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ON SOME ČEBYŠEV TYPE INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE (h_1, h_2)-CONVEX ON THE CO-ORDINATES

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Abstract. The aim of this paper is to establish some new Čebyšev type inequalities involving functions whose mixed partial derivatives are (h_1, h_2)-convex on the co-ordinates.

1. Introduction

In 1882, Čebyšev [4] gave the following inequality :

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty \tag{1.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and g' are bounded,

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\, g(x)\, dx - \left(\frac{1}{b-a}\int_a^b f(x)\, dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\, dx\right), \tag{1.2}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess \sup_{t\in[a,b]} |f(t)|$.

During the past few years many researchers have given considerable attention to the inequality (1.1), various generalizations, extensions and variants of this inequality have appeared in the literature, see $[1,3,6,8,9,10]$. Recently, Guezane-Lakoud and Aissaoui [6] established new Čebyšev type inequalities similar to (1.1) for functions f, g defined on bidimensional intervals $\Delta = [a,b] \times [c,d] \subset [0,\infty)^2$ whose mixed partial derivatives f_{st} and g_{st} are integrable and bounded. The authors of the paper [12] further extend these results in special cases when the mixed partial derivatives belong to certain classes of functions that generalize convex function on the co-ordinates.

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The main purpose of this work is to obtain new Čebyšev type inequalities involving functions whose mixed partial derivatives are (h_1, h_2)-convex on the co-ordinates.

2. Preliminaries

Throughout this paper we denote by Δ the bidimensional interval in $[0,\infty)^2$, $\Delta =: [a,b] \times [c,d]$ with $a < b$ and $c < d$, $k = (b-a)(d-c)$ and $f_{\lambda\alpha}$ for $\frac{\partial^2 f}{\partial\lambda\partial\alpha}$.

Definition 2.1 ([5])**.** A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) \leq\ & \lambda\alpha f(x,y) + \lambda(1-\alpha) f(x,v) \\ & + (1-\lambda)\alpha f(t,y) + (1-\lambda)(1-\alpha) f(t,v) \end{aligned} \tag{2.1}$$

holds for all $\lambda, \alpha \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex.

Definition 2.2 ([2])**.** A function $f : \Delta \to \mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) \leq\ & \lambda^s\alpha^s f(x,y) + \lambda^s (1-\alpha)^s f(x,v) \\ & + (1-\lambda)^s \alpha^s f(t,y) + (1-\lambda)^s (1-\alpha)^s f(t,v) \end{aligned} \tag{2.2}$$

holds for all $\lambda, \alpha \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$,
for some fixed $s \in (0,1]$.

s-convexity on the co-ordinates does not imply the s-convexity, that is there exist functions which are s-convex on the co-ordinates but are not s-convex.

Definition 2.3 ([7])**.** Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. A mapping $f : \Delta \to \mathbb{R}$ is said to be h-convex on Δ, if the following inequality

$$f(\alpha x + (1-\alpha)t, \alpha y + (1-\alpha)v) \leq h(\alpha) f(x,y) + h(1-\alpha) f(t,v) \tag{2.3}$$

holds, for all $(x,y), (t,v) \in \Delta$ and $\alpha \in (0,1)$.

Definition 2.4 ([7])**.** A function $f : \Delta \to \mathbb{R}$ is said to be (h_1, h_2)-convex on the coordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) \leq\ & h_1(\lambda)h_2(\alpha) f(x,y) + h_1(\lambda)h_2(1-\alpha) f(x,v) \\ & + h_1(1-\lambda) h_2(\alpha) f(t,y) \\ & + h_1(1-\lambda) h_2(1-\alpha) f(t,v) \end{aligned} \tag{2.4}$$

holds for all $\lambda, \alpha \in]0,1[$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

h-convexity on the co-ordinates does not imply the h-convexity, that is there exist functions which are h-convex on the co-ordinates but are not h-convex.

