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# $L^p$ LOCAL UNCERTAINTY PRINCIPLE FOR THE DUNKL TRANSFORM

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Abstract. In this paper, we establish $L^p$ local uncertainty principle for the Dunkl transform on $\mathbb{R}^d$; and we deduce $L^p$ version of the Heisenberg-Pauli-Weyl uncertainty principle for this transform. We use also the $L^p$ local uncertainty principle for the Dunkl transform and the techniques of Donoho-Stark, we obtain uncertainty principles of concentration type in the $L^p$ theory, when $1 < p \leq 2$.

## 1. Introduction

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y\rangle}$. For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let $\sigma_\alpha$ be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to $\alpha$:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y\rangle}{|\alpha|^2}\alpha.$$

A finite set $\Re \subset \mathbb{R}^d \backslash \{0\}$ is called a root system, if $\Re \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$ for all $\alpha \in \Re$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \Re$. For a root system $\Re$, the reflections $\sigma_\alpha$, $\alpha \in \Re$, generate a finite group $G$. The Coxeter group $G$ is a subgroup of the orthogonal group $O(d)$. All reflections in $G$, correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha\in\Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+$ or $-\alpha \in \Re_+$.

Let $k : \Re \to \mathbb{C}$ be a multiplicity function on $\Re$ (a function which is constant on the orbits under the action of $G$). As an abbreviation, we introduce the index $\gamma = \gamma_k := \sum_{\alpha\in\Re_+} k(\alpha)$.

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Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \Re$. Moreover, let $w_k$ denote the weight function $w_k(y) := \prod_{\alpha \in \Re_+} |\langle \alpha, y\rangle|^{2k(\alpha)}$, for all $y \in \mathbb{R}^d$, which is $G$-invariant and homogeneous of degree $2\gamma$.

Let $c_k$ be the Mehta-type constant given by $c_k := (\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y)\mathrm{d}y)^{-1}$. We denote by $\mu_k$ the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu_k(y) := c_k w_k(y)\mathrm{d}y$; and by $L^p(\mu_k)$, $1 \leq p \leq \infty$, the space of measurable functions $f$ on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(y)|^p \mathrm{d}\mu_k(y)\Big)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{y \in \mathbb{R}^d} |f(y)| < \infty,$$

and by $L^p_{rad}(\mu_k)$ the subspace of $L^p(\mu_k)$ consisting of radial functions.

For $f \in L^1(\mu_k)$ the Dunkl transform is defined (see [4]) by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y) \mathrm{d}\mu_k(y), \quad x \in \mathbb{R}^d,$$

where $E_k(-ix, y)$ denotes the Dunkl kernel (for more details, see the next section).

Many uncertainty principles have already been proved for the Dunkl transform, namely by Rösler [10] and Shimeno [11] who established the Heisenberg-Pauli-Weyl inequality for the Dunkl transform, by showing that for every $f \in L^2(\mu_k)$,

$$\|f\|^2_{L^2(\mu_k)} \leq \frac{2}{2\gamma + d} \| \, |x| f\|_{L^2(\mu_k)} \| \, |y| \mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{1.1}$$

Recently the author [17] proved the following $L^p$ version of the Heisenberg-Pauli-Weyl inequality for the Dunkl transform $\mathcal{F}_k$. Let $0 < a < (2\gamma + d)/q$, $b > 0$, if $1 < p \leq 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq C(a,b) \| \, |x|^a f\|^{\frac{b}{a+b}}_{L^p(\mu_k)} \| \, |y|^b \mathcal{F}_k(f)\|^{\frac{a}{a+b}}_{L^q(\mu_k)}, \tag{1.2}$$

where $C(a,b)$ is a positive constant.

Building on the ideas of Faris [5] and Price [8, 9] for the Fourier transform, we show a local uncertainty principles for the Dunkl transform $\mathcal{F}_k$. More precisely, we will show the following results. Let $E$ be a measurable subset of $\mathbb{R}^d$ such that $0 < \mu_k(E) < \infty$, and $a > 0$. If $1 < p \leq 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then

$$\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \begin{cases} K_1(a)(\mu_k(E))^{\frac{a}{2\gamma+d}} \| \, |x|^a f\|_{L^p(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a)(\mu_k(E))^{1/q} \|f\|^{1 - \frac{2\gamma+d}{qa}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{2\gamma+d}{qa}}_{L^p(\mu_k)}, & a > \frac{2\gamma+d}{q}, \\ 2K_1(\frac{a}{2})(\mu_k(E))^{\frac{1}{2q}} \|f\|^{1/2}_{L^p(\mu_k)} \| \, |x|^a f\|^{1/2}_{L^p(\mu_k)}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

where $\chi_E$ is the characteristic function of the set $E$ and $K_1(a)$, $K_2(a)$ are positive constants given explicitly by Theorem 2.1.

