{k^3} \left( \int_0^1 h_1(\lambda)d\lambda \right)^2 \left( \int_0^1 h_2(\alpha)d\alpha \right)^2 \times \int_c^d \int_a^b \left( \int_a^b |x - t||y - v| \right) dvdt \right) dydx
\]
\[
= \frac{MN}{k^3} \left( \int_0^1 h_1(\lambda)d\lambda \right)^2 \left( \int_0^1 h_2(\alpha)d\alpha \right)^2 \times \left[ \int_a^b \left( \int_a^b |x - t| \right) dx \right] \left[ \int_c^d \left( \int_c^d |y - v| \right) dy \right]
\]
\[
= \frac{49}{3600} k^2 \left( \int_0^1 h(\lambda)d\lambda \right)^2 \left( \int_0^1 h(\alpha)d\alpha \right)^2 MN.
\]

This completes the proof of Theorem 3.1. \(\square\)

**Corollary 3.1.** Let \(h : J \subseteq \mathbb{R} \to \mathbb{R}\) be a positive function, \(f, g : \Delta \to \mathbb{R}\) be partially differentiable functions, such that their second derivatives \(f_{\lambda\alpha}\) and \(g_{\lambda\alpha}\) are integrable on \(\Delta\). If \(|f_{\lambda\alpha}|\) and \(|g_{\lambda\alpha}|\) are \(h\)-convex on the co-ordinates, then we have

(3.5) \[
|T(f,g)| \leq \frac{49}{3600} k^2 \left( \int_0^1 h(\lambda)d\lambda \right)^4 MN,
\]

where \(T(f,g), M, N, k\) are defined as in Theorem 3.1.

**Proof.** Applying Theorem 3.1, for \(h_1(v) = h_2(v) = h(v)\), we obtain the desired inequality. \(\square\)
Corollary 3.2. Let \( f, g : \Delta \to \mathbb{R} \) be partially differentiable functions, such that their second derivatives \( f_{\lambda} \) and \( g_{\alpha} \) are integrable on \( \Delta \). If \( |f_{\lambda}| \) and \( |g_{\alpha}| \) are convex on the co-ordinates, then we have

\[
T(f, g) \leq \frac{49}{57600} k^2 MN,
\]

where \( T(f, g), M, N, k \) are defined as in Theorem 3.1.

Proof. In Theorem 3.1, if we replace \( h_1 \) and \( h_2 \) by the identity, we obtain

\[
|T(f, g)| \leq \frac{49}{3600} k^2 \left( \int_0^1 \lambda d\lambda \right)^2 \left( \int_0^1 \alpha d\alpha \right)^2 MN
\]

\[
= \frac{49}{3600} k^2 \left( \frac{\lambda^2}{2} \bigg|_{\lambda=1} \right)^2 \left( \frac{\alpha^2}{2} \bigg|_{\alpha=1} \right)^2 MN
\]

\[
= \frac{49}{3600} k^2 \times \frac{1}{4} \times \frac{1}{4} MN
\]

\[
= \frac{49}{57600} k^2 MN.
\]

This is the desired inequality in (3.6). The proof is completed.

\[\square\]

Remark 3.1. The result of Corollary 3.2 is similar to the inequality (6) of Theorem 2.1 in [12].

Corollary 3.3. Let \( f, g : \Delta \to \mathbb{R} \) be partially differentiable functions, such that their second derivatives \( f_{\lambda} \) and \( g_{\alpha} \) are integrable on \( \Delta \). If \( |f_{\lambda}| \) and \( |g_{\alpha}| \) are \((s_1, s_2)\)-convex in the second sense on the co-ordinates, then

\[
T(f, g) \leq \frac{49}{3600} k^2 \frac{1}{(1 + s_1)^2} \frac{1}{(1 + s_2)^2} MN,
\]

where \( T(f, g), M, N, k \) are defined as in Theorem 3.1 and \( s_1, s_2 \in (0, 1] \).

