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ON THE ISOMETRIES OF 3-DIMENSIONAL MAXIMUM SPACE

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Abstract. In this article, the hexahedron associated to metric geometry fullfiled by the metric of which unit sphere is hexahedron. We have analytically proved that the isometry group of the space with respect to this metric is the semi direct product of the Euclidean symmetry group of the cube and $T(3)$ which is all translations of analytical $3-$space.

1. Introduction

Many geometric studies and investigations are concerned with transformations of geometric objects on various spaces. Some of the transformations form group. Many of these groups consist simply of the symmetries of those spaces. The Platonic solids provide an excellent model for the investigation of symmetries. Also, Platonic solids are very important in the sense that they can be used not only in studies on properties of geometric structures, but also investigations on physical and chemical properties of the system under consideration. The isometry group have extensive applications in the theory of molecular and crystalline structure [1], [6]. The importance of isometries is that they preserve some of geometric properties; distance, angle measure, congruence, betweenness, and incidence [4], [5], [7], [8]. The isometry group is a fundamental concept in art as well as science. To develop this concept, it must be given a precise mathematical formulation.

Through the article we will use the definitions, explanations, propositions and the methods of proofs in the main reference [3].

2. The Maximum Metric

It is important to work on concepts related to the distance in geometric studies, because change of metric can reveals interesting results. What appears to be essential here is the way in which the lengths are to be measured. The present study aims to present isometry group of $\mathbb{R}^3$ by achieving the measuring process via the maximum metric $d_{\mathbf{M}}$ in preference to the usual Euclidean metric $d_{\mathbf{E}}$.

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For the sake of simple, $\mathbb{R}^3$ fullfiled by maximum metric is denoted $\mathbb{R}^3_{\mathbf{M}}$ in the rest of the article. Linear structure except distance function in the $\mathbb{R}^3_{\mathbf{M}}$ is the same as Euclidean analytical space [9]. This distance function $d_{\mathbf{M}}$ is defined as following.

Definition 2.1. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{\mathbf{M}} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow [0, \infty)$ defined by

$$d_{\mathbf{M}}(P_1, P_2) := \max\{|x_2 - x_1|, |y_2 - y_1|, |z_2 - z_1|\}$$

is called maximum distance function.

According to this distance function, the unit sphere is a hexahedron in the $\mathbb{R}^3_{\mathbf{M}}$.

Proposition 2.1. *The distance function $d_{\mathbf{M}}$ is a metric of which unit sphere is cube in $\mathbb{R}^3$ (see Figure 2.1).*

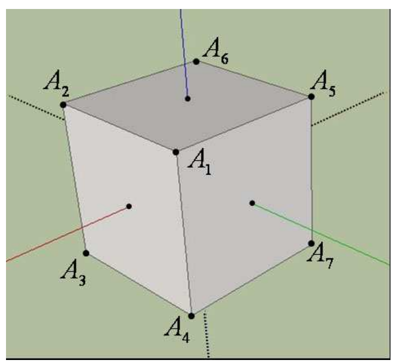

Figure 2.1

Proposition 2.2. *Given any two points A and B in $\mathbb{R}^3_{\mathbf{M}}$. Let direction vector of the line l through A and B be (p, q, r). Then,*

$$d_{\mathbf{E}}(A, B) = \mu(AB)\, d_{\mathbf{M}}(A, B)$$

where

$$\mu(AB) = \frac{\max\{|p|, |q|, |r|\}}{\sqrt{p^2 + q^2 + r^2}}.$$

Proof. Let $A=(x_1, y_1, z_1)$ and $B=(x_2, y_2, z_2)$ be two points in $\mathbb{R}^3_{\mathbf{M}}$. If line l with direction vector (p, q, r) passes through the points A and B, then $\overleftrightarrow{AB} \parallel (p, q, r)$. Therefore $\overrightarrow{AB} = \lambda(p, q, r)$ such that $\lambda \in \mathbb{R}\setminus\{0\}$. So

$$d_{\mathbf{M}}(A, B) = |\lambda| \max\{|p|, |q|, |r|\}$$

and similarly,

$$d_{\mathbf{E}}(A, B) = |\lambda| \sqrt{(p)^2 + (q)^2 + (r)^2}.$$

Consequently $\dfrac{d_{\mathbf{M}}(A, B)}{d_{\mathbf{E}}(A, B)} = \dfrac{\max\{|p|, |q|, |r|\}}{\sqrt{p^2 + q^2 + r^2}}$ is obtained. □

3. Isometries of the $\mathbb{R}^3_{\mathbf{M}}$

We want to show that isometry group of the maximum space $\mathbb{R}^3_{\mathbf{M}}$ in this section. At the end of this section we are going to show isometry group of $\mathbb{R}^3_{\mathbf{M}}$ is the semi direct product of " Euclidean symmetry group of cube " and " all translations of $\mathbb{R}^3$ ". Also, O_h consist of identity, reflections, rotations, inversion, rotary reflection and rotary inversions. Before we give isometries of $\mathbb{R}^3_{\mathbf{M}}$, we introduce elements of the set O_h.