We shall use the $L^p$ local uncertainty principle to show $L^p$ version of the Heisenberg-Pauli-Weyl uncertainty principle for the Dunkl transform $\mathcal{F}_k$. Let $a, b > 0$, if $1 < p \leq 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \begin{cases} K_1(a,b) \| \, |x|^a f\|^{\frac{b}{a+b}}_{L^p(\mu_k)} \| \, |y|^b \mathcal{F}_k(f)\|^{\frac{a}{a+b}}_{L^q(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a,b) \|f\|^{\frac{b(qa-2\gamma-d)}{a(qb+2\gamma+d)}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{b(2\gamma+d)}{a(2\gamma+d+qb)}}_{L^p(\mu_k)} \| \, |y|^b \mathcal{F}_k(f)\|^{\frac{2\gamma+d}{2\gamma+d+qb}}_{L^q(\mu_k)}, & a > \frac{2\gamma+d}{q}, \\ K_3(a,b) \|f\|^{\frac{b}{a+2b}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{b}{a+2b}}_{L^p(\mu_k)} \| \, |y|^b \mathcal{F}_k(f)\|^{\frac{a}{a+2b}}_{L^q(\mu_k)}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

where $K_1(a,b)$, $K_2(a,b)$ and $K_3(a,b)$ are positive constants given explicitly by Theorem 2.2. The inequalities which generalize the Heisenberg-Pauli-Weyl inequalities given by (1.1) and (1.2). In the case $k=0$ and $q=2$, these inequalities are due to Cowling-Price [1] and Hirschman [6].

We shall use also the local uncertainty principle, and building on the techniques of Donoho-Stark [2, 14, 15, 16, 18], we show uncertainty principles of concentration type in the $L^p$ theory, when $1 < p \le 2$.

This paper is organized as follows. In Section 2 we show a local uncertainty principle for the Dunkl transform $\mathcal{F}_k$; and we deduce $L^p$ version of the Heisenberg-Pauli-Weyl uncertainty principle for this transform. The last section is devoted to present uncertainty principles of concentration type in the $L^p$ theory, when $1 < p \le 2$.

## 2. $L^p$ UNCERTAINTY PRINCIPLES

The Dunkl operators $\mathcal{D}_j$; $j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group $G$ and multiplicity function $k$ are given, for a function $f$ of class $C^1$ on $\mathbb{R}^d$, by

$$\mathcal{D}_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha \in \Re_+} k(\alpha)\alpha_j \frac{f(y) - f(\sigma_\alpha y)}{\langle \alpha, y\rangle}.$$

For $y \in \mathbb{R}^d$, the initial problem $\mathcal{D}_j u(.,y)(x) = y_j u(x,y)$, $j = 1, ..., d$, with $u(0,y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x,y)$ and called Dunkl kernel [3, 7]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$. In our case (see [3, 4]),

$$|E_k(-ix,y)| \le 1, \quad x, y \in \mathbb{R}^d. \tag{2.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [4], where already many basic properties were established. Dunkl's results were completed and extended later by De Jeu [7]. The Dunkl transform of a function $f$ in $L^1(\mu_k)$, is defined by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix,y) f(y) \mathrm{d}\mu_k(y), \quad x \in \mathbb{R}^d.$$

We notice that $\mathcal{F}_0$ agrees with the Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y) \mathrm{d}y, \quad x \in \mathbb{R}^d.$$

Some of the properties of Dunkl transform $\mathcal{F}_k$ are collected bellow (see [4, 7]).

(a) $L^1 - L^\infty$-boundedness. *For all $f \in L^1(\mu_k)$, $\mathcal{F}_k(f) \in L^\infty(\mu_k)$ and*

$$\|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \le \|f\|_{L^1(\mu_k)}. \tag{2.2}$$

(b) Inversion theorem. *Let $f \in L^1(\mu_k)$, such that $\mathcal{F}_k(f) \in L^1(\mu_k)$. Then*

$$f(x) = \mathcal{F}_k(\mathcal{F}_k(f))(-x), \quad \text{a.e.} \quad x \in \mathbb{R}^d. \tag{2.3}$$

(c) Plancherel theorem. *The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular,*

$$\|f\|_{L^2(\mu_k)} = \|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{2.4}$$

Using relations (2.2) and (2.4) with Marcinkiewicz's interpolation theorem [19, 20], we deduce that for every $1 \leq p \leq 2$, and for every $f \in L^p(\mu_k)$, the function $\mathcal{F}_k(f)$ belongs to the space $L^q(\mu_k)$, $q = p/(p-1)$, and

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \|f\|_{L^p(\mu_k)}. \tag{2.5}$$

If $f \in L^1_{rad}(\mu_k)$ with $f(x) = F(|x|)$, then

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_k(x) = \frac{1}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})} \int_0^\infty F(r) r^{2\gamma+d-1}\mathrm{d}r. \tag{2.6}$$

In the following we use the inequality (2.5) to establish $L^p$ local uncertainty principle for the Dunkl transform $\mathcal{F}_k$, more precisely, we will show the following theorem.