Proof. Taking in Theorem 3.1, \( h_1(\lambda) = \lambda^{s_1} \) and \( h_2(\alpha) = \alpha^{s_2} \), we obtain

\[
|T(f, g)| \leq \frac{49}{3600} k^2 \left( \int_0^1 \lambda^{s_1} d\lambda \right)^2 \left( \int_0^1 \alpha^{s_2} d\alpha \right)^2 MN
\]

\[
= \frac{49}{3600} k^2 \frac{1}{(1 + s_1)^2} \frac{1}{(1 + s_2)^2} MN.
\]

This is the desired inequality in (3.7). The proof is completed.

\[\square\]

Corollary 3.4. Let \( f, g : \Delta \to \mathbb{R} \) be partially differentiable functions, such that their second derivatives \( f_{\lambda} \) and \( g_{\alpha} \) are integrable on \( \Delta \). If \( |f_{\lambda}| \) and \( |g_{\alpha}| \) are \( s \)-convex in the second sense on the co-ordinates, then

\[
T(f, g) \leq \frac{49}{3600} k^2 \frac{1}{(1 + s)^4} MN,
\]

where \( T(f, g), M, N, k \) are defined as in Theorem 3.1 and \( s \in (0, 1] \).
Proof. Putting in Theorem 3.1, $h_1(\lambda) = \lambda^s$ and $h_2(\alpha) = \alpha^s$, we get

$$|T(f, g)| \leq \frac{49}{3600}k^2 \left( \int_0^1 \lambda^s d\lambda \right)^2 \left( \int_0^1 \alpha^s d\alpha \right)^2 MN$$

$$= \frac{49}{3600}k^2 \frac{1}{(1 + s)^4} MN.$$  

(3.9)

This is the required inequality in (3.8). The proof is completed. \qed

**Theorem 3.2.** Let $h_i : J_i \subset \mathbb{R} \to \mathbb{R}$ be positive functions, for $i = 1, 2$, $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on $\Delta$. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are $(h_1, h_2)$-convex on the co-ordinates, then we have

$$|T(f, g)| \leq \frac{1}{8k^2} \left( \int_0^1 h_1(\lambda) d\lambda \right) \left( \int_0^1 h_2(\alpha) d\alpha \right)$$

$$\times \int_a^b \int_c^d \left[ M |g(x, y)| + N |f(x, y)| \right]$$

$$\times \left( (x-a)^2 + (b-x)^2 \right) \left( (y-c)^2 + (d-y)^2 \right) dy dx.$$  

(3.10)

where $T(f, g)$, $M$, $N$, $k$ are defined as in Theorem 3.1.

Proof. By Lemma 2.1, we have

$$f(x, y) = \frac{1}{b-a} \int_a^b f(t, y) dt + \frac{1}{d-c} \int_c^d f(x, s) ds - \frac{1}{k} \int_a^b \int_c^d f(t, v) dv dt$$

$$+ \frac{1}{k} \int_a^b \int_c^d (x-t)(y-v)$$

$$\times \left( \int_0^1 \int_0^1 f_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v) d\alpha d\lambda \right) dv dt,$$

(3.11)
and

\[
g(x, y) = \frac{1}{b-a} \int_a^b g(t, y) dt + \frac{1}{d-c} \int_c^d g(x, v) ds - \frac{1}{k} \int_a^b \int_c^d g(t, v) dv dt
\]

\[
+ \frac{1}{k} \int_a^b \int_c^d (x - t)(y - v)
\]

\[
\times \left( \int_0^1 \int_0^1 g_{\alpha \lambda} (\lambda x + (1 - \lambda)t, \alpha y - (1 - \alpha)v) d\alpha d\lambda \right) d\alpha d\lambda
\]

(3.12)

Multiplying (3.11) by \(\frac{1}{2k^2} g(x, y)\) and (3.12) by \(\frac{1}{2k^2} f(x, y)\), summing the resultant equalities, then integrating on \(\Delta\), we get

\[
T(f, g) = \frac{1}{2k^2} \left[ \int_a^b \int_c^d g(x, y) \left( \int_a^b \int_c^d (x - t)(y - v) \right) \int_0^1 \int_0^1 f_{\alpha \lambda} (\lambda x + (1 - \lambda)t, \alpha y - (1 - \alpha)v) d\alpha d\lambda \right] dy dx
\]

\[
+ \int_a^b \int_c^d f(x, y) \left( \int_a^b \int_c^d (x - t)(y - v) \right) \int_0^1 \int_0^1 g_{\alpha \lambda} (\lambda x + (1 - \lambda)t, \alpha y - (1 - \alpha)v) d\alpha d\lambda \right) d\alpha d\lambda
\]