A transformation is any function mapping a set to itself in $\mathbb{R}^3$. A figure in $\mathbb{R}^3$ is any subset of $\mathbb{R}^3$. An isometry of $\mathbb{R}^3_{\mathbf{M}}$ is a transformation from $\mathbb{R}^3$ onto $\mathbb{R}^3$ that preserves distance. This means $d_{\mathbf{M}}(X,Y) = d_{\mathbf{M}}(\alpha(X),\alpha(Y))$ for each points X and Y in $\mathbb{R}^3_{\mathbf{M}}$. A symmetry of a figure F in $\mathbb{R}^3$ is an isometry mapping F onto itself—that is, an isometry $f:\mathbb{R}^3 \to \mathbb{R}^3$ such that $f(F) = F$. The identity function I is a transformation is given $I(X) = X$ for each point X in $\mathbb{R}^3$. If Δ represents a plane, then the reflection σ_Δ across the plane Δ fixes every point on Δ, and takes every point X not on Δ to Y, where Δ is the perpendicular bisector of X and Y. A rotation is an isometric transformation which can be written as the composition of two distinc reflections. That is, a rotation about axis l is defined by $\sigma_\Delta\sigma_\Gamma$ where two planes Γ and Δ intersect at line l. A rotary reflection is an transformation which is the combination of a rotation about an axis and a reflection in a plane. That is, a rotary reflection is defined by $\sigma_\Pi\sigma_\Delta\sigma_\Gamma$ such that Γ and Δ are two intersecting planes each perpendicular to plane Π. A inversion according to a point P can be written as the $\sigma_P(X) = Y$ such that P is the midpoint of X and Y for X , $Y \in \mathbb{R}^3$. Rotary inversion is the combination of a rotation and an inversion in a point.

Proposition 3.1. *All Euclidean translation in $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3_{\mathbf{M}}$.*

Proof. Given a points $A = (a_1, a_2, a_3)$ in $\mathbb{R}^3_{\mathbf{M}}$. The translation $T_A : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ is a mapping such that $T_A(X) = A + X$.

Let $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$ be any two points in $\mathbb{R}^3_{\mathbf{M}}$, then

$$\begin{aligned} d_{\mathbf{M}}(T_A(X), T_A(Y)) &= \max\left\{ \begin{array}{r} |(a_1+x_2)-(a_1+x_1)|, |(a_2+y_2)-(a_2+y_1)| \\ , |(a_3+z_2)-(a_3+z_1)| \end{array} \right\} \\ &= d_{\mathbf{M}}(X,Y). \end{aligned}$$

This means that translation T_A is an isometry. □

Therefore, we now consider planes passing through the origin for all calculations in the rest of the article.

The following proposition gives reflections which preserve distance $\mathbb{R}^3_{\mathbf{M}}$.

Proposition 3.2. *Given the plane Δ having equation $ax+by+cz = 0$ in $\mathbb{R}^3_{\mathbf{M}}$. Reflection σ_Δ is a isometry iff unit normal vector of the plane Δ is written as $\lambda.\vec{V}$ where λ is a scalar and $\vec{V} \in D$ such that*

$$D = \{(1,0,0), (0,1,0), (0,0,1), (\pm1,1,0), (\pm1,0,1), (0,\pm1,1)\}.$$

Proof. Euclidean reflection $\sigma_\Delta : \mathbb{R}^3 \to \mathbb{R}^3$ can be defined by

$$\sigma_\Delta(x,y,z) = \left(\begin{array}{r} (1-2a^2)x - 2aby - 2acz, -2abx + (1-2b^2)y - 2bcz, \\ , -2acx - 2bcy + (1-2c^2)z \end{array} \right)$$

such that (a,b,c) is the unit normal vector of the plane Δ.

We know that the reflection preserving a base of $\mathbb{R}^3$ is a isometry. If we take vector set $T=\{A_1=(1,1,1), A_2=(1,-1,1), A_3=(1,-1,-1)\}$ as base of $\mathbb{R}^3$, we shall find that reflections which preserve vectors of this base. To find reflections, we shall calculate $a,\ b,\ c$. If we calculate image of set T under Euclidean reflection, we get

$$\begin{aligned}
\sigma_\Delta(A_1) &= \left(1-2a^2-2ab-2ac, -2ab+1-2b^2-2bc, -2ac-2bc+1-2c^2\right),\\
\sigma_\Delta(A_2) &= \left(1-2a^2+2ab-2ac, -2ab-1+2b^2-2bc, -2ac+2bc+1-2c^2\right),\\
\sigma_\Delta(A_3) &= \left(1-2a^2+2ab+2ac, -2ab-1+2b^2+2bc, -2ac+2bc-1+2c^2\right).
\end{aligned}$$

If reflection preserves $d_{\mathbf{M}}-$distance, we have three equations;

$$\begin{aligned}
d_{\mathbf{M}}(O,A_1) &= d_{\mathbf{M}}(\sigma_\Delta(O), \sigma_\Delta(A_1)) = 1\\
d_{\mathbf{M}}(O,A_2) &= d_{\mathbf{M}}(\sigma_\Delta(O), \sigma_\Delta(A_2)) = 1\\
d_{\mathbf{M}}(O,A_3) &= d_{\mathbf{M}}(\sigma_\Delta(O), \sigma_\Delta(A_3)) = 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\max\left\{\left|1\text{-}2a^2\text{-}2ab\text{-}2ac\right|, \left|\text{-}2ab+1\text{-}2b^2\text{-}2bc\right|, \left|\text{-}2ac\text{-}2bc+1\text{-}2c^2\right|\right\}=1\\
&\max\left\{\left|1\text{-}2a^2+2aby\text{-}2ac\right|, \left|\text{-}2ab\text{-}1+2b^2y\text{-}2bc\right|, \left|\text{-}2ac+2bc+1\text{-}2c^2\right|\right\}=1\\
&\max\left\{\left|1\text{-}2a^2+2ab+2ac\right|, \left|\text{-}2ab\text{-}1+2b^2+2bc\right|, \left|\text{-}2ac+2bc\text{-}1+2c^2\right|\right\}=1
\end{aligned}$$