Lemma 2.1 (Lemma 1. [11])**.** *Let $f : \Delta \to \mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$. If $f_{\lambda\alpha} \in L_1(\Delta)$, then for any*

$(x,y)\in\Delta$, we have the equality:

$$\begin{aligned} f(x,y) &= \frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{d-c}\int_c^d f(x,v)dv-\frac{1}{k}\int_a^b\int_c^d f(t,v)dvdt \\ &\quad +\frac{1}{k}\int_a^b\int_c^d (x-t)(y-v) \\ &\quad \times\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v)\,d\alpha d\lambda\right)dvdt \end{aligned} \tag{2.5}$$

3. Main result

Theorem 3.1. *Let $h_i : J_i \subseteq \mathbb{R}\to\mathbb{R}$ be positive functions, for $i=1,2$. $f,g:\Delta\to\mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (h_1,h_2)-convex on the co-ordinates, then we have*

$$|T(f,g)|\leq\frac{49}{3600}k^2\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 MN \tag{3.1}$$

where

$$\begin{aligned} T(f,g) &= \frac{1}{k}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx-\frac{(d-c)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_a^b f(t,y)\,dt\right)dydx \\ &\quad -\frac{(b-a)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_c^d f(x,v)\,dv\right)dydx \\ &\quad +\frac{1}{k^2}\left(\int_a^b\int_c^d f(x,y)\,dydx\right)\left(\int_a^b\int_c^d g(t,v)\,dvdt\right) \end{aligned} \tag{3.2}$$

$M=\underset{x,t\in[a,b],y,v\in[c,d]}{ess\sup}\left[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,v)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,v)|\right],$

$N=\underset{x,t\in[a,b],y,v\in[c,d]}{ess\sup}\left[|g_{\lambda\alpha}(x,y)|+|g_{\lambda\alpha}(x,v)|+|g_{\lambda\alpha}(t,y)|+|g_{\lambda\alpha}(t,v)|\right]$

and $k=(b-a)(d-c)$.

Proof. Let F, G, $\widetilde{F}$ and $\widetilde{G}$ be defined as follows

$$F=f(x,y)-\frac{1}{b-a}\int_a^b f(t,y)dt-\frac{1}{d-c}\int_c^d f(x,v)dv+\frac{1}{k}\int_a^b\int_c^d f(t,v)dvdt$$

$$G=g(x,y)-\frac{1}{b-a}\int_a^b g(t,y)dt-\frac{1}{d-c}\int_c^d g(x,v)dv+\frac{1}{k}\int_a^b\int_c^d g(t,v)dvdt$$

$$\widetilde{F}=\frac{1}{k}\int_a^b\int_c^d (x-t)(y-v)\times\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v)\,d\alpha d\lambda\right)dvdt$$

$$\widetilde{G}=\frac{1}{k}\int_a^b\int_c^d (x-t)(y-v)\times\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt.$$

By Lemma 2.1, we have

$$F=\widetilde{F} \text{ and } G=\widetilde{G},$$

then

$$FG=\widetilde{F}\widetilde{G}. \tag{3.3}$$

Integrating (3.3) over Δ, with respect to x,y, multiplying the resultant equality by $\frac{1}{k}$, using Fubini's Theoerm and modulus, we get

$$\begin{aligned}
|T(f,g)| &= \frac{1}{k^3}\left|\int_a^b\int_c^d \left[\int_a^b\int_c^d (x-t)(y-v)\right.\right.\\
&\quad\times\left.\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt\right]\\
&\quad\times\left[\int_a^b\int_c^d (x-t)(y-v)\right.\\
&\quad\times\left.\left.\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt\right]dydx\right|\\
&\le \frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\\
&\quad\times\left.\left(\int_0^1\int_0^1 \left|f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)\right|d\alpha d\lambda\right)dvdt\right]\\
&\quad\times\left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\\
&\quad\times\left.\left(\int_0^1\int_0^1 \left|g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y-(1-\alpha)v\right)\right|d\alpha d\lambda\right)dvdt\right]dydx.
\end{aligned} \tag{3.4}$$

Using the (h_1,h_2)-convexity and taking into account that

$$\begin{aligned}
\int_a^b\left(\int_a^b |x-t|\,dt\right)^2 dx &= \frac{7}{60}(b-a)^5,\\
\int_c^d\left(\int_c^d |y-v|\,dv\right)^2 dy &= \frac{7}{60}(d-c)^5,
\end{aligned}$$

$$\int_0^1 h_1(1-\lambda)d\lambda = \int_0^1 h_1(\ \lambda)d\lambda \text{ and } \int_0^1 h_2(1-\alpha)d\alpha = \int_0^1 h_2(\ \alpha)d\alpha,$$