(3.13)

using the properties of modulus, (3.13) becomes

\[
|T(f, g)| \leq \frac{1}{2k^2} \left[ \int_a^b \int_c^d |g(x, y)| \left( \int_a^b \int_c^d |x - t||y - v| \right) \int_0^1 \int_0^1 |f_{\alpha \lambda} (\lambda x + (1 - \lambda)t, \alpha y - (1 - \alpha)v)| d\alpha d\lambda \right] dy dx
\]

\[
+ \int_a^b \int_c^d |f(x, y)| \left( \int_a^b \int_c^d |x - t||y - v| \right) \int_0^1 \int_0^1 |g_{\alpha \lambda} (\lambda x + (1 - \lambda)t, \alpha y - (1 - \alpha)v)| d\alpha d\lambda \right) d\alpha d\lambda
\]

(3.14)
Using the \((h_1, h_2)\)-convexity, (3.14) gives

\[
|T(f, g)| \leq \frac{1}{2k^2} \left[ \int_{a}^{b} \int_{c}^{d} |g(x, y)| \left( \int_{0}^{1} h_1(\lambda)d\lambda \right) \left( \int_{0}^{1} h_2(\alpha)d\alpha \right) \right.
\]
\[
\times \left[ \int_{a}^{b} \int_{c}^{d} |x - t| |y - v| \left[ |f_{\lambda_0}(x, y)| + |f_{\lambda_0}(x, v)| \right.
\]
\[
+ |f_{\lambda_0}(t, y)| + |f_{\lambda_0}(t, v)| \right] dvdt \right] dydx
\]
\[
\left. + \int_{a}^{b} \int_{c}^{d} |f(x, y)| \left( \int_{0}^{1} h_1(\lambda)d\lambda \right) \left( \int_{0}^{1} h_2(\alpha)d\alpha \right) \right]
\]
\[
\times \left[ \int_{a}^{b} \int_{c}^{d} |x - t| |y - v| \left[ |g_{\lambda_0}(x, y)| + |g_{\lambda_0}(x, v)| \right.
\]
\[
+ |g_{\lambda_0}(t, y)| + |g_{\lambda_0}(t, v)| \right] dvdt \right] dydx,
\]
(3.15)

By a simple calculation we get

\[
|T(f, g)| \leq \frac{1}{2k^2} \left( \int_{0}^{1} h_1(\lambda)d\lambda \right) \left( \int_{0}^{1} h_2(\alpha)d\alpha \right)
\]
\[
\times \left[ \int_{a}^{b} \int_{c}^{d} M |g(x, y)| \left( \int_{a}^{b} \int_{c}^{d} |x - t| |y - v| dvdt \right) \right.
\]
\[
+ N |f(x, y)| \left( \int_{a}^{b} \int_{c}^{d} |x - t| |y - v| dvdt \right) \right] dydx
\]
\[
= \frac{1}{8k^2} \left( \int_{0}^{1} h_1(\lambda)d\lambda \right) \left( \int_{0}^{1} h_2(\alpha)d\alpha \right)
\]
\[
\times \left[ \int_{a}^{b} \int_{c}^{d} [M |g(x, y)| + N |f(x, y)|]
\]
\[
\times \left( (x - a)^2 + (b - x)^2 \right) \left( (y - c)^2 + (d - y)^2 \right) dydx.
\]
(3.16)

This completes the proof of Theorem 3.2. \(\square\)

**Corollary 3.5.** Let \(h : J \subseteq \mathbb{R} \to \mathbb{R}\) be a positive function, \(f, g : \Delta \to \mathbb{R}\) be partially differentiable functions, such that their second derivatives \(f_{\lambda_0}\) and \(g_{\lambda_0}\) are integrable...
on $\Delta$. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are $h$-convex on the co-ordinates, then we have

$$
|T(f, g)| \leq \frac{1}{8k^2} \left( \int_0^1 h(\lambda) d\lambda \right)^2 \int_a^b \int_c^d \left( (x-a)^2 + (b-x)^2 \right) \left( (y-c)^2 + (d-y)^2 \right) dy dx.
$$

where $T(f, g)$, $M$, $N$, $k$ are defined as in Theorem 3.1.