is obtained. Consequently, we have the system of equations with three unknows $a,\ b$ and c. Solving these system of equations for $a,\ b$ and c, we get

$$\begin{gathered}
(\mp1,0,0),\ (0,\mp1,0),\ (0,0,\mp1),\\
\left(0,\mp\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2}\right),\ \left(\mp\frac{\sqrt{2}}{2},0,\mp\frac{\sqrt{2}}{2}\right),\ \left(\mp\frac{\sqrt{2}}{2},\mp\frac{\sqrt{2}}{2},0\right).
\end{gathered}$$

Conversely, we shall show that reflections σ_Δ preserve distance $d_{\mathbf{M}}$. Given reflection σ_Δ such that $\sigma_\Delta(X)=Y$ for $X,\ Y\in\mathbb{R}^3_{\mathbf{M}}$. Let (p_1,q_1,r_1) and (p_2,q_2,r_2) be the direction vectors of the lines OX and OY, respectively. If $\mu(OX)=\mu(OY)$, then $d_{\mathbf{M}}(O,X)=d_{\mathbf{M}}(O,Y)$ is obtained by Proposition 2. 2.

To show $d_{\mathbf{M}}(O,X)=d_{\mathbf{M}}(O,Y)$, we must check for all possible;

Δ	(p_2,q_2,r_2)
$x=0$	$(-p_1,q_1,r_1)$
$y=0$	$(p_1,-q_1,r_1)$
$z=0$	$(p_1,q_1,-r_1)$
$x+y=0$	$(-q_1,-p_1,r_1)$
$x-y=0$	(q_1,p_1,r_1)

Δ	(p_2,q_2,r_2)
$x+z=0$	$(-r_1,q_1,-p_1)$
$x-z=0$	(r_1,q_1,p_1)
$y+z=0$	$(p_1,-r_1,-q_1)$
$y-z=0$	(p_1,r_1,q_1)

□

Corollary 3.1. *In $\mathbb{R}^3_{\mathbf{M}}$, nine Euclidean reflections according to the planes having equations $x=0,\ y=0,\ z=0,\ x+y=0,\ x-y=0,\ x+z=0,\ x-z=0,\ y+z=0,\ y-z=0$ are isometric reflections.*

Following proposition tell us isometric rotations in $\mathbb{R}^3_{\mathbf{M}}$.

Proposition 3.3. *Given a rotation $r_\theta:\mathbb{R}^3_{\mathbf{M}}\to\mathbb{R}^3_{\mathbf{M}}$ according to l having equation $\frac{x}{p}=\frac{y}{q}=\frac{z}{r}$. Rotation r_θ is an isometry iff $r_\theta\in R_M=R_1\cup R_2\cup R_3$ such that*

$$R_1=\left\{r_\theta:\theta\in\left\{\frac{\pi}{2},\pi,\frac{3\pi}{2}\right\},\ \textit{rotation axis has a direction vector in } D_1\right\},$$

$R_2 = \left\{ r_\theta : \theta \in \left\{ \frac{2\pi}{3}, \frac{4\pi}{3} \right\},\ \textit{rotation axis has a direction vector in } D_2 \right\},$

$R_3 = \{ r_\theta : \theta \in \{\pi\},\ \textit{rotation axis has a direction vector in } D_3 \},$

where

$D_1 = \{(1,0,0), (0,1,0), (0,0,1)\},$

$D_2 = \{(1,1,1), (-1,1,1), (1,-1,1), (1,1,-1)\},$

and

$D_3 = \{(1,1,0), (1,-1,0), (1,0,1), (1,0,-1), (0,1,1), (0,1,-1)\}.$

Proof. Recall that if $r_\theta : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ according to l having equation $\frac{x}{p} = \frac{y}{q} = \frac{z}{r}$ where (p,q,r) is a unit vector is a Euclidean rotation, then r_θ has following matrix representation:

$$\begin{bmatrix} \cos\theta + p^2(1-\cos\theta) & pq(1-\cos\theta) - r\sin\theta & pr(1-\cos\theta) + q\sin\theta \\ pq(1-\cos\theta) + r\sin\theta & \cos\theta + q^2(1-\cos\theta) & qr(1-\cos\theta) - p\sin\theta \\ pr(1-\cos\theta) - q\sin\theta & qr(1-\cos\theta) + p\sin\theta & \cos\theta + r^2(1-\cos\theta) \end{bmatrix}$$