**Theorem 2.1.** *Let $E$ be a measurable subset of $\mathbb{R}^d$ such that $0 < \mu_k(E) < \infty$, and $a > 0$. If $1 < p \leq 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then*

$$\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \begin{cases} K_1(a)(\mu_k(E))^{\frac{a}{2\gamma+d}} \| \, |x|^a f\|_{L^p(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a)(\mu_k(E))^{1/q} \|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{qa}} \| \, |x|^a f\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{qa}}, & a > \frac{2\gamma+d}{q}, \\ 2K_1(\frac{a}{2})(\mu_k(E))^{\frac{1}{2q}} \|f\|_{L^p(\mu_k)}^{1/2} \| \, |x|^a f\|_{L^p(\mu_k)}^{1/2}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

*where*

$$K_1(a) = \frac{2\gamma+d}{2\gamma+d-qa}\left[\frac{(2\gamma+d-qa)^{q-1}}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})(qa)^q}\right]^{\frac{a}{2\gamma+d}},$$

$$K_2(a) = \frac{qa}{qa-2\gamma-d}\left(\frac{qa}{2\gamma+d}-1\right)^{\frac{2\gamma+d}{pqa}} \left[\frac{(qa-2\gamma-d)\Gamma(\frac{qa-2\gamma-d}{pa})\Gamma(\frac{2\gamma+d}{pa})}{2^{\gamma+\frac{d}{2}-1}pqa^2\Gamma(\gamma+\frac{d}{2})\Gamma(\frac{q}{p})}\right]^{\frac{1}{q}}.$$

**Proof.** (i) The first inequality holds if $\| \, |x|^a f\|_{L^p(\mu_k)} = \infty$. Assume that $\| \, |x|^a f\|_{L^p(\mu_k)} < \infty$. For $r > 0$, let $B_r = \{x : \ |x| < r\}$ and $B_r^c = \mathbb{R}^d \backslash B_r$. Denote by $\chi_E$, $\chi_{B_r}$ and $\chi_{B_r^c}$ the characteristic functions. Let $f \in L^p(\mu_k)$, $1 < p \leq 2$ and let $q = p/(p-1)$. By Minkowski's inequality, for all $r > 0$,

$$\begin{aligned} \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} &\leq \|\chi_E \mathcal{F}_k(\chi_{B_r} f)\|_{L^q(\mu_k)} + \|\chi_E \mathcal{F}_k(\chi_{B_r^c} f)\|_{L^q(\mu_k)} \\ &\leq (\mu_k(E))^{1/q} \|\mathcal{F}_k(\chi_{B_r} f)\|_{L^\infty(\mu_k)} + \|\mathcal{F}_k(\chi_{B_r^c} f)\|_{L^q(\mu_k)}; \end{aligned}$$

hence it follows from (2.2) and (2.5) that

$$\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq (\mu_k(E))^{1/q} \|\chi_{B_r} f\|_{L^1(\mu_k)} + \|\chi_{B_r^c} f\|_{L^p(\mu_k)}. \tag{2.7}$$

On the other hand, by Hölder's inequality,

$$\|\chi_{B_r} f\|_{L^1(\mu_k)} \leq \| \, |x|^{-a} \chi_{B_r}\|_{L^q(\mu_k)} \| \, |x|^a f\|_{L^p(\mu_k)}.$$

By (2.6) and hypothesis $a < (2\gamma+d)/q$,

$$\| \, |x|^{-a}\chi_{B_r}\|_{L^q(\mu_k)} = a_k r^{-a+(2\gamma+d)/q},$$

where

$$a_k = \left[(2\gamma+d-qa)2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})\right]^{-1/q},$$

and therefore,

$$\|\chi_{B_r} f\|_{L^1(\mu_k)} \leq a_k r^{-a+(2\gamma+d)/q} \| \, |x|^a f\|_{L^p(\mu_k)}. \tag{2.8}$$

Moreover,

$$\|\chi_{B_r^c} f\|_{L^p(\mu_k)} \leq \| \, |x|^{-a}\chi_{B_r^c}\|_{L^\infty(\mu_k)} \| \, |x|^a f\|_{L^p(\mu_k)} \leq r^{-a} \| \, |x|^a f\|_{L^p(\mu_k)}. \tag{2.9}$$

Combining the relations (2.7), (2.8) and (2.9), we deduce that

$$\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \Big[ r^{-a} + a_k(\mu_k(E))^{1/q} r^{-a+(2\gamma+d)/q} \Big] \| \, |x|^a f\|_{L^p(\mu_k)}.$$

We choose $r = \left(\frac{qa}{(2\gamma+d-qa)a_k}\right)^{\frac{q}{2\gamma+d}} (\mu_k(E))^{-\frac{1}{2\gamma+d}}$, we obtain the first inequality.