we obtain

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{k^3}\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 \\
&\quad \times\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-v| \times [|f_{\lambda\alpha}(x,y)| + |f_{\lambda\alpha}(x,v)|\right. \\
&\quad + |f_{\lambda\alpha}(t,y)| + |f_{\lambda\alpha}(t,v)|]\, dvdt \\
&\quad \times\left[\int_a^b\int_c^d |x-t|\,|y-v| \times [|g_{\lambda\alpha}(x,y)| + |g_{\lambda\alpha}(x,v)|\right. \\
&\quad + |g_{\lambda\alpha}(t,y)| + |g_{\lambda\alpha}(t,v)|]\, dvdt]\, dydx \\
&\leq \frac{MN}{k^3}\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 \\
&\quad \times\int_a^b\int_c^d\left(\int_a^b\int_c^d |x-t|\,|y-v|\, dvdt\right)^2 dydx \\
&= \frac{MN}{k^3}\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 \\
&\quad \times\left[\int_a^b\left(\int_a^b |x-t|\ \ dt\right)^2 dx\right]\left[\int_c^d\left(\int_c^d |y-v|\, dv\right)^2 dy\right] \\
&= \frac{49}{3600}k^2\left(\int_0^1 h_1(\lambda)d\lambda\right)^2\left(\int_0^1 h_2(\alpha)d\alpha\right)^2 MN.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

Corollary 3.1. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be positive function, $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are h-convex on the co-ordinates, then we have*

$$|T(f,g)| \leq \frac{49}{3600}k^2\left(\int_0^1 h(\lambda)d\lambda\right)^4 MN, \tag{3.5}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. Applying Theorem 3.1, for $h_1(v) = h_2(v) = h(v)$, we obtain the desired inequality. □

Corollary 3.2. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are convex on the co-ordinates, then we have*

$$|T(f,g)| \leq \frac{49}{57600}k^2 MN, \tag{3.6}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. In Theorem 3.1, if we replace h_1 and h_2 by the identity, we obtain

$$\begin{aligned} |T(f,g)| &\leq \frac{49}{3600}k^2 \left(\int_0^1 \lambda d\lambda\right)^2 \left(\int_0^1 \alpha d\alpha\right)^2 MN \\ &= \frac{49}{3600}k^2 \left(\frac{\lambda^2}{2}\Big|_{\lambda=0}^{\lambda=1} \right)^2 \left(\frac{\alpha^2}{2}\Big|_{\alpha=0}^{\alpha=1} \right)^2 MN \\ &= \frac{49}{3600}k^2 \times \frac{1}{4} \times \frac{1}{4}MN \\ &= \frac{49}{57600}\ k^2 MN. \end{aligned}$$

This is the desired inequality in (3.6). The proof is completed. □

Remark 3.1. The result of Corollary 3.2 is similar to the inequality (6) of Theorem 2.1 in [12].

Corollary 3.3. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (s_1, s_2)-convex in the second sense on the co-ordinates, then*

$$|T(f,g)| \leq \frac{49}{3600}k^2\ \frac{1}{(1+s_1)^2}\ \frac{1}{(1+s_2)^2}\ MN, \tag{3.7}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $s_1, s_2 \in (0,1]$.

Proof. Taking in Theorem 3.1, $h_1(\lambda) = \lambda^{s_1}$ and $h_2(\alpha) = \alpha^{s_2}$, we obtain

$$\begin{aligned} |T(f,g)| &\leq \frac{49}{3600}k^2 \left(\int_0^1 \lambda^{s_1} d\lambda\right)^2 \left(\int_0^1 \alpha^{s_2} d\alpha\right)^2 MN \\ &= \frac{49}{3600}\ k^2 \frac{1}{(1+s_1)^2}\ \frac{1}{(1+s_2)^2}\ MN. \end{aligned}$$

This is the desired inequality in (3.7). The proof is completed. □

Corollary 3.4. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are s-convex in the second sense on the co-ordinates, then*

$$|T(f,g)| \leq \frac{49}{3600}k^2\ \frac{1}{(1+s)^4}\ \ MN, \tag{3.8}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $s \in (0,1]$.

Proof. Putting in Theorem 3.1, $h_1(\lambda) = \lambda^s$ and $h_2(\alpha) = \alpha^s$, we get

$$\begin{aligned} |T(f,g)| &\leq \frac{49}{3600}k^2 \left(\int_0^1 \lambda^s d\lambda\right)^2 \left(\int_0^1 \alpha^s d\alpha\right)^2 MN \\ &= \frac{49}{3600}k^2 \frac{1}{(1+s)^4} \quad MN. \end{aligned} \tag{3.9}$$

This is the required inequality in (3.8). The proof is completed. □

Theorem 3.2. *Let $h_i : J_i \subseteq \mathbb{R} \to \mathbb{R}$ be positive functions, for $i = 1, 2$, $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (h_1, h_2)-convex on the co-ordinates, then we have*