A rotation can be written as the combination of two distinc reflections. So, a rotation with axis l can be defined by $\sigma_\Delta \sigma_\Gamma$ where l is line of intersection between planes Γ and Δ. Consequently, vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$ $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, $(1,\pm1,0)$, $(1,0,\pm1)$, $(0,1,\pm1)$, $(0,1,-1)$ can be take as direction vector of the line l by Corollary 3.1. To show isometric rotations in $\mathbb{R}^3_{\mathbf{M}}$, our next step is to give that rotations which preserve the lenghts of the edges of the unit sphere. To do this it will be enough to find isometric rotations. Let $A_1 = (1,1,1)$ and $A_2 = (1,-1,1)$ be points on the unit sphere. If we find image of A_1 and A_2 under r_θ, we get

$$r_\theta(A_1) = \begin{pmatrix} \cos\theta + p^2(1-\cos\theta) + pq(1-\cos\theta) - r\sin\theta + pr(1-\cos\theta) + q\cos\theta \\ ,\ pq(1-\cos\theta) + r\sin\theta + \cos\theta + q^2(1-\cos\theta) + qr(1-\cos\theta) - p\sin\theta \\ ,\ pr(1-\cos\theta) - q\sin\theta + qr(1-\cos\theta) + p\sin\theta + \cos\theta + r^2(1-\cos\theta) \end{pmatrix}$$

$$r_\theta(A_2) = \begin{pmatrix} \cos\theta + p^2(1-\cos\theta) - pq(1-\cos\theta) + r\sin\theta + pr(1-\cos\theta) + q\cos\theta \\ ,\ pq(1-\cos\theta) + r\sin\theta - \cos\theta - q^2(1-\cos\theta) + qr(1-\cos\theta) - p\sin\theta \\ ,\ pr(1-\cos\theta) - q\sin\theta - qr(1-\cos\theta) - p\sin\theta + \cos\theta + r^2(1-\cos\theta) \end{pmatrix}$$

If r_θ preserves $d_{\mathbf{M}}$−distance, we have equation;

$$d_{\mathbf{M}}(A_1, A_2) = d_{\mathbf{M}}(r_\theta(A_1), r_\theta(A_2)) = 2.$$

Let $(1,0,0)$ can be taken the direction vector of l in D_1. Then $(p,q,r) = (1,0,0)$. Setting $p = 1$, $q = 0$ and $r = 0$ in the equation $d_{\mathbf{M}}(r_\theta(A_1), r_\theta(A_2)) = 2$, we get $\max\{|\cos\theta|, |\sin\theta|\} = 1$. Solving this equation for $\theta \neq 0$, we obtain $\theta = \pi/2$, π or $3\pi/2$. Consequently, all Euclidean rotation about the $x-$axis with $\theta = \pi/2$, π or $3\pi/2$ is an isometry of $\mathbb{R}^3_{\mathbf{M}}$. Similarly, if the direction vector of l is one of $(0,1,0)$, $(0,0,1)$, then $\theta = \pi/2$, π or $3\pi/2$.

Let $(1,1,1)$ can be taken the direction vector of l in D_2. Then $(p,q,r) = \frac{1}{\sqrt{3}}(1,1,1)$.

Setting p, q and $r{=}1/\sqrt{3}$ in the equation $d_{\mathbf{M}}\left(r_\theta\left(A_1\right), r_\theta\left(A_2\right)\right)=2$, we get:

$$\begin{aligned} d_{\mathbf{M}}\left(r_\theta\left(A_1\right), r_\theta\left(A_2\right)\right) &= \left\{\begin{array}{r} \left|\frac{1}{3}(1-\cos\theta)-\frac{1}{\sqrt{3}}\sin\theta\right|, \left|\cos\theta+\frac{1}{3}(1-\cos\theta)\right| \\ , \left|\frac{1}{3}(1-\cos\theta)+\frac{1}{\sqrt{3}}\sin\theta\right| \end{array}\right\} \\ &=1. \end{aligned}$$

Solving above equation for $\theta \neq 0$, we get $\theta=2\pi/3$ or $4\pi/3$. Consequently, rotations r_θ according to the line l having direction $(1,1,1)$ with $\theta=2\pi/3$ or $4\pi/3$ is an isometry of $\mathbb{R}^3_{\mathbf{M}}$. Similarly, if the direction vector of l is one of (-1, 1, 1) , (1, -1, 1) , (1, 1, -1) , then $\theta=2\pi/3$ or $4\pi/3$.

Let $(1,1,0)$ can be taken the direction vector of l in D_3. Then $(p,q,r){=}\frac{1}{\sqrt{2}}(1,1,0)$. Setting $p=1/\sqrt{2},\ q=1/\sqrt{2}$ and $r=0$ in the equation $d_{\mathbf{M}}\left(r_\theta\left(A_1\right), r_\theta\left(A_2\right)\right){=}2$, we get

$$\begin{aligned} d_{\mathbf{M}}\left(r_\theta\left(A_1\right), r_\theta\left(A_2\right)\right) &= \max\left\{\left|1-\cos\theta\right|, \left|\cos\theta\right|, \left|\sin\theta\right|\right\} \\ &= 1. \end{aligned}$$

Solving above equation for $\theta \neq 0$, we get $\theta=\pi$. That is, every Euclidean rotation about the line l that has the direction $(1,1,0)$ with $\theta=\pi$ is an isometry of $\mathbb{R}^3_{\mathbf{M}}$. Similarly, if the direction vector of l is one of $(1,-1,0)$, $(1,0,1)$, $(1,0,-1)$, $(0,1,1)$, $(0,1,-1)$, then $\theta=\pi$.