(ii) The second inequality holds if $\|f\|_{L^p(\mu_k)} = \infty$ or $\| \, |x|^a f\|_{L^p(\mu_k)} = \infty$. Assume that $\|f\|_{L^p(\mu_k)} + \| \, |x|^a f\|_{L^p(\mu_k)} < \infty$. From the hypothesis $a > (2\gamma+d)/q$, we deduce that the function $x \to (1+|x|^{pa})^{-1/p}$ belongs to $L^q(\mu_k)$, and by Hölder's inequality,

$$\begin{aligned} \|f\|^p_{L^1(\mu_k)} &= \left( \int_{\mathbb{R}^d} (1+|x|^{pa})^{1/p} |f(x)| (1+|x|^{pa})^{-1/p} \mathrm{d}\mu_k(x) \right)^p \\ &= \left( \int_{\mathbb{R}^d} \frac{\mathrm{d}\mu_k(x)}{(1+|x|^{pa})^{q/p}} \right)^{p/q} \Big[ \|f\|^p_{L^p(\mu_k)} + \| \, |x|^a f\|^p_{L^p(\mu_k)} \Big]. \end{aligned}$$

Then the function $f$ belongs to $L^1(\mu_k)$. Replacing $f(x)$ by $f(rx)$, $r > 0$, in the last inequality gives

$$\|f\|^p_{L^1(\mu_k)} \leq \left( \int_{\mathbb{R}^d} \frac{\mathrm{d}\mu_k(x)}{(1+|x|^{pa})^{q/p}} \right)^{p/q} \Big[ r^{(2\gamma+d)(p-1)} \|f\|^p_{L^p(\mu_k)} + r^{(2\gamma+d)(p-1)-pa} \| \, |x|^a f\|^p_{L^p(\mu_k)} \Big].$$

We choose $r = \left(\frac{qa}{2\gamma+d} - 1\right)^{\frac{1}{pa}} \left( \frac{\| \, |x|^a f\|_{L^p(\mu_k)}}{\|f\|_{L^p(\mu_k)}} \right)^{1/a}$ and the fact that

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu_k(x)}{(1+|x|^{pa})^{q/p}} = \frac{1}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})} \int_0^\infty \frac{r^{2\gamma+d-1}\mathrm{d}r}{(1+r^{pa})^{q/p}} = \frac{\Gamma(\frac{qa-2\gamma-d}{pa})\Gamma(\frac{2\gamma+d}{pa})}{2^{\gamma+\frac{d}{2}-1} pa \Gamma(\gamma+\frac{d}{2})\Gamma(\frac{q}{p})},$$

we deduce that

$$\|f\|_{L^1(\mu_k)} \leq K_2(a) \|f\|^{1-\frac{2\gamma+d}{qa}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{2\gamma+d}{qa}}_{L^p(\mu_k)}.$$

Thus,

$$\begin{aligned} \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} &\leq (\mu_k(E))^{1/q} \|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \\ &\leq (\mu_k(E))^{1/q} \|f\|_{L^1(\mu_k)} \\ &\leq K_2(a)(\mu_k(E))^{1/q} \|f\|^{1-\frac{2\gamma+d}{qa}}_{L^p(\mu_k)} \| \, |x|^a f\|^{\frac{2\gamma+d}{qa}}_{L^p(\mu_k)}, \end{aligned}$$

which gives the second inequality.

(iii) Let $r > 0$. From the inequality $\left(\frac{|x|}{r}\right)^{\frac{2\gamma+d}{2q}} \leq 1 + \left(\frac{|x|}{r}\right)^{\frac{2\gamma+d}{q}}$, it follows that

$$\| \, |x|^{\frac{2\gamma+d}{2q}} f\|_{L^p(\mu)} \leq r^{\frac{2\gamma+d}{2q}} \|f\|_{L^p(\mu)} + r^{-\frac{2\gamma+d}{2q}} \| \, |x|^{\frac{2\gamma+d}{q}} f\|_{L^p(\mu)}.$$

Optimizing in $r$, we get

$$\| \, |x|^{\frac{2\gamma+d}{2q}} f\|_{L^p(\mu)} \leq 2 \|f\|^{1/2}_{L^p(\mu)} \| \, |x|^{\frac{2\gamma+d}{q}} f\|^{1/2}_{L^p(\mu)}.$$

Thus, we deduce that

$$\begin{aligned} \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)} &\leq K_1(\frac{2\gamma+d}{2q})(\mu_k(E))^{\frac{1}{2q}} \| \, |x|^{\frac{2\gamma+d}{2q}} f\|_{L^p(\mu)} \\ &\leq 2K_1(\frac{2\gamma+d}{2q})(\mu_k(E))^{\frac{1}{2q}} \|f\|^{1/2}_{L^p(\mu)} \| \, |x|^{\frac{2\gamma+d}{q}} f\|^{1/2}_{L^p(\mu)}, \end{aligned}$$

which gives the result for $a = (2\gamma + d)/q$. □

*Remark* 2.1. Let $a > 0$. If $1 < p \le 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then

$$\|f\|_{L^{\frac{q(2\gamma+d)}{2\gamma+d-qa},q}(\mu_k)} \le K_1(a)\| \,|x|^a f\|_{L^p(\mu_k)}, \quad 0 < a < (2\gamma+d)/q,$$

$$\|f\|_{L^{\infty,q}(\mu_k)} \le K_2(a)\|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{2a}} \| \,|x|^a f\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{2a}}, \quad a > (2\gamma+d)/q,$$