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{8k^2} \left(\int_0^1 h_1(\lambda) d\lambda\right) \left(\int_0^1 h_2(\alpha) d\alpha\right) \\ &\times \int_a^b \int_c^d [M |g(x,y)| + N |f(x,y)|] \\ &\times \left((x-a)^2 + (b-x)^2\right) \left((y-c)^2 + (d-y)^2\right) dydx. \end{aligned} \tag{3.10}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. By Lemma 2.1, we have

$$\begin{aligned} f(x,y) &= \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{d-c}\int_c^d f(x,s)dv - \frac{1}{k}\int_a^b\int_c^d f(t,v)dvdt \\ &+ \frac{1}{k}\int_a^b\int_c^d (x-t)(y-v) \\ &\times \left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y - (1-\alpha)v)\, d\alpha d\lambda\right) dvdt, \end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
g(x,y) &= \frac{1}{b-a}\int_a^b g(t,y)dt + \frac{1}{d-c}\int_c^d g(x,v)ds - \frac{1}{k}\int_a^b\int_c^d g(t,v)dvdt \\
&+\frac{1}{k}\int_a^b\int_c^d (x-t)(y-v) \\
&\times\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt.
\end{aligned}
\tag{3.12}
$$

Multiplying (3.11) by $\frac{1}{2k}g(x,y)$ and (3.12) by $\frac{1}{2k}f(x,y)$, summing the resultant equalities, then integrating on Δ, we get

$$
\begin{aligned}
T(f,g) &= \frac{1}{2k^2}\left[\int_a^b\int_c^d g(x,y)\left[\int_a^b\int_c^d (x-t)(y-v)\right.\right. \\
&\times\left.\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt\right]dydx \\
&+\int_a^b\int_c^d f(x,y)\left[\int_a^b\int_c^d (x-t)(y-v)\right. \\
&\times\left.\left.\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)d\alpha d\lambda\right)dvdt\right]dydx\right],
\end{aligned}
\tag{3.13}
$$

using the properties of modulus, (3.13) becomes

$$
\begin{aligned}
|T(f,g)| &\leq \frac{1}{2k^2}\left[\int_a^b\int_c^d |g(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\right. \\
&\times\left.\left(\int_0^1\int_0^1 \left|f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)\right|d\alpha d\lambda\right)dvdt\right]dydx \\
&+\int_a^b\int_c^d |f(x,y)|\left[\int_a^b\int_c^d |x-t|\,|y-v|\right. \\
&\times\left.\left.\left(\int_0^1\int_0^1 \left|g_{\lambda\alpha}\left(\lambda x+(1-\lambda)t, \alpha y-(1-\alpha)v\right)\right|d\alpha d\lambda\right)dvdt\right]dydx\right].
\end{aligned}
\tag{3.14}
$$

Using the (h_1, h_2)-convexity, (3.14) gives

$$
\begin{aligned}
|T(f,g)| &\leq \frac{1}{2k^2}\left[\int_a^b\int_c^d |g(x,y)| \left(\int_0^1 h_1(\lambda)d\lambda\right)\left(\int_0^1 h_2(\alpha)d\alpha\right)\right. \\
&\times \left[\int_a^b\int_c^d |x-t|\,|y-v|\,[|f_{\lambda\alpha}(x,y)| + |f_{\lambda\alpha}(x,v)| \right. \\
&+ |f_{\lambda\alpha}(t,y)| + |f_{\lambda\alpha}(t,v)|]\,dvdt\ \Big]\,dydx \\
&+\int_a^b\int_c^d |f(x,y)| \left(\int_0^1 h_1(\lambda)d\lambda\right)\left(\int_0^1 h_2(\alpha)d\alpha\right) \\
&\times \left[\int_a^b\int_c^d |x-t|\,|y-v|\,[|g_{\lambda\alpha}(x,y)| + |g_{\lambda\alpha}(x,v)| \right. \\
&+ |g_{\lambda\alpha}(t,y)| + |g_{\lambda\alpha}(t,v)|]\,dvdt]\,dydx\Big],
\end{aligned}
\tag{3.15}
$$