Conversely, we must show that rotations $r_\theta \in R_M = R_1 \cup R_2 \cup R_3$ preserve distance $d_{\mathbf{M}}$. To show $d_{\mathbf{M}}(O,X) = d_{\mathbf{M}}(O,Y)$, we shall consider the following cases to check $\mu(OX)=\mu(OY)$. One can easily calculate $\mu(OX)=\mu(OY)$ for all possible cases as in Proposition 3. 2. For example:

rotation	$(1,0,0)$ $\theta=\pi/2$	$\frac{1}{\sqrt{3}}(1,1,1)$ $\theta=2\pi/3$	$\frac{1}{\sqrt{2}}(1,1,0)$ $\theta=\pi$	$\cdots$
(p_2,q_2,r_2)	$(p_1,-r_1,q_1)$	(r_1,p_1,q_1)	(q_1,p_1,r_1)	$\cdots$

□

Corollary 3.2. *Twenty three Euclidean rotations about the lines passing through origin are isometric rotations in* $\mathbb{R}^3_{\mathbf{M}}$.

Note that the inversion σ_O about $O=(0,0,0)$ is the transformation such that $\sigma_O(x,y,z)=(-x,-y,-z)$ for each point (x,y,z) in $\mathbb{R}^3_{\mathbf{M}}$. Also, inversion σ_O is a isometry in $\mathbb{R}^3_{\mathbf{M}}$. We use σ_O to prove following propositions.

Proposition 3.4. *There are only six rotary reflections about* O *that preserve the* $d_{\mathbf{M}}-$*distances.*

Proof. We know that the composition of a reflection in a plane and a rotation about an axis orthogonal to the plane is called a rotary reflection. A rotary reflection is determined by the reflection and an angle of rotation [2]. So, rotary reflection can be written briefly as $\rho := \sigma_\Pi\sigma_\Delta\sigma_\Gamma = \sigma_\Pi r_\theta$ such that $r_\theta \in R_M$, Γ and Δ perdendicular to Π [7]. This means that 9 rotation axes can be selected from 13 rotation axes are given in Proposition 3. 3, because vectors of the set D_2 are not normal vectors of the planes is given Corrollary 3.1. Let $A_1 = (1,1,1)$ and $A_2 = (1,-1,1)$ be two points in $\mathbb{R}^3_{\mathbf{M}}$. Then $d_{\mathbf{M}}(A_1, A_2) = 2$.

If Π is the plane having equation $x = 0$, then $(1,0,0)$ is unit direction vector of r_θ and $\rho(x,y,z) = \sigma_\Pi r_\theta(x,y,z) = (-x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta)$.

$$\begin{aligned}\rho(A_1) &= (-1, \cos\theta - \sin\theta, \sin\theta + z\cos\theta)\\ \rho(A_2) &= (-1, -\cos\theta - \sin\theta, -\sin\theta + z\cos\theta).\end{aligned}$$

Therefore,

$$d_{\mathbf{M}}(\rho(A_1), \rho(A_2)) = 2 \Leftrightarrow |2\cos\theta| + |2\sin\theta| = 2 \Leftrightarrow \theta \in \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\},$$

but one can easily obtain that $\sigma_\Pi r_\pi$ is equals to the inversiyon σ_O about $O = (0,0,0)$. Therefore, there are only two rotary reflections according to the plane $x = 0$. Similarly, two rotary reflections are obtained using the planes $y = 0$ and $z = 0$.

If Π is the plane having equation $x+y = 0$, then $\left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$ is unit direction vector of r_θ and

$$\rho(x,y,z) = \sigma_\Pi r_\theta(x,y,z) = \left(\begin{array}{r}\left(\frac{-1+\cos\theta}{2}\right)x-\left(\frac{1+\cos\theta}{2}\right)y+\left(\frac{\sin\theta}{\sqrt{2}}\right)z,\\ ,\left(\frac{-1-\cos\theta}{2}\right)x+\left(\frac{-1+\cos\theta}{2}\right)y-\left(\frac{\sin\theta}{\sqrt{2}}\right)z,\\ ,\frac{-\sin\theta}{\sqrt{2}}x+\frac{\sin\theta}{\sqrt{2}}y+(\cos\theta)z\end{array}\right).$$

Clearly

$$\begin{aligned}\rho(A_1) &= \left(-1 + \frac{\sin\theta}{\sqrt{2}}, -1 - \frac{\sin\theta}{\sqrt{2}}, \cos\theta\right)\\ \rho(A_2) &= \left(\cos\theta + \frac{\sin\theta}{\sqrt{2}}, -\cos\theta - \frac{\sin\theta}{\sqrt{2}}, \cos\theta - \sqrt{2}\sin\theta\right)\end{aligned}$$

and

$$d_{\mathbf{M}}(\rho(A_1), \rho(A_2)) = 2 \Leftrightarrow \max\left\{|1+\cos\theta|, |-1+\cos\theta|, \left|\sqrt{2}\sin\theta\right|\right\} = 2 \Leftrightarrow \theta \in \{0, \pi\},$$

but one can easily obtain that $\sigma_\Pi r_\pi$ is equals to the inversiyon σ_O about $O = (0,0,0)$. This means that if $\theta = \pi$, then there is no new rotary reflection. Similarly, if Π are the planes having equations $x-y=0$, $x+z=0$, $x-z=0$, $y+z=0$, $y-z=0$, there is no new rotary reflection. □

Proposition 3.5. *There are only eight rotary inversions about O that preserve the $d_{\mathbf{M}}$−distances.*

Proof. We know that a rotary inversions is defined by $\rho := \sigma_O\sigma_\Delta\sigma_\Gamma = \sigma_O r_\theta$ such that $r_\theta \in R_M$. To show isometric rotary inversions, we have to consider 13 axes of rotations is given in Proposition 3. 3.