$$\|f\|_{L^{2q,q}(\mu_k)} \le 2K_1(\frac{a}{2})\|f\|_{L^p(\mu_k)}^{1/2} \| \,|x|^a f\|_{L^p(\mu_k)}^{1/2}, \quad a = (2\gamma+d)/q,$$

where $L^{s,q}(\mu_k)$ is the Lorentz-space defined by the norm

$$\|f\|_{L^{s,q}(\mu_k)} := \sup_{\substack{E\subset\mathbb{R}^d \\ 0<\mu_k(E)<\infty}} \Big( (\mu_k(E))^{\frac{1}{s}-\frac{1}{q}} \|\chi_E f\|_{L^q(\mu_k)} \Big).$$

In the next part of this section, we shall use the $L^p$ local uncertainty principle (Theorem 2.1) to extend the Heisenberg-Pauli-Weyl uncertainty principles (1.1) and (1.2) to more general case.

**Theorem 2.2.** *Let $a, b > 0$,If $1 < p \le 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then*

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le \begin{cases} K_1(a,b)\| \,|x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+b}} \| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+b}}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a,b)\|f\|_{L^p(\mu_k)}^{\frac{b(qa-2\gamma-d)}{a(qb+2\gamma+d)}} \| \,|x|^a f\|_{L^p(\mu_k)}^{\frac{b(2\gamma+d)}{a(2\gamma+d+qb)}} \| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{2\gamma+d}{2\gamma+d+qb}}, & a > \frac{2\gamma+d}{q}, \\ K_3(a,b)\|f\|_{L^p(\mu_k)}^{\frac{b}{a+2b}} \| \,|x|^a f\|_{L^p(\mu_k)}^{\frac{b}{a+2b}} \| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)}^{\frac{a}{a+2b}}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

*where*

$$K_1(a,b) = \frac{\left[ \left(\frac{b}{a}\right)^{\frac{a}{a+b}} + \left(\frac{a}{b}\right)^{\frac{b}{a+b}} \right]^{1/q}}{\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{\frac{ab}{(2\gamma+d)(a+b)}}} (K_1(a))^{\frac{b}{a+b}},$$

$$K_2(a,b) = \frac{\left[ \left(\frac{qb}{2\gamma+d}\right)^{\frac{2\gamma+d}{2\gamma+d+qb}} + \left(\frac{2\gamma+d}{qb}\right)^{\frac{qb}{2\gamma+d+qb}} \right]^{1/q}}{\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{\frac{b}{2\gamma+d+qb}}} (K_2(a))^{\frac{qb}{2\gamma+d+qb}},$$

*and*

$$K_3(a,b) = \frac{\left[ \left(\frac{2b}{a}\right)^{\frac{a}{a+2b}} + \left(\frac{a}{2b}\right)^{\frac{2b}{a+2b}} \right]^{1/q}}{\left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{\frac{b}{2\gamma+d+2qb}}} (2K_1(\frac{a}{2}))^{\frac{2b}{a+2b}}.$$

**Proof.** (i) Let $0 < a < (2\gamma+d)/q$, $b > 0$ and $r > 0$. Then

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}^q = \|\chi_{B_r}\mathcal{F}_k(f)\|_{L^q(\mu_k)}^q + \|\chi_{B_r^c}\mathcal{F}_k(f)\|_{L^q(\mu_k)}^q. \tag{2.10}$$

Firstly,

$$\|\chi_{B_r^c}\mathcal{F}_k(f)\|_{L^q(\mu_k)}^q \le r^{-qb}\| \,|y|^b \mathcal{F}_k(f)\|_{L^q(\mu_k)}^q. \tag{2.11}$$

By (2.6) and Theorem 2.1, we get

$$\|\chi_{B_r}\mathcal{F}_k(f)\|_{L^q(\mu_k)}^q \le K_1 r^{qa}\| \,|x|^a f\|_{L^p(\mu_k)}^q, \tag{2.12}$$

where

$$K_1 = \Big(K_1(a)\Big)^q \left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{-\frac{qa}{2\gamma+d}}.$$

Combining the relations (2.10), (2.11) and (2.12), we obtain

$$\|\mathcal{F}_k(f)\|^q_{L^q(\mu_k)} \le K_1 r^{qa} \|\,|x|^a f\|^q_{L^p(\mu_k)} + r^{-qb}\|\,|y|^b \mathcal{F}_k(f)\|^q_{L^q(\mu_k)}.$$

We choose $r = \left(\frac{b}{aK_1}\right)^{\frac{1}{q(a+b)}} \left(\frac{\|\,|y|^b\mathcal{F}_k(f)\|_{L^q(\mu_k)}}{\|\,|x|^a f\|_{L^p(\mu_k)}}\right)^{\frac{1}{a+b}}$, we get the first inequality.