**Proof.** Applying Theorem 3.2, for $h_1(\lambda) = h_2(\lambda)$, we obtain the desired inequality. $\square$

**Corollary 3.6.** Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on $\Delta$. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are convex on the co-ordinates, then we have

$$
|T(f, g)| \leq \frac{1}{32k^2} \int_a^b \int_c^d \left( (x-a)^2 + (b-x)^2 \right) \left( (y-c)^2 + (d-y)^2 \right) dy dx.
$$

(3.17)

where $T(f, g)$, $M$, $N$, $k$ are defined as in Theorem 3.1.

**Proof.** In Theorem 3.2, if we replace $h_1$ and $h_2$ by the identity, we obtain

$$
|T(f, g)| \leq \frac{1}{8k^2} \left( \int_0^1 h(\lambda) d\lambda \right)^2 \left( \int_0^1 h(\alpha) d\alpha \right) \int_a^b \int_c^d \left( (x-a)^2 + (b-x)^2 \right) \left( (y-c)^2 + (d-y)^2 \right) dy dx.
$$

This is the desired inequality in (3.17). The proof is completed. $\square$

**Remark 3.2.** The result of Corollary 3.6, is similar to the inequality (7) of Theorem 2.1 in [12].

**Corollary 3.7.** Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on $\Delta$. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are
(s₁, s₂)-convex in the second sense on the co-ordinates, then we have

\[
|T(f, g)| \leq \frac{1}{8k²(1 + s₁)(1 + s₂)} \\
\times \int_a^b \int_c^d [M|g(x, y)| + N|f(x, y)|] \\
\times \left( (x - a)^2 + (b - x)^2 \right) \left( (y - c)^2 + (d - y)^2 \right) \, dy \, dx,
\]

(3.18)

where T(f, g), M, N, k are defined as in Theorem 3.1 and s₁, s₂ ∈ (0, 1].

Proof. Putting in Theorem 3.2, \( h₁(\lambda) = \lambda^{s₁} \) and \( h₂(\alpha) = \alpha^{s₂} \), we get

\[
|T(f, g)| \leq \frac{1}{8k²} \left( \int_0^{λ₁} \lambda^{s₁} \, dλ \right) \left( \int_0^{α₂} \alpha^{s₂} \, dα \right) \\
\times \int_a^b \int_c^d [M|g(x, y)| + N|f(x, y)|] \\
\times \left( (x - a)^2 + (b - x)^2 \right) \left( (y - c)^2 + (d - y)^2 \right) \, dy \, dx.
\]

This is the required inequality in (3.18). The proof is completed. \( \Box \)

**Corollary 3.8.** Let \( f, g : Δ \to \mathbb{R} \) be partially differentiable functions, such that their second derivatives \( f_{λα} \) and \( g_{λα} \) are integrable on \( Δ \). If \(|f_{λα}|\) and \(|g_{λα}|\) are \( s \)-convex in the second sense on the co-ordinates, then we have

\[
|T(f, g)| \leq \frac{1}{8k²(1 + s)^2} \\
\times \int_a^b \int_c^d [M|g(x, y)| + N|f(x, y)|] \\
\times \left( (x - a)^2 + (b - x)^2 \right) \left( (y - c)^2 + (d - y)^2 \right) \, dy \, dx,
\]

(3.19)

where T(f, g), M, N, k are defined as in Theorem 3.1 and \( s \in (0, 1] \).
Proof. Taking in Theorem 3.2, $h_1(\lambda) = \lambda^s$ and $h_2(\alpha) = \alpha^s$, we get

$$|T(f,g)| \leq \frac{1}{8k^2} \left( \int_0^1 \lambda^s d\lambda \right) \left( \int_0^1 \alpha^s d\alpha \right)$$

$$\times \int_a^b \int_c^d \left[ M |g(x,y)| + N |f(x,y)| \right]$$

$$\times \left( (x-a)^2 + (b-x)^2 \right) \left( (y-c)^2 + (d-y)^2 \right) dy dx.$$

$$= \frac{1}{8k^2 (1+s)^2}$$

$$\times \int_a^b \int_c^d \left[ M |g(x,y)| + N |f(x,y)| \right]$$

$$\times \left( (x-a)^2 + (b-x)^2 \right) \left( (y-c)^2 + (d-y)^2 \right) dy dx.$$  

This is the desired inequality in (3.19). The proof is completed. □

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University of Guelma. Guelma, Algeria.
E-mail address: khaledv2004@yahoo.fr