If r_θ represents the rotations about the x−axis, then $(1,0,0)$ is the unit direction vector of r_θ and

$$\rho(x,y,z) = \sigma_O r_\theta(x,y,z) = (-x, -y\cos\theta + z\sin\theta, -y\sin\theta - z\cos\theta).$$

Consequently,

$$\rho(A_1) = (-1, -\cos\theta + \sin\theta, -\sin\theta - \cos\theta)$$
$$\rho(A_2) = (-1, \cos\theta + \sin\theta, \sin\theta - \cos\theta)$$

and

$$d_{\mathbf{M}}(\rho(A_1), \rho(A_2)) = 2 \Leftrightarrow \max\{|2\cos\theta|, |2\sin\theta|\} = 2 \Leftrightarrow \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}.$$

One can easily obtain that $\sigma_\Pi r_\pi$ is equals to a rotary reflection or a reflection. This means that if $\theta = \frac{\pi}{2}$, π and $\frac{3\pi}{2}$, then there is no new rotary inversion. Similarly, if r_θ represents the rotations about the y, $z-$axis, then there is no new rotary inversion.

If r_θ represents the rotations about the parallel to $(1,1,0)$, then $\left(1/\sqrt{2}, 1/\sqrt{2}, 0\right)$ is unit direction vector r_θ and

$$\rho(x,y,z) = \sigma_O r_\theta(x,y,z) = \begin{pmatrix} \left(\frac{-1-\cos\theta}{2}\right)x + \left(\frac{-1+\cos\theta}{2}\right)y - \left(\frac{\sin\theta}{\sqrt{2}}\right)z, \\ ,\left(\frac{-1+\cos\theta}{2}\right)x + \left(\frac{-1-\cos\theta}{2}\right)y + \left(\frac{\sin\theta}{\sqrt{2}}\right)z, \\ ,\frac{\sin\theta}{\sqrt{2}}x - \frac{\sin\theta}{\sqrt{2}}y - \cos\theta z \end{pmatrix}.$$

Clearly

$$\rho(A_1) = \left(-1 - \frac{\sin\theta}{\sqrt{2}}, -1 + \frac{\sin\theta}{\sqrt{2}}, -\cos\theta\right)$$
$$\rho(A_2) = \left(-\cos\theta - \frac{\sin\theta}{\sqrt{2}}, \cos\theta + \frac{\sin\theta}{\sqrt{2}}, -\cos\theta + \sqrt{2}\sin\theta\right)$$

and

$$d_{\mathbf{M}}(\rho(A_1), \rho(A_2)) = 2 \Leftrightarrow \max\left\{|-1+\cos\theta|, |1+\cos\theta|, \left|\sqrt{2}\sin\theta\right|\right\} = 2 \Leftrightarrow \theta \in \{0, \pi\}.$$

but one can easily obtain that $\sigma_O r_\pi$ is equals to a reflection. This means that if $\theta = \pi$, then there is no new rotary inversion. Similarly, it is easily seen that there is no new rotary inversion if r_θ is any of the remaining rotation axes parallel to $(1,-1,0)$, $(1,0,1)$, $(1,0,-1)$, $(0,1,1)$, $(0,1,-1)$.

If r_θ represents the rotations about the parallel to $(1,1,1)$, then $\frac{1}{\sqrt{3}}(1,1,1)$ is the unit direction vector of r_θ and $\rho(x,y,z) = \sigma_O r_\theta(x,y,z)$ is equals to

$$\begin{pmatrix} \left(\frac{-1-2\cos\theta}{3}\right)x + \left(\frac{-1+\cos\theta+\sqrt{3}\sin\theta}{3}\right)y + \left(\frac{-1+\cos\theta-\sqrt{3}\sin\theta}{3}\right)z, \\ \left(\frac{-1+\cos\theta-\sqrt{3}\sin\theta}{3}\right)x + \left(\frac{-1-2\cos\theta}{3}\right)y + \left(\frac{-1+\cos\theta+\sqrt{3}\sin\theta}{3}\right)z, \\ \left(\frac{-1+\cos\theta+\sqrt{3}\sin\theta}{3}\right)x + \left(\frac{-1+\cos\theta-\sqrt{3}\sin\theta}{3}\right)y + \left(\frac{-1-2\cos\theta}{3}\right)z \end{pmatrix}$$

Clearly

$$\rho(A_1) = (-1,-1,-1)$$
$$\rho(A_2) = \left(\frac{-1-2\cos\theta-2\sqrt{3}\sin\theta}{3}, \frac{-1+4\cos\theta}{3}, \frac{-1-2\cos\theta+2\sqrt{3}\sin\theta}{3}\right)$$

and

$$d_{\mathbf{M}}(\rho(A_1),\rho(A_2))=2 \quad \Leftrightarrow \max\left\{\left|\frac{-2+2\cos\theta+2\sqrt{3}\sin\theta}{3}\right|, \left|\frac{4-4\cos\theta}{3}\right|, \left|\frac{4+2\cos\theta-2\sqrt{3}\sin\theta}{3}\right|\right\}=2$$
$$\Leftrightarrow \theta \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

Therefore, we obtain that only two rotary inversion according to rotation about the axis parallel to $(1,1,1)$. Similarly, it is easily obtained that there are two new rotary inversions each of the remaining rotation axes parallel to $(-1,1,1), (1,-1,1), (1,1,-1)$. That is, there are eight rotary inversions that preserve $d_{\mathbf{M}}-$distances. □

It can be easily check that $\sigma_O\sigma_\Delta = r_\pi$, $r_\pi \in R_1 \cup R_3$. Thus we have the **octahedral group**, O_h, consisting of nine reflections about planes, twenty-three rotations, six rotary reflections, eight rotary inversions, one inversion and the identity. That is, the Euclidean symmetry group of the cube.