(ii) Let $a > (2\gamma+d)/q$, $b > 0$ and $r > 0$. By (2.6) and Theorem 2.1, we get

$$\|\chi_{B_r}\mathcal{F}_k(f)\|^q_{L^q(\mu_k)} \le K_2 r^{2\gamma+d}\|f\|^{q-\frac{2\gamma+d}{a}}_{L^p(\mu_k)}\|\,|x|^a f\|^{\frac{2\gamma+d}{a}}_{L^p(\mu_k)}, \tag{2.13}$$

where

$$K_2 = (K_2(a))^q \left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{-1}.$$

Combining the relations (2.10), (2.11) and (2.13), we obtain

$$\|\mathcal{F}_k(f)\|^q_{L^q(\mu_k)} \le K_2 r^{2\gamma+d}\|f\|^{q-\frac{2\gamma+d}{a}}_{L^p(\mu_k)}\|\,|x|^a f\|^{\frac{2\gamma+d}{a}}_{L^p(\mu_k)} + r^{-qb}\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}.$$

We choose $r = \left(\frac{qb}{(2\gamma+d)K_2}\right)^{\frac{1}{2\gamma+d+qb}} \left(\frac{\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}}{\|f\|^{q-\frac{2\gamma+d}{a}}_{L^p(\mu_k)}\|\,|x|^a f\|^{\frac{2\gamma+d}{a}}_{L^p(\mu_k)}}\right)^{\frac{1}{2\gamma+d+qb}}$, we get the second inequality.

(iii) Let $a = (2\gamma+d)/q$, $b > 0$ and $r > 0$. From Theorem 2.1, we get

$$\int_{B_r}|\mathcal{F}_k(f)(y)|^q \mathrm{d}\mu_k(y) \le K_3 r^{\gamma+\frac{d}{2}}\|f\|^{q/2}_{L^p(\mu_k)}\|\,|x|^{\frac{2\gamma+d}{q}}f\|^{q/2}_{L^p(\mu_k)},$$

where

$$K_3 = (K_1(\frac{2\gamma+d}{2q}))^q \left[2^{\gamma+\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)\right]^{-1/2}.$$

Therefore,

$$\|\mathcal{F}_k(f)\|^q_{L^q(\mu_k)} \le K_3 r^{\gamma+\frac{d}{2}}\|f\|^{q/2}_{L^p(\mu_k)}\|\,|x|^{\frac{2\gamma+d}{q}}f\|^{q/2}_{L^p(\mu_k)} + r^{-qb}\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}.$$

We choose $r = \left(\frac{2qb}{(2\gamma+d)K_3}\right)^{\frac{2}{2\gamma+d+2qb}} \left(\frac{\|\,|y|^b\mathcal{F}_k(f)\|^q_{L^q(\mu_k)}}{\|f\|^{1/2}_{L^p(\mu_k)}\|\,|x|^{\frac{2\gamma+d}{q}}f\|^{1/2}_{L^p(\mu_k)}}\right)^{\frac{2q}{2\gamma+d+2qb}}$, we get the third inequality. □

*Remark* 2.2. The inequalities of Theorem 2.2 generalize the results of the papers [12, 13, 17]. Furthermore, we have explicitly given the values of the constants $K_1(a,b)$, $K_2(a,b)$ and $K_3(a,b)$. In particular case, if $q = 2$, the inequalities of Theorem 2.2 are given by

$$\|f\|_{L^2(\mu_k)} \le K(a,b)\|\,|x|^a f\|^{\frac{b}{a+b}}_{L^2(\mu_k)}\|\,|y|^b\mathcal{F}_k(f)\|^{\frac{a}{a+b}}_{L^2(\mu_k)},$$

where

$$K(a,b) = \begin{cases} K_1(a,b), & 0 < a < (2\gamma+d)/2,\ b > 0, \\ (K_2(a,b))^{\frac{a(2\gamma+d+2b)}{(2\gamma+d)(a+b)}}, & a > (2\gamma+d)/2,\ b > 0, \\ (K_3(a,b))^{\frac{a+2b}{a+b}}, & a = (2\gamma+d)/2,\ b > 0. \end{cases}$$

Here $K_1(a,b)$, $K_2(a,b)$ and $K_3(a,b)$ are the constants given by Theorem 2.2 with $p=q=2$.

## 3. $L^p$ Donoho-Stark uncertainty principles

Let $T$ and $E$ be a measurable subsets of $\mathbb{R}^d$. We introduce the time-limiting operator $P_T$ by

$$P_T f := \chi_T f,$$

and, we introduce the partial Dunkl integral $S_E f$ by

$$\mathcal{F}_k(S_E f) = \chi_E \mathcal{F}_k(f). \tag{3.1}$$

We shall use the $L^p$ local uncertainty principle (Theorem 2.1) to obtain the following results for the partial Dunkl integral $S_E f$.