By a simple calculation we get

$$
\begin{aligned}
|T(f,g)| &\leq \frac{1}{2k^2}\left(\int_0^1 h_1(\lambda)d\lambda\right)\left(\int_0^1 h_2(\alpha)d\alpha\right) \\
&\times\int_a^b\int_c^d \left[M\,|g(x,y)| \left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right)\right. \\
&\left. +N\,|f(x,y)| \left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right)\right] dydx \\
&= \frac{1}{8k^2}\left(\int_0^1 h_1(\lambda)d\lambda\right)\left(\int_0^1 h_2(\alpha)d\alpha\right) \\
&\times\int_a^b\int_c^d [M\,|g(x,y)| + N\,|f(x,y)|] \\
&\times\left((x-a)^2 + (b-x)^2\right)\left((y-c)^2 + (d-y)^2\right) dydx.
\end{aligned}
\tag{3.16}
$$

This completes the proof of Theorem 3.2. □

Corollary 3.5. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be positive function, $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable* *on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are h-convex on the co-ordinates, then we have*

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{8k^2}\left(\int_0^1 h(\lambda)d\lambda\right)^2 \int_a^b\int_c^d \left[(M\left|g(x,y)\right| + N\left|f(x,y)\right|)\right. \\ &\quad \left.\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)\right] dydx. \end{aligned}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. Applying Theorem 3.2, for $h_1(\lambda) = h_2(\lambda)$, we obtain the desired inequality. □

Corollary 3.6. *Let $f,g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are convex on the co-ordinates, then we have*

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{32k^2}\int_a^b\int_c^d \left[M\left|g(x,y)\right| + N\left|f(x,y)\right|\right] \\ &\quad \times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right) dydx. \end{aligned} \tag{3.17}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1.

Proof. In Theorem 3.2, if we replace h_1 and h_2 by the identity, we obtain

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{8k^2}\left(\int_0^1 \lambda d\lambda\right)\left(\int_0^1 \alpha d\alpha\right) \\ &\quad \times\int_a^b\int_c^d \left[M\left|g(x,y)\right| + N\left|f(x,y)\right|\right] \\ &\quad \times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right) dydx. \\ &= \frac{1}{32k^2}\int_a^b\int_c^d \left[M\left|g(x,y)\right| + N\left|f(x,y)\right|\right] \\ &\quad \times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right) dydx. \end{aligned}$$

This is the desired inequality in (3.17). The proof is completed. □

Remark 3.2. The result of Corollary 3.6, is similar to the inequality (7) of Theorem 2.1 in [12].

Corollary 3.7. *Let $f,g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are*

(s_1, s_2)-convex in the second sense on the co-ordinates, then we have

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{8k^2(1+s_1)(1+s_2)} \\ &\quad\times\int_a^b\int_c^d [M|g(x,y)|+N|f(x,y)|] \\ &\quad\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx,\end{aligned} \tag{3.18}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $s_1, s_2 \in (0,1]$.

Proof. Putting in Theorem 3.2, $h_1(\lambda)=\lambda^{s_1}$ and $h_2(\alpha)=\alpha^{s_2}$, we get

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{8k^2}\left(\int_0^1 \lambda^{s_1}d\lambda\right)\left(\int_0^1 \alpha^{s_2}d\alpha\right) \\ &\quad\times\int_a^b\int_c^d [M|g(x,y)|+N|f(x,y)|] \\ &\quad\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx. \\ &= \frac{1}{8(1+s_1)(1+s_2)k^2} \\ &\quad\times\int_a^b\int_c^d [M|g(x,y)|+N|f(x,y)|] \\ &\quad\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx.\end{aligned}$$

This is the required inequality in (3.18). The proof is completed. □

Corollary 3.8. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are s-convex in the second sense on the co-ordinates, then we have*

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{8k^2(1+s)^2} \\ &\quad\times\int_a^b\int_c^d [M|g(x,y)|+N|f(x,y)|] \\ &\quad\times\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)dydx,\end{aligned} \tag{3.19}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $s \in (0,1]$.

Proof. Taking in Theorem 3.2, $h_1(\lambda) = \lambda^s$ and $h_2(\alpha) = \alpha^s$, we get

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{8k^2}\left(\int_0^1 \lambda^s d\lambda\right)\left(\int_0^1 \alpha^s d\alpha\right) \\ &\times \int_a^b \int_c^d \left[M\,|g(x,y)| + N\,|f(x,y)|\right] \\ &\times \left((x-a)^2 + (b-x)^2\right)\left((y-c)^2 + (d-y)^2\right) dydx. \\ &= \frac{1}{8k^2\,(1+s)^2} \\ &\times \int_a^b \int_c^d \left[M\,|g(x,y)| + N\,|f(x,y)|\right] \\ &\times \left((x-a)^2 + (b-x)^2\right)\left((y-c)^2 + (d-y)^2\right) dydx. \end{aligned}$$

This is the desired inequality in (3.19). The proof is completed. □

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