Now, let us show that all isometries of $\mathbb{R}^3_{\mathbf{M}}$ are in $T(3).O_h$.

Definition 3.1. Given $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ points in $\mathbb{R}^3_{\mathbf{M}}$. The minimum distance set of A, B is defined by

$$\{X : d_{\mathbf{M}}(A,X) + d_{\mathbf{M}}(X,B) = d_{\mathbf{M}}(A,B)\}$$

and is denoted by $[AB]$.

Proposition 3.6. *If $\phi : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ is an isometry, then*

$$\phi([AB]) = [\phi(A)\,\phi(B)] \ .$$

Proof. Let $Y \in \phi([AB])$. Then,

$$\begin{aligned} Y \in \phi([AB]) \quad &\Leftrightarrow \exists X \ni Y = \phi(X) \\ &\Leftrightarrow d_{\mathbf{M}}(A,X) + d_{\mathbf{M}}(X,B) = d_{\mathbf{M}}(A,B) \\ &\Leftrightarrow d_{\mathbf{M}}(\phi(A),\phi(X)) + d_{\mathbf{M}}(\phi(X),\phi(B)) = d_{\mathbf{M}}(\phi(A),\phi(B)) \\ &\Leftrightarrow Y = \phi(X) \in [\phi(A)\,\phi(B)] \ . \end{aligned}$$

□

Corollary 3.3. *Let $\phi : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ be an isometry. Then ϕ maps vertices to vertices and preserves the lengths of edges of $[AB]$.*

Proposition 3.7. *Let $f : \mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ be an isometry such that $f(O) = O$. Then f is in O_h.*

Proof. Let $A_1 = (1,1,1)$, $A_2 = (1,-1,1)$, $A_5 = (-1,1,1)$, $A_6 = (-1,-1,1)$ and $D = (0,0,2)$. Consider the minimum distance set $[OD]$ with corner point D (see

Figure 3.1).

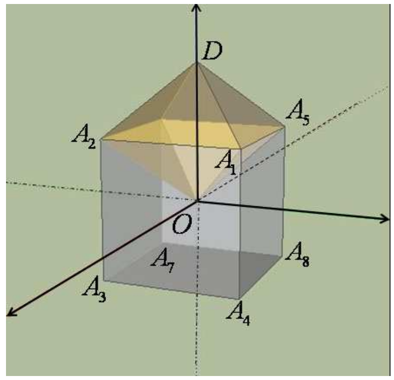

Figure 3.1

So, $f(A_1) \in A_iA_j,\ i \neq j,\ i,j \in \{1,2,3,4,5,6,7,8\}$. Here the points A_i and A_j are not on the same coordinate axis. Since f is an isometry by Corollary 3.3, $f(A_1),\ f(A_2),\ f(A_5)$ and $f(A_6)$ must be the vertices of the minimum distance set with corner point D and origin. Therefore, if $f(A_1) \in A_iA_j$, then $f(A_1) = A_i$ or $f(A_1) = A_j$. Similarly $f(A_2) = A_i$ or $f(A_2) = A_j,\ f(A_5) = A_i$ or $f(A_5) = A_j$ and $f(A_6) = A_i$ or $f(A_6) = A_6$. Also any three of $f(A_1),\ f(A_2),\ f(A_5)$ or $f(A_6)$ is not on the same coordinate axis. Now the following eight cases are possible;

$$f(A_1) = A_1 \Rightarrow \begin{cases} f(A_2) = A_2 & , f(A_5) = A_5 & , f(A_6) = A_6 \\ f(A_2) = A_4 & , f(A_5) = A_5 & , f(A_6) = A_6 \\ f(A_2) = A_4 & , f(A_5) = A_2 & , f(A_6) = A_3 \\ f(A_2) = A_5 & , f(A_5) = A_2 & , f(A_6) = A_6 \\ f(A_2) = A_5 & , f(A_5) = A_4 & , f(A_6) = A_8 \end{cases}$$

$$f(A_1) = A_2 \Rightarrow \begin{cases} f(A_2) = A_1 & , f(A_5) = A_6 & , f(A_6) = A_5 \\ f(A_2) = A_1 & , f(A_5) = A_3 & , f(A_6) = A_4 \\ f(A_2) = A_3 & , f(A_5) = A_1 & , f(A_6) = A_4 \\ f(A_2) = A_3 & , f(A_5) = A_6 & , f(A_6) = A_7 \\ f(A_2) = A_6 & , f(A_5) = A_1 & , f(A_6) = A_5 \\ f(A_2) = A_6 & , f(A_5) = A_3 & , f(A_6) = A_7 \end{cases}$$

$$f(A_1) = A_3 \Rightarrow \begin{cases} f(A_2) = A_2 & , f(A_5) = A_4 & , f(A_6) = A_1 \\ f(A_2) = A_2 & , f(A_5) = A_7 & , f(A_6) = A_6 \\ f(A_2) = A_4 & , f(A_5) = A_2 & , f(A_6) = A_1 \\ f(A_2) = A_4 & , f(A_5) = A_7 & , f(A_6) = A_8 \\ f(A_2) = A_7 & , f(A_5) = A_2 & , f(A_6) = A_6 \\ f(A_2) = A_7 & , f(A_5) = A_4 & , f(A_6) = A_8 \end{cases}$$