**Lemma 3.1.** *(i) If $\mu_k(E) < \infty$ and $f \in L^p(\mu_k)$, $1 \le p \le 2$,*

$$S_E f(x) = \mathcal{F}_k^{-1}(\chi_E \mathcal{F}_k(f))(x).$$

*(ii) If $0 < \mu_k(E) < \infty$, $a > 0$, $1 < p \le 2$, $q = p/(p-1)$ and $f \in L^p(\mu_k)$, then*

$$\|S_E f\|_{L^q(\mu_k)} \le \begin{cases} K_1(a)(\mu_k(E))^{\frac{2}{p}+\frac{a}{2\gamma+d}-1}\| \, |x|^a f\|_{L^p(\mu_k)}, & 0 < a < \frac{2\gamma+d}{q}, \\ K_2(a)(\mu_k(E))^{1/p}\|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{qa}} \| \, |x|^a f\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{qa}}, & a > \frac{2\gamma+d}{q}, \\ 2K_1(\frac{a}{2})(\mu_k(E))^{\frac{3}{2p}-\frac{1}{2}}\|f\|_{L^p(\mu_k)}^{1/2} \| \, |x|^a f\|_{L^p(\mu_k)}^{1/2}, & a = \frac{2\gamma+d}{q}, \end{cases}$$

*where $K_1(a)$ and $K_2(a)$ are the constants given by Theorem 2.1.*

**Proof.** (i) Let $f \in L^p(\mu_k)$, $1 \le p \le 2$ and let $q = p/(p-1)$. Then by Hölder's inequality and (2.5), we have

$$\|\chi_E \mathcal{F}_k(f)\|_{L^1(\mu_k)} \le (\mu_k(E))^{1/p}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le (\mu_k(E))^{1/p}\|f\|_{L^p(\mu_k)},$$

and

$$\|\chi_E \mathcal{F}_k(f)\|_{L^2(\mu_k)} \le (\mu_k(E))^{\frac{q-2}{2q}}\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \le (\mu_k(E))^{\frac{q-2}{2q}}\|f\|_{L^p(\mu_k)}.$$

Thus $\chi_E \mathcal{F}_k(f) \in L^1(\mu_k) \cap L^2(\mu_k)$. Then by (2.3) and (3.1), we obtain

$$S_E f = \mathcal{F}_k^{-1}(\chi_E \mathcal{F}_k(f)).$$

(ii) Let $f \in L^p(\mu_k)$, $1 < p \le 2$ and let $q = p/(p-1)$. By (2.5) and Hölder's inequality, we have

$$\|S_E f\|_{L^q(\mu_k)} \le \|\chi_E \mathcal{F}_k(f)\|_{L^p(\mu_k)} \le (\mu_k(E))^{\frac{2}{p}-1}\|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Then we obtain the results from Theorem 2.1. □

Let $T$ be a measurable subset of $\mathbb{R}^d$. We say that a function $f \in L^p(\mu_k)$, $1 \le p \le 2$, is $\varepsilon$-concentrated to $T$ in $L^p(\mu_k)$-norm, if

$$\|f - P_T f\|_{L^p(\mu_k)} \le \varepsilon_T \|f\|_{L^p(\mu_k)}. \tag{3.2}$$

Let $E$ be a measurable subset of $\mathbb{R}^d$, and $f \in L^p(\mu_k)$, $1 \le p \le 2$. We say that $\mathcal{F}_k(f)$ is $\varepsilon_E$-concentrated to $E$ in $L^q(\mu_k)$-norm, $q = p/(p-1)$, if

$$\|\mathcal{F}_k(f) - \mathcal{F}_k(S_E f)\|_{L^q(\mu_k)} \le \varepsilon_E \|\mathcal{F}_k(f)\|_{L^q(\mu_k)}. \tag{3.3}$$

Let $B_p(E)$, $1 \le p \le 2$, be the set of functions $f \in L^p(\mu_k)$ that are bandlimited to $E$ (i.e. $f \in B_p(E)$ implies $S_E f = f$).

Then, the space $B_p(E)$ satisfies the following property.

**Lemma 3.2.** *Let $T$ and $E$ be a measurable subsets of $\mathbb{R}^d$ such that $0<\mu_k(E)<\infty$, and $a>0$. If $1<p\le 2$, $q=p/(p-1)$ and $f\in B_p(E)$, then*

$$\|P_Tf\|_{L^p(\mu_k)}\le\begin{cases} K_1(a)(\mu_k(T))^{1/p}(\mu_k(E))^{\frac{1}{p}+\frac{a}{2\gamma+d}}\|\,|x|^af\|_{L^p(\mu_k)}, & 0<a<\frac{2\gamma+d}{q},\\ K_2(a)(\mu_k(T))^{1/p}\mu_k(E)\|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{qa}}\|\,|x|^af\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{qa}}, & a>\frac{2\gamma+d}{q},\\ 2K_1(\frac{a}{2})(\mu_k(T))^{1/p}(\mu_k(E))^{\frac{1}{2p}+\frac{1}{2}}\|f\|_{L^p(\mu_k)}^{1/2}\|\,|x|^af\|_{L^p(\mu_k)}^{1/2}, & a=\frac{2\gamma+d}{q},\end{cases}$$