$$f(A_1) = A_4 \Rightarrow \begin{cases} f(A_2) = A_1 & , f(A_5) = A_3 & , f(A_6) = A_2 \\ f(A_2) = A_1 & , f(A_5) = A_8 & , f(A_6) = A_5 \\ f(A_2) = A_3 & , f(A_5) = A_1 & , f(A_6) = A_2 \\ f(A_2) = A_3 & , f(A_5) = A_8 & , f(A_6) = A_7 \\ f(A_2) = A_8 & , f(A_5) = A_1 & , f(A_6) = A_5 \\ f(A_2) = A_8 & , f(A_5) = A_3 & , f(A_6) = A_7 \end{cases}$$

$$f(A_1)=A_5 \Rightarrow \begin{cases} f(A_2)=A_1 & , f(A_5)=A_6 & , f(A_6)=A_2 \\ f(A_2)=A_1 & , f(A_5)=A_8 & , f(A_6)=A_4 \\ f(A_2)=A_6 & , f(A_5)=A_1 & , f(A_6)=A_2 \\ f(A_2)=A_6 & , f(A_5)=A_8 & , f(A_6)=A_7 \\ f(A_2)=A_8 & , f(A_5)=A_6 & , f(A_6)=A_7 \\ f(A_2)=A_8 & , f(A_5)=A_1 & , f(A_6)=A_4 \end{cases}$$

$$f(A_1)=A_6 \Rightarrow \begin{cases} f(A_2)=A_2 & , f(A_5)=A_5 & , f(A_6)=A_1 \\ f(A_2)=A_2 & , f(A_5)=A_7 & , f(A_6)=A_3 \\ f(A_2)=A_5 & , f(A_5)=A_2 & , f(A_6)=A_1 \\ f(A_2)=A_5 & , f(A_5)=A_7 & , f(A_6)=A_8 \\ f(A_2)=A_7 & , f(A_5)=A_2 & , f(A_6)=A_3 \\ f(A_2)=A_7 & , f(A_5)=A_5 & , f(A_6)=A_8 \end{cases}$$

$$f(A_1)=A_7 \Rightarrow \begin{cases} f(A_2)=A_3 & , f(A_5)=A_6 & , f(A_6)=A_2 \\ f(A_2)=A_3 & , f(A_5)=A_8 & , f(A_6)=A_4 \\ f(A_2)=A_6 & , f(A_5)=A_3 & , f(A_6)=A_2 \\ f(A_2)=A_6 & , f(A_5)=A_8 & , f(A_6)=A_5 \\ f(A_2)=A_8 & , f(A_5)=A_6 & , f(A_6)=A_5 \\ f(A_2)=A_8 & , f(A_5)=A_3 & , f(A_6)=A_4 \end{cases}$$

$$f(A_1)=A_8 \Rightarrow \begin{cases} f(A_2)=A_4 & , f(A_5)=A_5 & , f(A_6)=A_1 \\ f(A_2)=A_4 & , f(A_5)=A_7 & , f(A_6)=A_3 \\ f(A_2)=A_5 & , f(A_5)=A_4 & , f(A_6)=A_1 \\ f(A_2)=A_5 & , f(A_5)=A_7 & , f(A_6)=A_6 \\ f(A_2)=A_7 & , f(A_5)=A_5 & , f(A_6)=A_6 \\ f(A_2)=A_7 & , f(A_5)=A_4 & , f(A_6)=A_3 \end{cases}$$

In each case it is easy to show that f is unique and is O_h. For instance in the first case:

If $f(A_1)=A_1,\ f(A_2)=A_2,\ f(A_5)=A_5,\ f(A_6)=A_6$, then f is the identity.

If $f(A_1)=A_1,\ f(A_2)=A_4,\ f(A_5)=A_5,\ f(A_6)=A_6$, then $f=\sigma_\Delta$ such that $\Delta : y-z=0$.

If $f(A_1)=A_1,\ f(A_2)=A_4,\ f(A_5)=A_2,\ f(A_6)=A_3$, then $f=r_{2\pi/3}$ with rotation axis $\parallel (1,1,1)$.

If $f(A_1)=A_1,\ f(A_2)=A_5,\ f(A_5)=A_2,\ f(A_6)=A_6$, then $f=\sigma_\Delta$ such that $\Delta : x-y=0$.

If $f(A_1)=A_1,\ f(A_2)=A_5,\ f(A_5)=A_4,\ f(A_6)=A_8$, then $f=r_{4\pi/3}$ with rotation axis $\parallel (1,1,1)$.

The proofs of the remaining cases are quite similar to that of the first case. □

Theorem 3.1. *Let $f:\mathbb{R}^3_{\mathbf{M}} \to \mathbb{R}^3_{\mathbf{M}}$ be an isometry. Then there exists a unique $T_A \in T(3)$ and $g \in O_h$ such that $f = T_A \circ g$.*

Proof. Let $f(O)=A$ where $A=(a_1,a_2,a_3)$. Define $g=T_{-A}\circ f$. We know that g is an isometry and $g(O)=O$. Thus, $g\in O_h$ and $f=T_A\circ g$ by Proposition 3.7. The proof of uniqueness is trivial. □

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