**Proof.** If $\mu_k(T)=\infty$, the inequality is clear. Assume that $\mu_k(T)<\infty$. For $f\in B_p(E)$, $1<p\le 2$, from Lemma 3.1 (i), we have

$$S_Ef(x)=\mathcal{F}_k^{-1}(\chi_E\mathcal{F}_k(f))(x).$$

By (2.1) and Hölder's inequality, we obtain

$$|f(x)|\le(\mu_k(E))^{1/p}\|\chi_E\mathcal{F}_k(f)\|_{L^q(\mu_k)},\quad q=p/(p-1).$$

Hence,

$$\|P_Tf\|_{L^p(\mu_k)}\le(\mu_k(T))^{1/p}(\mu_k(E))^{1/p}\|\chi_E\mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Then we obtain the results Theorem 2.1. □

The following theorem, states an uncertainty principle of concentration type for the $L^p$ theory.

**Theorem 3.1.** *Let $T$ and $E$ be a measurable subsets of $\mathbb{R}^d$ such that $0<\mu_k(E)<\infty$, and $a>0$. If $1<p\le 2$, $q=p/(p-1)$, $f\in B_p(E)$ and $f$ is $\varepsilon_T$-concentrated to $T$ in $L^p(\mu_k)$-norm, then*

$$\|f\|_{L^p(\mu_k)}\le\begin{cases} \frac{K_1(a)}{1-\varepsilon_T}(\mu_k(T))^{1/p}(\mu_k(E))^{\frac{1}{p}+\frac{a}{2\gamma+d}}\|\,|x|^af\|_{L^p(\mu_k)}, & 0<a<\frac{2\gamma+d}{q},\\ \left(\frac{K_2(a)}{1-\varepsilon_T}\right)^{\frac{qa}{2\gamma+d}}(\mu_k(T))^{\frac{qa}{p(2\gamma+d)}}(\mu_k(E))^{\frac{qa}{2\gamma+d}}\|\,|x|^af\|_{L^p(\mu_k)}, & a>\frac{2\gamma+d}{q},\\ \left(\frac{2K_1(\frac{a}{2})}{1-\varepsilon_T}\right)^2(\mu_k(T))^{2/p}(\mu_k(E))^{\frac{1}{p}+1}\|\,|x|^af\|_{L^p(\mu_k)}, & a=\frac{2\gamma+d}{q},\end{cases}$$

**Proof.** Let $f\in B_p(E)$, $1<p\le 2$. Since $f$ is $\varepsilon_T$-concentrated to $T$ in $L^p(\mu_k)$-norm, then by (3.2), we have

$$\|f\|_{L^p(\mu_k)}\le\varepsilon_T\|f\|_{L^p(\mu_k)}+\|P_Tf\|_{L^p(\mu_k)}.$$

Thus,

$$\|f\|_{L^p(\mu_k)}\le\frac{1}{1-\varepsilon_T}\|P_Tf\|_{L^p(\mu_k)}.$$

Then we obtain the results from Lemma 3.2. □

Another uncertainty principle of concentration type for the $L^p$ theory is given by the following theorem.

**Theorem 3.2.** *Let $E$ be a measurable subset of $\mathbb{R}^d$ such that $0<\mu_k(E)<\infty$, and $a>0$. If $1<p\le 2$, $q=p/(p-1)$, $f\in L^p(\mu_k)$ and $\mathcal{F}_k(f)$ is $\varepsilon_E$-concentrated to $E$ in $L^q(\mu_k)$-norm, then*

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)}\le\begin{cases} \frac{K_1(a)}{1-\varepsilon_E}(\mu_k(E))^{\frac{a}{2\gamma+d}}\|\,|x|^af\|_{L^p(\mu_k)}, & 0<a<\frac{2\gamma+d}{q},\\ \frac{K_2(a)}{1-\varepsilon_E}(\mu_k(E))^{1/q}\|f\|_{L^p(\mu_k)}^{1-\frac{2\gamma+d}{qa}}\|\,|x|^af\|_{L^p(\mu_k)}^{\frac{2\gamma+d}{qa}}, & a>\frac{2\gamma+d}{q},\\ \frac{2K_1(\frac{a}{2})}{1-\varepsilon_E}(\mu_k(E))^{\frac{1}{2q}}\|f\|_{L^p(\mu_k)}^{1/2}\|\,|x|^af\|_{L^p(\mu_k)}^{1/2}, & a=\frac{2\gamma+d}{q},\end{cases}$$

**Proof.** Let $f \in L^p(\mu_k)$, $1 < p \leq 2$. Since $\mathcal{F}_k(f)$ is $\varepsilon_E$-concentrated to $E$ in $L^q(\mu_k)$-norm, $q = p/(p-1)$, then by (3.3), we deduce that

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \varepsilon_E \|\mathcal{F}_k(f)\|_{L^q(\mu_k)} + \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Thus,

$$\|\mathcal{F}_k(f)\|_{L^q(\mu_k)} \leq \frac{1}{1-\varepsilon_E} \|\chi_E \mathcal{F}_k(f)\|_{L^q(\mu_k)}.$$

Then we obtain the results from Theorem 2.1. □

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